

# Optimal Selling Mechanisms with Countervailing Positive Externalities and an Application to Tradable Retaliation in the WTO \*

Bo Chen<sup>†</sup> and Tanapong Potipiti<sup>‡</sup>

March 20, 2010

## Abstract

We study revenue-maximizing mechanisms for a seller who sells an indivisible good to several buyers with *positive, type-dependent and countervailing allocative externalities*. To cope with the difficulty of types obtaining reservation utilities being endogenously determined, we first solve a minimax version of the seller's problem by generalizing Myerson's characterization techniques for the non-regular case. The solution is then shown to solve the seller's original maximin problem as well in our setting. We find that the seller's optimal mechanism features bunching even in the regular case and the type with the lowest expected payoff is typically not an extreme type. As an important illustration of our characterization procedures, we apply our results to the problem of selling retaliation rights in the WTO.

## 1 Introduction

This article characterizes optimal selling mechanisms for an auction design problem with private information and *positive* type-dependent countervailing externalities.

---

\*We are especially grateful to an anonymous referee for very constructive and helpful comments. We are also grateful to Kyle Bagwell and Charles Zheng for invaluable discussions and encouragement, to Robert Staiger for his advice and guidance on an early version of the paper, and to Yeon-Cho Che, John Kennan, Kiriya Kulkolkarn, Vijit Kunapongkul, and Kamal Saggi for comments. A previous version of the paper was titled "Selling Retaliation Rights in the WTO — How Difficult Is It?"

<sup>†</sup>Department of Economics, Southern Methodist University, 3300 Dyer Street Suite 301, Umphrey Lee Center, Dallas TX 75275-0496; E-mail: bochen@smu.edu; Tel.: 1-214-768-2715; Fax: 1-214-768-1821.

<sup>‡</sup>Faculty of Economics, Chulalongkorn University, Bangkok, Thailand. E-mail: Tanapong.P@chula.ac.th, Tel.: 662-218-6188 ext. 2, Fax: 662-218-6235.

Our motivation for studying such auction design problems mainly comes from Mexico's 2002 proposal of making retaliation right tradable in the World Trade Organization (WTO).<sup>1</sup> The main purpose of Mexico's proposal is to enforce multilateral retaliation, by making retaliation rights tradeable, so as to address a well-known problem in the WTO dispute settlement procedures that small countries find it difficult to retaliate against large countries when the latter violates trade agreements. A distinguishing feature of making retaliation right tradeable in the WTO, as first pointed out by Bagwell, Mavroidis and Staiger (2007), is that such auctions are characterized by positive externalities, as retaliation exercised by the country that wins the retaliation right affects the world price, and hence the terms of trade for another bidding country. Such a novel form of positive externalities creates interesting countervailing incentives for bidders in the auction, as well as difficulties in characterizing the optimal selling mechanisms.

We develop a fairly general mechanism design model with such positive countervailing externalities. To be specific, we consider a seller sells an indivisible good to several potential buyers who enjoys (allocative) positive externalities when another buyer receives the good. Different from the previous literature, our allocative positive externalities are *not* generated by outside options, and the intensity of the externalities is closely related to each bidder's private type (type-dependent). In particular, buyers' payoffs, always non-negative in feasible allocations, are specified as follows: a winning buyer's payoff is *increasing in her type*, while a losing buyer's payoff is *decreasing in her type*.<sup>2</sup> Such a seemingly minor feature, not captured previously as detailed in Section 1.1, results in an enormous difficulty in the characterization of the seller's optimal mechanisms as illustrated below.

First observe that our externality specification generates countervailing incentives for types in the reporting stage of the mechanism, as winning and losing payoffs vary in types in opposite directions. Consequently, the type receiving reservation utility, or an expected payoff of zero, in the mechanism has to be endogenously determined as a solution of some minimization problem of the allocation function, which is itself a choice variable for the seller to design optimal mechanisms. This distinguishing feature creates a non-trivial technical difficulty in characterizing the seller's optimal mechanisms, and renders the problem not solvable under previous characterization techniques, including Myerson's ironing approach for the general non-regular case.

To address this difficulty in the characterization, we consider first the minimax version of the seller's optimization problem: For each *exogenously* given type with payoff zero, we modify Myerson's ironing technique to characterize the corresponding "optimal" allocation rule. This rule is then used to derive the type with reservation

---

<sup>1</sup>Our results apply also to auction problems where the outcome of the auction affects future interactions of the current bidders, generating allocative externalities among the bidders in the current auction. See Jehiel and Moldovanu (2000) and Jehiel, Moldovanu and Stacchetti (1999) for further details and examples.

<sup>2</sup>A buyer wins if she obtains the good and a buyer loses if another buyer receives the good.

utility as a *fixed point* of some minimization problem.<sup>3</sup> As a second step, we invoke and apply an important minimax theorem by Terkelsen (1972), a generalization of von Neumann’s minimax theorem, to show that the solution obtained from the minimax problem solves exactly the seller’s original optimization problem, which is observed to be a *maximin* problem in our setting.

We find such optimal mechanisms as characterized above feature bunching even in the *regular* case where the monotone hazard rate assumption holds.<sup>4</sup> Such bunching occurs in the optimal mechanism to serve the following purposes: First, bunching arises in a set of intermediate types in the regular case (as a result of countervailing externalities) so as to address the conflict between individual rationality constraint and minimization of informational rents; Second, in the non-regular case, bunching also arises in regions where virtual surplus is non-increasing so as to relax the incentive constraints of the buyers. Consequently, in the optimal mechanism, the type with zero payoff is typically an *interior* type and each buyer’s payoff is in general non-monotonic in types. Importantly, the optimal allocation rule also features *ex post inefficiency* for the following reasons: First, as in the standard case without externalities, the seller may sometimes keep the good (for example, when positive externalities are very small so that we are close to the standard case in Myerson (1981)); Second, as bunching *always* arises in the optimal mechanism, the seller may randomly assign the good to the buyers when types are in some set with a positive measure.<sup>5</sup>

We next apply our characterization procedures to the problem of selling retaliation rights in the WTO. We first follow Bagwell, Mavroidis and Staiger (2007) to develop a three-country trade model where an exporting country exports a common good to two importing countries. When the exporting country violates its WTO obligations with some (unmodelled) small country, the unmodelled country is granted a retaliation right by the WTO. This country can then sell the retaliation right to the two importing countries, who are able and willing to exercise the right effectively. As exercising the retaliation right against the exporting country by a winning (importing) country will decrease the world price of the common good, a losing (importing) country hence also benefits from not obtaining the retaliation right, with the magnitude of the benefit being *negatively* related to the losing country’s *private political-economy parameter*, which captures how much a government cares about income distribution between consumers and producers. The unmodelled country’s problem of selling the retaliation right hence exhibits the type of countervailing positive externalities studied in our general model.

---

<sup>3</sup>Notice that even in the general non-regular case in Myerson (1981), the seller’s “virtual surplus” function for each buyer does not depend on another buyer’s type, which is *not* the case in our setting with allocative positive externalities. This creates a second difficulty in our characterization as we shall see.

<sup>4</sup>In our characterization, we allow for both the regular case and the non-regular case as distinguished in Myerson (1981).

<sup>5</sup>These features can be seen more clearly in Section 4 where we apply our characterization procedures to solve explicitly a country’s optimal mechanism in selling retaliation rights.

Our (workable) theory on such mechanism design problems enables us to obtain a closed form solution of the optimal selling mechanism for selling retaliation rights in the WTO under the uniform distribution assumption. We find the optimal allocation rule is deterministic except in a neighborhood around *intermediate* types (who receive the reservation utility) of the two buyers, where the seller allocates the good to the buyers *randomly*. Such randomizations enable the seller to extract more surplus from extreme types, who have larger incentives to misreport.

## 1.1 Related Literature

This paper straddles two separate strands of the previous literature: optimal mechanism design with private information and externalities, and the international trade theory on the WTO.

First, there have been several important studies on optimal mechanism design with externalities, where the existing literature has mainly focused on *negative* externality environments (for example, Jehiel, Moldovanu and Stacchetti (1996), Jehiel, Moldovanu and Stacchetti (1999) and Brocas (2007)). Jehiel, Moldovanu and Stacchetti (1999) characterize optimal multidimensional incentive compatible mechanisms in an auction with (mainly negative) externalities where a buyer's type is multidimensional and specifies the allocative externalities of the buyer for each possible allocation outcomes. Although Jehiel, Moldovanu and Stacchetti (1999) also allow for the possibility of positive externalities, the positive payoff a losing buyer enjoys is modelled as a separate random variable that is *independent* of the buyer's intrinsic (private) value of the good. In our model, however, the positive externality of a losing buyer is dependent on and, more importantly, decreasing in, the buyer's intrinsic value of the good. This, as mentioned, creates countervailing incentives for buyers in reporting their private information and technical difficulties in the characterization of the optimal mechanisms, which are not present in their study.

Aseff and Chade (2008) analyze an optimal auction problem where a seller sells multiple identical units of a good to several buyers with unit demand and identity-dependent (positive and negative) externalities. In their study of the case with positive externalities, each buyer's valuation depends on her intrinsic value of one unit of the good, as well as who obtains the other unit of the good, unlike our specification where a buyer only receives positive externalities when she does not win the good. More importantly, the (identity-dependent) positive externalities in Aseff and Chade (2008) are strictly increasing in a buyer's valuation of the good. The countervailing incentives to misreport are consequently not present in their model either, rendering their optimal mechanisms always deterministic.

Figueroa and Skreta (2009a) and Brocas (2007) analyze optimal auctions with (negative) externalities. Figueroa and Skreta (2009b) study auction design with both positive and negative externalities in a very general setting to identify interesting classes of problems that are *solvable* via Myerson-like techniques. Additionally, a

common feature shared by these studies is an emphasis on the role played by (privately known) *outside options* in the optimal allocation, which is different from the countervailing allocative positive externalities in our setting.

Finally, our paper is also related to the literature of optimal contracting with type-dependent reservation utilities, most notably, by Lewis and Sappington (1989), Maggi and Rodriguez (1995) and Jullien (2000). Some features of our optimal selling mechanisms also appear in the optimal contracts with type-dependent outside options. The main difference between our paper and this literature is that we study optimal selling mechanisms where the seller sells an indivisible good to *multiple* buyers and we adopt a qualitatively different approach in our characterization.

For the strand of international trade literature on the WTO, we first provide a brief background on the WTO issues that are related to our application section, and then discuss Bagwell, Mavroidis and Staiger (2007), which is the most closely related paper in this literature.

As an international organization, the main purpose of the WTO is to promote international trade cooperation and to offer means to solve the terms-of-trade problem.<sup>6</sup> For such cooperation to be sustained, the design of dispute settlement procedures is crucial. Although the WTO is a multilateral organization, its enforcement system is mainly bilateral. However, a well-known problem in the WTO dispute settlement procedures is that small countries find it difficult to retaliate against large countries: Even when authorized to do so, small countries such as the Netherlands and Ecuador, have not actually implemented retaliatory responses. Those concerned with this matter have supported moves to multilateral retaliation, e.g., Lawrence (2003) and Maggi (1999). A natural way to multilateralize retaliation is to allow such retaliations to be tradeable, as proposed by Mexico in WTO (2002). In this proposal, if a country is authorized to retaliate by increasing its tariff against a country, it would be able to sell the retaliation right to another interested country.

Bagwell, Mavroidis and Staiger (2007) is the first analytical study on outcomes of selling retaliation rights in the WTO in first-price auctions. They identify the type of countervailing positive externalities among buyers (importing countries) and they also show that because of such externalities, retaliation rights sometimes cannot be sold and allowing retaliation to be tradable might therefore not serve its purposes.<sup>7</sup> We follow Bagwell, Mavroidis and Staiger (2007) to construct a trade model in our application section. Our focus, however, is on characterizing the features of an optimal selling procedure in the WTO, which can be regarded as a natural question arising from their findings.

The rest of the article proceeds as follows. Section 2 presents our general model. Section 3 simplifies the seller's problem and characterizes the optimal mechanisms.

---

<sup>6</sup>This view was formalized in Bagwell and Staiger (1999).

<sup>7</sup>Bagwell, Mavroidis and Staiger (2007) also extend the basic first-price auction to allow the exporting country to bid to retire the retaliation right. They show that in this setting, there is no auction failure: the exporting country always wins and the retaliation right is always retired.

Section 4 applies our characterization results to the tradeable retaliation problem in the WTO. Section 5 concludes. All proofs are relegated to an Appendix.

## 2 The Model

Consider a mechanism design problem where a seller sells an indivisible object to two buyers. The value of the object to the seller is  $x_0$ , which is assumed to be 0. For each buyer  $i$ , nature draws a type  $x_i$ ,  $i \in \{1, 2\}$ , from a commonly known distribution  $F$ , and support  $X = [\underline{x}, \bar{x}] \subset \mathbb{R}_+$ , where we assume that types are independent across  $i$  and  $F$  has a continuous strictly positive density  $f$  on  $X$ . We denote  $x := (x_i)_{i \in \{1, 2\}} \in X^2$ . The defining feature of our optimal mechanism design problem is that each buyer receives *positive allocative externalities* when the other obtains the object. To be specific, buyer  $i$ 's payoff when  $i$  receives the object is  $\omega(x_i) = A + ax_i$ , while buyer  $i$ 's payoff when buyer  $j$  receives the object is  $\lambda(x_i) = B - bx_i$ , where  $a$  and  $b$  are positive constants.<sup>8</sup> To focus on the case of positive externalities, we assume that constants  $A$  and  $B$  are such that  $\lambda(x_i), \omega(x_i) \geq 0$  for all  $x_i \in X$ . Finally, if the seller keeps the good, both buyers receive zero payoff.

As mentioned in the introduction, our (unidimensional) specification of the positive externalities is motivated by selling retaliation rights in the WTO, where if an importing country wins the retaliation right, another importing country benefits from a decrease of the world price, resulting from the exercise of the retaliation right. In particular, such externality specification is different from that in Jehiel, Moldovanu and Stacchetti (1999) where the allocative externality (a buyer's payoff when another buyer obtains the auctioned object) is modeled as a *separate* random variable that is *independent* of the buyer's private value of the object. In our specification, however, buyers' outside option yields a zero payoff, and a losing buyer's payoff is *perfectly correlated* with and *decreasing in* his private intrinsic value of the auctioned object. Such type of allocative externalities is hence *not* subsumed by the general specification in Jehiel, Moldovanu and Stacchetti (1999).

The seller is assumed to be able to perfectly commit to any outcome of the selling mechanism. Hence, the revelation principle holds. It is then without loss of generality to restrict our search for the optimal selling scheme to direct revelation mechanisms that are incentive compatible and individually rational. Let the set of possible allocations be  $\Phi := \{(y, z) \mid y, z \geq 0 \text{ and } y + z \leq 1\}$ . A direct mechanism, denoted as a pair  $(\mathbf{Q}, \mathbf{M})$ , then consists of an allocation rule  $\mathbf{Q} : X^2 \rightarrow \Phi$  and a payment function  $\mathbf{M} : X^2 \rightarrow \mathbb{R}^2$ , where  $\mathbf{Q}(x) = (Q_1(x), Q_2(x))$ ,  $\mathbf{M}(x) = (M_1(x), M_2(x))$ ,  $Q_i(x)$  is the *ex post* probability that buyer  $i$  receives the object, and  $M_i(x)$  is the *ex post* payment buyer  $i$  pays to the seller.

---

<sup>8</sup>Hereafter, we sometimes use  $i$  and  $j$  to denote the two buyers, where  $i, j \in \{1, 2\}$  and  $i \neq j$ .

Given the above, buyer  $i$ 's *interim* expected utility can be written as

$$u_i(x_i) = \int_{\underline{x}}^{\bar{x}} [Q_i(x)\omega(x_i) + Q_j(x)\lambda(x_i) - M_i(x)] f(x_j) dx_j.$$

Next, we define

$$\begin{aligned} q_i(x_i) &: = \int_{\underline{x}}^{\bar{x}} Q_i(x) f(x_j) dx_j, \\ \bar{q}_i(x_i) &: = \int_{\underline{x}}^{\bar{x}} Q_j(x) f(x_j) dx_j, \\ m_i(x_i) &: = \int_{\underline{x}}^{\bar{x}} M_i(x) f(x_j) dx_j, \end{aligned} \tag{1}$$

where  $q_i(x_i)$  (resp.,  $\bar{q}_i(x_i)$ ) is the *interim* probability that buyer  $i$  (resp., buyer  $j$ ) receives the object, and  $m_i(x_i)$  is buyer  $i$ 's *interim* expected payment, from the mechanism  $(\mathbf{Q}, \mathbf{M})$  when buyer  $i$  reports  $x_i$ . Given these definitions,  $i$ 's *interim* expected utility can be alternatively written as:

$$u_i(x_i) = q_i(x_i)\omega(x_i) + \bar{q}_i(x_i)\lambda(x_i) - m_i(x_i). \tag{2}$$

The seller's maximization problem can then be written as:

$$\begin{aligned} P_1 : \max_{\mathbf{Q}, \mathbf{M}} & \int_{\underline{x}}^{\bar{x}} \int_{\underline{x}}^{\bar{x}} \left[ x_0 \left( 1 - \sum_i Q_i(x) \right) + \sum_i M_i(x) \right] f(x_1) f(x_2) dx_1 dx_2 \\ & \mathbf{Q}(x) \in \Phi, \forall x \quad (F) \\ \text{s.t. } & u_i(x_i) \geq 0, \forall (i, x_i) \quad (IR) \\ & u_i(x_i) \geq q_i(s_i)\omega(x_i) + \bar{q}_i(s_i)\lambda(x_i) - m_i(s_i), \forall (i, x_i, s_i) \quad (IC) \end{aligned},$$

where the three constraints are, respectively, feasibility ( $F$ ), individual rationality ( $IR$ ), and incentive compatibility ( $IC$ ). Furthermore, we call selling mechanisms that satisfy these three constraints *feasible mechanisms*.

### 3 Optimal Selling Mechanisms

We now characterize the seller's optimal selling mechanisms. As is standard in optimal auction theory, we start with some simplifications of the maximization problem  $P_1$ .

Our first result, Lemma 1, is a standard result in optimal auction design and it characterizes the set of feasible mechanisms.

**Lemma 1** *A selling mechanism  $(\mathbf{Q}, \mathbf{M})$  is feasible if and only if the following conditions hold:*

$$aq_i(x_i) - b\bar{q}_i(x_i) \text{ is non-decreasing in } x_i, \forall i, \quad (3)$$

$$u_i(x_i) = u_i(\underline{x}) + \int_{\underline{x}}^{x_i} [aq_i(t) - b\bar{q}_i(t)] dt \geq 0, \forall x_i, \forall i, \quad (4)$$

$$\mathbf{Q}(x) \in \Phi, \forall x. \quad (5)$$

Notice that Lemma 1, though standard, marks two essential differences from the standard case of no externalities ( $b = 0$ ) in Myerson (1981):

First, in the standard case, incentive compatibility reduces to the associated interim winning probability being non-decreasing in types, as buyers only care about winning the object. In our case of positive externality, incentive compatibility is, however, equivalent to the monotonicity (in types) of some “net weighted winning” probability, where the weights can be regarded as the measures of (opposing) payoff and hence incentive intensities. This is a result of the fact that buyers also care about losing the auction, which brings positive payoffs as well.

Second, in the standard case, individual rationality constraint binds for the lowest type. With countervailing positive externalities, however, the type whose individual rationality constraint is binding has to be *endogenously determined* and is typically not the lowest type  $\underline{x}$ , as we shall see (Lemma 2).

Given Lemma 1, the seller’s expected revenue from buyer  $i$  is now written as:

$$E[m_i(x_i)] = \int_X [q_i(z)\omega(z) + \bar{q}_i(z)\lambda(z)] dF(z) - u_i(\underline{x}) - \int_X \int_{\underline{x}}^z (aq_i(t) - b\bar{q}_i(t)) dt dF(z).$$

This can be more compactly expressed as<sup>9</sup>

$$E[m_i(x_i)] = \int_X [q_i(z)v_\omega(z) + \bar{q}_i(z)v_\lambda(z)] f(z) dz - u_i(\underline{x}),$$

where we define *the virtual surplus of winning* as  $v_\omega(x_i)$  and *the virtual surplus of losing* as  $v_\lambda(x_i)$ :

$$\begin{aligned} v_\omega(x_i) &= \omega(x_i) - a \frac{1 - F(x_i)}{f(x_i)}, \\ v_\lambda(x_i) &= \lambda(x_i) + b \frac{1 - F(x_i)}{f(x_i)}. \end{aligned}$$

---

<sup>9</sup>A standard interchanging the order of integration in the last term gives that

$$E[m_i(x_i)] = \int_X [q_i(z)\omega(z) + \bar{q}_i(z)\lambda(z)] dF(z) - u_i(\underline{x}) - \int_X (1 - F(z))(aq_i(z) - b\bar{q}_i(z)) dz.$$

Given the above results, the seller's maximization problem  $P_1$  can be equivalently written as  $P_2$  :<sup>10</sup>

$$\begin{aligned}
P_2 : \quad & \max_{\mathbf{Q}, u_i(\underline{x})} \sum_i \int_X \int_X [v_\omega(x_i) + v_\lambda(x_j)] Q_i(x) f(x_i) f(x_j) dx_i dx_j - \sum_i u_i(\underline{x}) \\
& \mathbf{Q}(x) \in \Phi, \forall x \quad (F) \\
& \text{s.t. } aq_i(x_i) - b\bar{q}_i(x_i) \text{ is non-decreasing in } x_i, \forall i \quad (ND) \\
& u_i(x_i) = u_i(\underline{x}) + \int_{\underline{x}}^{x_i} (aq_i(t) - b\bar{q}_i(t)) dt \geq 0, \forall (i, x_i) \quad (IR)
\end{aligned}$$

To further simplify the maximization problem, we present a simple lemma with respect to  $u_i(\underline{x})$  :

**Lemma 2** *In an optimal mechanism  $(\mathbf{Q}, \mathbf{M})$ , the payoff for type  $\underline{x}$  is written as  $u_i(\underline{x}) = - \int_{\underline{x}}^{x_i^*} (aq_i(t) - b\bar{q}_i(t)) dt$ , where  $x_i^* \in \arg \min_z \int_{\underline{x}}^z (aq_i(t) - b\bar{q}_i(t)) dt \subseteq X$ .*

Lemma 2 is a direct result from the incentive constraint (3) and the individually rational constraint (4), which also jointly imply that there exists at least one type for each buyer where (4) is binding. As defined in Lemma 2, type  $x_i^*$  is also the type who receives zero payoff in the optimal mechanism, with the possibility that more than one type can receive zero payoff.

An immediate implication of  $P_2$  and Lemma 2 is that finding an optimal mechanism reduces to searching for an optimal choice of the allocation rule  $\mathbf{Q}(x)$ , and the type receiving zero payoff in the optimal mechanism,  $x_i^*$ , hence  $u_i(\underline{x})$ , which is implicitly determined by  $\mathbf{Q}(x)$ . To ease our characterization, we redefine the objective function in  $P_2$  using Lemma 2 as follows:

$$\begin{aligned}
& \sum_i \int_X \int_X [v_\omega(x_i) + v_\lambda(x_j)] Q_i(x) f(x_i) f(x_j) dx_i dx_j - \sum_i u_i(\underline{x}) \tag{6} \\
& = \sum_i \int_X \int_X [v_\omega(x_i) + v_\lambda(x_j)] Q_i(x) f(x_i) f(x_j) dx_i dx_j + \sum_i \int_{\underline{x}}^{x_i^*} (aq_i(t) - b\bar{q}_i(t)) dt \\
& = \sum_i \int_X \int_X [v_\omega(x_i) + v_\lambda(x_j) + a \frac{I(x_i < x_i^*)}{f(x_i)} - b \frac{I(x_j < x_j^*)}{f(x_j)}] Q_i(x) f(x_i) f(x_j) dx_i dx_j,
\end{aligned}$$

where  $I(\cdot)$  is the indicator function

$$I(x_i < y) = \begin{cases} 1, & \text{if } x_i < y \\ 0, & \text{if } x_i \geq y \end{cases}$$

Given the symmetry between the buyers, it is natural for us to consider the case where the type with zero payoff in the optimal mechanism is identical for the buyers.

<sup>10</sup>As before, we denote  $i, j \in \{1, 2\}$  and  $i \neq j$ .

We thus focus on optimal mechanism where  $x_i^* = x_j^* = x^*$  hereafter.<sup>11</sup> Consequently, the seller's maximization problem can be equivalently expressed as the following  $P_3$ :

$$\begin{aligned}
P_3 \quad &: \max_{\mathbf{Q}, x^*} \sum_i \int_X \int_X [V(x_i; x^*) + J(x_i, x_j; x^*)] Q_i(x) f(x_i) f(x_j) dx_i dx_j \\
&V(x_i; x^*) = v_\omega(x_i) - v_\lambda(x_i) + \frac{aI(x_i < x^*)}{f(x_i)} + \frac{bI(x_i < x^*)}{f(x_i)} \\
&J(x_i, x_j; x^*) = v_\lambda(x_i) + v_\lambda(x_j) - \frac{bI(x_i < x^*)}{f(x_i)} - \frac{bI(x_j < x^*)}{f(x_j)} \\
s.t. \quad &\mathbf{Q}(x) \in \Phi, \forall x \quad (F) \\
&aq_i(x_i) - b\bar{q}_i(x_i) \text{ is non-decreasing in } x_i, \forall i \quad (ND) \\
&x^* \in \arg \min_t \int_{\underline{x}}^t (aq_i(z) - b\bar{q}_i(z)) dz, \forall i \quad (IR)
\end{aligned}$$

Several remarks are in order at this stage:

First, in the standard case in Myerson (1981), maximizing the seller's expected profit amounts to choosing a mechanism to maximize expected virtual surplus across all feasible mechanisms, where expected virtual surplus is the expected surplus adjusted to account for the buyers' information rent. In our setting, the appropriate virtual surplus the seller should care about for each buyer now contains two distinct parts, the virtual surplus of winning for  $i$ ,  $v_\omega(x_i)$  and the virtual surplus of losing for  $j$ ,  $v_\lambda(x_j)$ , which can be seen more clearly in  $P_2$ . This is natural given our externality specification: by awarding the object to buyer  $i$ , the seller can not only capture the surplus from  $i$ 's winning, but also the surplus from  $j$ 's losing, as a result of buyer  $j$  also obtaining positive payoff.

Second, in the standard case, a main incentive issue for a seller is to prevent a type from mimicking a lower type. Consequently, the participation constraint is only binding for the lowest type. In our setting, however, as each buyer also receives positive payoff when she loses the auction, it is pertinent for the seller to prevent each type from mimicking a lower type and/or from mimicking a higher type. The type getting zero expected payoff in the optimal mechanism is thus implicitly and jointly determined by *the intensities of incentives to misreport* and *the allocation rule*. As a result, the term  $\sum_i u_i(\underline{x})$  enters the seller's optimization problem in a non-trivial way. This fact also creates some difficulties for our characterization of the optimal mechanism, as Myerson's characterization techniques is not, at least directly, applicable.

Finally, compared to  $P_2$ , the objective function in  $P_3$  is rewritten to be the sum of  $V(x_i; x^*)$ , only a function of  $(x_i, x^*)$ , and  $J(x_i, x_j; x^*)$ , which is (importantly) a *symmetric* function of  $(x_i, x_j)$ . This rather awkward (but equivalent) transformation enables us to employ Myerson's technique in characterizing the optimal mechanism

---

<sup>11</sup>Observe that we can restrict our attention to symmetric/anonymous mechanisms because a symmetric optimal mechanism always exists as long as an optimal mechanism exists: Suppose  $(\mathbf{Q}, \mathbf{M})$  is an optimal mechanism. Then define  $(\mathbf{Q}', \mathbf{M}')$  to be the mirror of  $(\mathbf{Q}, \mathbf{M})$  such that the indices of the two buyers  $i$  and  $j$  are swapped. Finally, we can construct another mechanism  $\frac{(\mathbf{Q}, \mathbf{M}) + (\mathbf{Q}', \mathbf{M}')}{2}$  which is symmetric and optimal.

for the non-regular case without incurring multi-dimensional complications, as the symmetry of  $J(x_i, x_j; x^*)$  makes it play no role in comparing surpluses from the bidders.

We now characterize the optimal mechanism for the seller. Our characterization proceeds in two main steps: In the first, we characterize the optimal allocation rule  $\mathbf{Q}^*(\cdot; t)$  for each exogenously fixed  $x^* = t$ . In doing so, we employ and modify Myerson's characterization technique for the non-regular case.<sup>12</sup> In the second step, we use the derived optimal allocation rule  $\mathbf{Q}^*(\cdot; t)$  to “determine” the type that gets zero expected payoff,  $x^*$ , in the optimal mechanism. We show, somewhat surprisingly, such a derived pair  $(\mathbf{Q}^*, x^*)$  from this two-step characterization “coincides” with the true solution to problem  $P_3$ . Such an equivalence is due to our specification of the setting, which allows us to invoke an important minimax theorem in Terkelsen (1972) to establish the equivalence.

To present our result in the first step, we first denote  $P_3(t)$  as a revised version of  $P_3$ , where we ignore the individual rationality constraint ( $IR$ ) and take  $x^* = t$  as exogenously given. We next define several functions over the same domain  $[0, 1]$ , where we suppress the argument  $t$  whenever there is no danger of confusion:<sup>13</sup>

$$\begin{aligned} h_i(p) & : = V(F^{-1}(p); t), \\ H_i(p) & : = \int_0^p h_i(y) dy, \\ G_i(p) & : = \text{conv}H_i(p), \\ g_i(p) & : = G'_i(p), \end{aligned}$$

where  $G_i(\cdot)$  is the convex hull of the function  $H_i(\cdot)$ , with the interpretation of  $G_i(\cdot)$  being the largest convex function over  $[0, 1]$  such that  $G_i(p) \leq H_i(p), \forall p$ . This is illustrated in Figure 1.

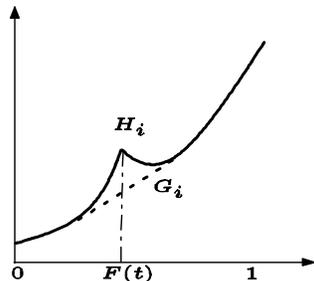


Figure 1. An illustration of  $H_i$  and  $G_i$ .

As it is convex,  $G_i(\cdot)$  is continuously differentiable almost everywhere, hence the resulting  $g_i(p)$  is defined and exists almost everywhere. Finally, define a “criterion”

<sup>12</sup>We discuss our motivation of using Myerson's characterization techniques for the non-regular case after Proposition 1.

<sup>13</sup>These definitions, while less essential in our symmetric setting, are important if the buyers' type distributions are asymmetric. We follow Myerson (1981) to define these functions so that the interested reader can compare our proof of Proposition 1 with that in Myerson (1981).

function  $\bar{c}_i(\cdot)$ , closely related to  $V(\cdot; t)$  after the transformation, as follows:

$$\bar{c}_i(x_i) := g_i(F(x_i)).$$

**Proposition 1** *Conditional on sale,  $\mathbf{Q}^{K(t)} = (Q_1^{K(t)}, Q_2^{K(t)})$  solves the problem  $P_3(t)$ , where*

$$k(x; t) = \{i | \bar{c}_i(x_i) = \max(\bar{c}_1(x_1), \bar{c}_2(x_2))\}$$

and

$$Q_i^{K(t)}(x) = \begin{cases} \frac{1}{|k(x; t)|}, & \text{if } i \in k(x; t) \\ 0, & \text{if } i \notin k(x; t) \end{cases}. \quad (7)$$

Notice that the allocation rule  $\mathbf{Q}^{K(t)}$  is the solution to  $P_3(t)$  when the seller *does* transfer the good to the buyers. In our setting, this happens when positive externalities are sufficiently large (see more detail in the proof of Proposition 1).

Interestingly, the optimal allocation rule  $\mathbf{Q}^{K(t)}$  for  $P_3(t)$  for an exogenous  $x^* = t$  features *bunching* for some (typically intermediate) types. Notice in particular that bunching happens in our positive externality setting even in the regular case where the hazard rate  $\frac{1-F(x_i)}{f(x_i)}$  is decreasing in  $x_i$  (monotone hazard rate). To see this, first observe that only the function  $V(x_i; x^*)$  is essential in determining the optimal allocation rule for the seller, as the other function  $J(x_i, x_j; x^*)$  is symmetric in  $x_i$  and  $x_j$ , hence playing no role in comparing surpluses from the two buyers. When the monotone hazard rate condition holds,  $V(x_i; x^*)$  is only decreasing in some neighborhood where  $x_i \geq x^*$ . By the optimal allocation rule  $\mathbf{Q}^{K(t)}$  in (7), the seller randomly allocates the object to the two buyers when both  $x_i$  and  $x_j$  are in some neighborhood around  $x^*$ . Intuitively, such bunching arises as a result of countervailing incentives intermediate types have when they report their private information. Such incentives compel the seller to specify an identical allocation for an entire set of intermediate types. With a monotone hazard rate, such bunching *only* arises in a neighborhood around  $x^*$  with the optimal  $\mathbf{Q}^{K(t)}$ . On the other hand, when the monotone hazard rate assumption fails,  $\mathbf{Q}^{K(t)}$  may feature bunching in a set around  $x^*$  as in the regular case, as well as in other regions where the virtual surplus is non-increasing so as to relax the incentive constraint. The latter is similar to the rationale of bunchings in the non-regular case in Myerson (1981).

Given  $\mathbf{Q}^{K(t)}$  in Proposition 1, define

$$\begin{aligned} q_i^{K(t)}(x_i) &: = \int_X Q_i^{K(t)}(x) f(x_j) dx_j, \\ \bar{q}_i^{K(t)}(x_i) &: = \int_X Q_j^{K(t)}(x) f(x_j) dx_j, \\ t^* &\in \arg \min_t \int_{\underline{x}}^t \left( a q_i^{K(t^*)}(z) - b \bar{q}_i^{K(t^*)}(z) \right) dz, \end{aligned} \quad (8)$$

where  $q_i^{K(t)}(x_i)$  and  $\bar{q}_i^{K(t)}(x_i)$  are, respectively,  $i$ 's interim probability of winning and interim probability of losing given type  $x_i$  under  $\mathbf{Q}^{K(t)}$ , while  $t^*$  is a type with zero expected payoff under  $\mathbf{Q}^{K(t)}$ . Moreover, observe that if there is an *interior* type ( $t^\dagger \in [\underline{x}, \bar{x}]$ ) such that:<sup>14</sup>

$$aq_i^{K(t^\dagger)}(t^\dagger) - b\bar{q}_i^{K(t^\dagger)}(t^\dagger) = 0, \quad (9)$$

then the type  $t^\dagger$  is the desired type  $t^*$  whose expected payoff is zero under  $\mathbf{Q}^{K(t^*)}$ .

We now present our second key step in our characterization (Proposition 2): the pair  $(\mathbf{Q}^{K(t^*)}, t^*)$  is exactly the solution we are looking for:

**Proposition 2** *Conditional on sale, the pair  $(\mathbf{Q}^{K(t^*)}, t^*)$  derived from Proposition 1 and (8) solves the optimal mechanism design problem  $P_3$ .*

The key argument for Proposition 2 is an application of an important minimax theorem established in Terkelsen (1972). First, notice that together with Lemma 2, the seller's optimization problem  $P_2$  can be equivalently written as (constraint  $(IR)$  has been incorporated into the objective function):

$$(P_2) : \max_{\mathbf{Q}} \left\{ \pi(\mathbf{Q}) + \min_{t \in X} \phi(\mathbf{Q}, t) \right\} \text{ subject to } (ND) \text{ and } (F),$$

where we denote

$$\begin{aligned} \pi(\mathbf{Q}) &= \sum_i \int_X \int_X (v_\omega(x_i) + v_\lambda(x_j)) Q_i(x) f(x_i) f(x_j) dx_i dx_j, \\ \phi(\mathbf{Q}, t) &= \sum_i \int_{\underline{x}}^t (aq_i(z) - b\bar{q}_i(z)) dz. \end{aligned} \quad (10)$$

The seller's optimization problem can hence be regarded as a *maximin problem* where the seller first chooses, for each allocation  $\mathbf{Q}$  satisfying  $(F)$  and  $(ND)$ , an optimal  $t(\mathbf{Q})$  to minimize  $[\pi(\mathbf{Q}) + \phi(\mathbf{Q}, t)]$ , and then finds an optimal allocation rule  $\mathbf{Q}$  to maximize the revenue represented by  $[\pi(\mathbf{Q}) + \phi(\mathbf{Q}, t(\mathbf{Q}))]$ .

On the other hand, our candidate solution pair  $(\mathbf{Q}^{K(t^*)}, t^*)$  can be seen as a solution to the *minimax problem* where for each given  $t$  we first maximize the seller's objective so as to derive the optimal allocation rule  $\mathbf{Q}^{K(t)}$ , and then we conduct the minimization problem given the solved allocation rule  $\mathbf{Q}^{K(t)}$ .<sup>15</sup>

It is known that in general for a real-valued function  $f$  defined on a product set  $Y \times Z$ , when solutions exist, we have  $\min_{y \in Y} \max_{z \in Z} f(y, z) \geq \max_{z \in Z} \min_{y \in Y} f(y, z)$ . For the reverse

<sup>14</sup>It can be shown that if the monotone hazard rate condition holds, then  $(aq_i^{K(t)}(x_i) - b\bar{q}_i^{K(t)}(x_i))$  is continuous and increases from a negative value when  $x_i = \underline{x}$  to a positive value when  $x_i = \bar{x}$  and there exists an interior  $t^*$  that solves (9). See Lemma 5 for more detail.

<sup>15</sup>We present this more rigorously in Lemma 4, which is presented explicitly in the proof of Proposition 2.

weak inequality to hold, different conditions have been imposed on  $f(y, z)$ ,  $Y$  and  $Z$  so as to obtain various minimax theorems. In our setting, the optimal allocation rule  $\mathbf{Q}$  (variable  $z$  in this case) is chosen from a functional space that satisfies the feasibility and the non-decreasing constraints. For our purpose, ideally one should invoke a minimax theorem that imposes little structure on the set  $Z$  so as to minimize technical issues. Therefore, we use the main theorem of Terkelsen (1972) in our proof of Proposition 2 to show that the specific structure of the seller's maximization problem implies that Terkelsen's minimax theorem holds in our setting, and consequently the solution pair  $(\mathbf{Q}^{K(t^*)}, t^*)$  also solves our original revenue maximization problem.

Finally, it is important to point out that, our optimal allocation rule  $\mathbf{Q}^{K(t^*)}$ , albeit very similar, is qualitatively different from Myerson's optimal allocation rule. The similarity is only the result that we modify Myerson's techniques so as to obtain a solution for the minimax version of the seller's problem. In general, such minimax solution, however, *may not* solve the maximization problem in optimal auction design.

## 4 An Application to Selling Retaliation Rights in the WTO

In this section, we apply our general results in Section 3 to solve for a specific optimal mechanism for selling retaliation rights in the WTO, which was first analyzed by Bagwell, Mavroidis and Staiger (2007). In Section 4.1, we describe our basic international trade model. In Section 4.2, we first apply Proposition 1 and Proposition 2 to solve for the optimal allocation rule and we then explicitly calculate players' payoffs, payment functions, and the seller's expected revenue in the optimal selling mechanism, with some discussion on interesting features of the optimal mechanism.

### 4.1 The Basic Trade Model

Our basic model is similar to that in Bagwell, Mavroidis and Staiger (2007). The model consists of three countries — Exporter and two ex-ante symmetric importers, denoted as Importer 1 and Importer 2, and a single good  $y$  such that Exporter exports good  $y$  to Importer 1 and Importer 2.

The domestic demand ( $D_i$ ) and domestic supply ( $S_i$ ) of Importer  $i$  are given as, respectively,  $D_i(P_i) = 1 - P_i$  and  $S_i(P_i) = 1/4$ , where  $P_i$  is the domestic price of good  $y$  sold in Importer  $i$ ,  $i \in \{1, 2\}$ . Denoting  $\tau_i$  as Importer  $i$ 's import tariff, the domestic price of Importer  $i$  is thus determined by  $P_i = P_W + \tau_i$ , where  $P_W$  represents the world price of good  $y$ . The import demand faced by Importer  $i$  is given by  $IM_i(P_i) = D_i(P_i) - S_i(P_i) = 3/4 - P_i$ .

We assume that Exporter has no domestic demand for good  $y$ ,  $D^e(P^e) = 0$ , and Exporter supplies  $1/2$  units of good  $y$  to the two importing countries,  $S^e(P^e) = 1/2$ , where  $P^e$  denotes Exporter's domestic price of good  $y$ . We further assume that

Exporter employs no export policy and hence  $P^e = P_W$ . With the world-market-clearing condition  $IM_1(P_1) + IM_2(P_2) = S^e(P^e) - D^e(P^e)$ , the equilibrium world price and the equilibrium domestic prices are then determined as

$$P_W(\tau_1, \tau_2) = \frac{1}{2} - \frac{\tau_1 + \tau_2}{2} \text{ and } P_i = P_W + \tau_i = \frac{1}{2} + \frac{\tau_i - \tau_j}{2}, \quad (11)$$

respectively, for  $i, j \in \{1, 2\}$  and  $i \neq j$ .

We then consider the welfare functions of the importers. The government of Importer  $i$  is concerned with national income, as well as income distribution, which is captured by some political-economy parameter  $x_i$  that is private to Importer  $i$ . Specifically, the welfare of Importer  $i$ 's government is<sup>16</sup>

$$\begin{aligned} GW & : = CS(P_i) + TR_i(P_i, P_W) + x_i \Pi_i(P_i) \\ & = (1 - P_i)^2/2 + (P_i - P_W)IM_i(P_i) + x_i P_i/4, \end{aligned}$$

where  $CS_i$ ,  $TR_i$  and  $\Pi_i(P_i)$  represent, respectively, Importer  $i$ 's consumer surplus, tariff revenue, and producer surplus.<sup>17</sup> As mentioned, we denote  $x_i$  to be the political-economy parameter of Importer  $i$ . When  $x_i = 1$ , the government's welfare coincides with the national welfare. With respect to  $x_i$ , we assume

**Assumption 1** *The political-economy parameter  $x_i$  satisfies  $1 \leq x_i \leq 2$ ,  $\forall i$ .*

Specifically,  $x_i \geq 1$  implies that the government weighs producer surplus more than consumer surplus, while  $x_i \leq 2$  restricts our model to the common case where the welfare function of each government is decreasing in the price of its imports ( $y$ ).

With equilibrium prices  $P_i$  and  $P_W$  defined in expression (11), the welfare of Importer  $i$  in terms of the tariff rates and its political-economy parameter can then be represented as

$$GW = GW(\tau_i, \tau_j, x_i) = \frac{1}{8} - \frac{3\tau_i^2}{8} + \frac{\tau_j^2}{8} + \frac{\tau_i\tau_j}{4} + \frac{\tau_j}{4} + \frac{x_i}{8}(\tau_i - \tau_j + 1),$$

from which, the best-response tariff of Importer  $i$  is given as<sup>18</sup>

$$\tau^{br}(x_i, \tau_j) = \frac{x_i}{6} + \frac{\tau_j}{3}. \quad (12)$$

---

<sup>16</sup>This welfare function was first introduced in Baldwin (1987). It can also be considered as a reduced form in the model of Grossman and Helpman (1994).

<sup>17</sup>The welfare function of Exporter can be defined similarly, with the absence of consumer surplus and tariff revenue, as Exporter does not import from the importing countries. As our focus in the paper is on how an (unmodeled) outside country can optimally sell the retaliation right to the importing countries, we thus omit such descriptions for Exporter.

<sup>18</sup>See Bagwell, Mavroidis and Staiger (2007) for a detailed discussion on Nash tariffs and efficient tariffs.

We now introduce the central object of this section, retaliation rights. We first assume that according to a prior agreement in the WTO, initially the two importing countries have agreed to set their tariffs to be zero. Suppose that Exporter violates some prior agreement with an outside (unmodelled) country. Consequently, the WTO authorizes this outside country to retaliate by increasing its tariffs on goods it imports from Exporter. Suppose that this outside country does not import goods from Exporter and therefore cannot retaliate by itself. It can, however, sell the right to retaliate to Importer 1 or Importer 2 (the selling country is called the *seller* hereafter). The right to increase the tariff, called the retaliation right hereafter, is such that any Importer who obtains this retaliation right may increase its tariff against Exporter on good  $y$  from zero up to  $\Delta$ . The value of  $\Delta$  reflects the size of Exporter's violation. We further impose the following assumption on  $\Delta$ :

**Assumption 2** *Parameter  $\Delta$  is less than the best-response tariffs of the two importing countries:  $\Delta \in (0, \frac{1}{6}]$ .*

Under Assumption 2, it is thus always optimal for the importing country that obtains the retaliation right to increase its tariff to  $\Delta$ .

In a selling mechanism, when Importer  $i$  wins the right and retaliates, its *normalized* welfare can be represented as<sup>19</sup>

$$\omega(x_i) := GW(\Delta, 0, x_i) - GW(0, 0, x_i) = \frac{\Delta}{8}(x_i - 3\Delta), \quad (13)$$

where  $GW(\Delta, 0, x_i)$  is Importer  $i$ 's welfare when  $i$  retaliates and  $GW(0, 0, x_i)$  is  $i$ 's welfare when no retaliation occurs.

When Importer  $i$  loses and Importer  $j$  obtains the right and retaliates, Importer  $i$ 's normalized welfare, or externality payoff from losing, is

$$\lambda(x_i) := GW(0, \Delta, x_i) - GW(0, 0, x_i) = \frac{\Delta}{8}(2 + \Delta - x_i). \quad (14)$$

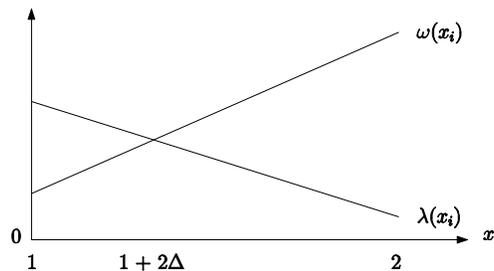


Figure 2. An illustration of functions  $\omega(x_i)$  and  $\lambda(x_i)$ .

The functions  $\omega(x_i)$  and  $\lambda(x_i)$  are shown in Figure 2. Notice that under Assumptions 1 and 2, our model features positive externalities:  $\omega(x_i), \lambda(x_i) > 0$  for all  $x_i \in [1, 2]$ .

<sup>19</sup>An importing country *wins* if it gets the retaliation right and then retaliates. An importing country *loses* if the other importing country wins.

## 4.2 Optimal Mechanisms in an Incomplete Information Environment

In this section, we analyze optimal (revenue-maximizing) mechanisms the seller (the unmodelled outside country) chooses to sell the (indivisible) retaliation right to the buyers, Importer  $i$  and Importer  $j$ . As the seller cannot retaliate by itself, the retaliation right authorized by the WTO is of value zero to the seller. We assume that the buyers' political-economy parameters  $x_1$  and  $x_2$  are buyers' private information and are independently and identically distributed. For illustrative simplicity, we assume that the types of each buyer are distributed according a continuous *uniform distribution*  $f$  where the support of  $f$  is  $[1, 2]$ .

We now apply Proposition 1 and Proposition 2 to solve for the seller's optimal selling mechanism explicitly. Interpreting the seller's problem in the light of the setup in Section 2 and Section 3, we present the following table to illustrate the specific parameter values and functional forms:

$$\left[ \begin{array}{l} \underline{x} = 1, \quad \bar{x} = 2, \quad X \in [1, 2], \quad F(x_i) = x_i - 1, \quad f(x_i) = 1; \\ A = -\frac{3\Delta^2}{8}, \quad B = \frac{\Delta}{4} + \frac{\Delta^2}{8}, \quad a = b = \frac{\Delta}{8}; \\ \omega(x_i) = -\frac{3\Delta^2}{8} + \frac{\Delta}{8}x_i, \quad \lambda(x_i) = \frac{\Delta}{4} + \frac{\Delta^2}{8} - \frac{\Delta}{8}x_i; \\ v_\omega(x_i) = -\frac{\Delta}{4} - \frac{3\Delta^2}{8} + \frac{\Delta}{4}x_i, \quad v_\lambda(x_i) = \frac{\Delta}{2} + \frac{\Delta^2}{8} - \frac{\Delta}{4}x_i; \\ V(x_i; t) = -\frac{3}{4}\Delta - \frac{\Delta^2}{2} + \frac{\Delta}{2}x_i + \frac{\Delta}{4}I(x_i \leq t); \\ J(x_1, x_2; t) = \Delta + \frac{\Delta^2}{4} - \frac{\Delta}{4}(x_1 + x_2) - \frac{\Delta}{8}(I(x_1 \leq t) + I(x_2 \leq t)). \end{array} \right]$$

To ease our exposition, we first *guess* that the type who obtains zero expected payoff in the optimal mechanism is  $t = \frac{3}{2}$ . Given such  $t$ , we then follow the procedures specified in Proposition 1 to obtain the optimal allocation rule  $\mathbf{Q}^{K(t)}$ . Finally, we verify that  $t = \frac{3}{2}$  is exactly the type we are looking for given the solved  $\mathbf{Q}^{K(t)}$ .

First, with  $t = \frac{3}{2}$ , we have

$$V(x_i; t) = \begin{cases} -\frac{1}{2}\Delta - \frac{\Delta^2}{2} + \frac{\Delta}{2}x_i & \text{for } x_i \leq \frac{3}{2} \\ -\frac{3}{4}\Delta - \frac{\Delta^2}{2} + \frac{\Delta}{2}x_i & \text{otherwise} \end{cases}.$$

Next, from  $H_i(p) = \int_0^p V(F^{-1}(y); t)dy = \int_0^p V(1 + y; t)dy$ , we derive

$$H_i(p) = \begin{cases} \frac{\Delta}{4}p^2 - \frac{\Delta^2}{2}p & \text{for } p \in [0, 1/2] \\ \frac{\Delta}{4}p^2 - (\frac{\Delta^2}{2} + \frac{\Delta}{4})p + \frac{\Delta}{8} & \text{for } p \in (1/2, 1] \end{cases},$$

which, using  $G_i = \text{conv}(H_i)$ , implies

$$G_i(p) = \begin{cases} -\frac{\Delta}{64} + (\frac{\Delta}{8} - \frac{\Delta^2}{2})p & \text{for } p \in [\frac{1}{4}, \frac{3}{4}] \\ H_i(p) & \text{for } p \in [0, \frac{1}{4}] \cup (\frac{3}{4}, 1] \end{cases}.$$

Functions  $V$ ,  $H_i$  and  $G_i$  are illustrated in the following figures:<sup>20</sup>

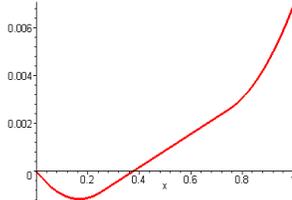
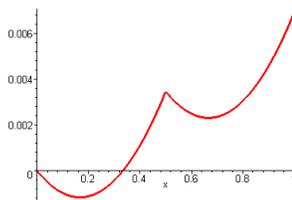
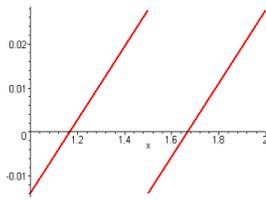


Figure 3. Function  $V(x_i; t)$ . Figure 4. Function  $H_i(p)$ . Figure 5. Function  $G_i(p)$ .

Finally, from  $\bar{c}_i(x_i) = G'_i(F(x_i))$ , we have

$$\bar{c}_i(x_i) = \begin{cases} \frac{\Delta}{8} - \frac{\Delta^2}{2} & \text{for } x_i \in [\frac{1}{4}, \frac{3}{4}] \\ V(x_i, t) & \text{otherwise} \end{cases} .$$

Such a ‘‘criterion’’ function for the seller is shown in Figure 6 below:

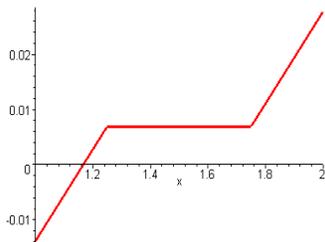


Figure 6. Function  $\bar{c}_i(x_i)$ .

To solve for the optimal allocation rule  $\mathbf{Q}^{K(t)}$ , we first verify that in our trade setting, the retaliation right is always sold:<sup>21</sup>

**Lemma 3** *In our trade setting, there is always sale. Specifically, for  $t = \frac{3}{2}$ , we have*

$$\max \{ \bar{c}_1(x_1) + J(x_1, x_2), \bar{c}_2(x_2) + J(x_2, x_1) \} \geq 0, \quad \forall x \in [1, 2]^2 .$$

Now a direct application of Proposition 1 implies that  $k(x; \frac{3}{2}) = \{i | \bar{c}_i(x_i) = \max(\bar{c}_1(x_1), \bar{c}_2(x_2))\}$  and  $Q_i^{K(t)}(x) = 1/|k(x; \frac{3}{2})|$  if  $i \in k(x; \frac{3}{2})$ , which yields the optimal allocation rule as

$$Q_i^{K(t)}(x) = \begin{cases} \frac{1}{2} & \text{if } x_i = x_j \text{ or } x \in [\frac{5}{4}, \frac{7}{4}]^2 \\ I(x_i > x_j) & \text{otherwise} \end{cases} .$$

Consequently, buyer  $i$ 's interim probability of winning is obtained as

$$q_i^{K(t)}(x_i) = \begin{cases} \frac{1}{2} & \text{if } x_i \in [\frac{5}{4}, \frac{7}{4}] \\ x_i - 1 & \text{otherwise} \end{cases} , \quad (15)$$

<sup>20</sup>All numerical figures were generated using  $\Delta = 1/6$  by Maple.

<sup>21</sup>The sufficiency of the inequality in Lemma 3 for always sale is standard and can be seen more clearly in the proof of Proposition 1 and in Footnote 24.

and buyer  $i$ 's interim probability of losing is  $\bar{q}_i^{K(t)} = 1 - q_i^{K(t)}$ . It follows that for  $t = \frac{3}{2}$ ,  $aq_i^{K(t)}(t) - b\bar{q}_i^{K(t)}(t) = a(2q_i^{K(t)}(t) - 1) = a(1 - 1) = 0$ . This verifies that type  $t = \frac{3}{2}$  is indeed the type that corresponds to the solved  $\mathbf{Q}^{K(t)}$ . Finally, by Proposition 2, the pair  $(\mathbf{Q}^{K(t)}, t)$  is also the solution to the seller's optimization problem. Figure 7 provides a pictorial description of the optimal allocation rule  $\mathbf{Q}^{K(t)}$  and the corresponding interim probability of winning  $q_i^{K(t)}$ .

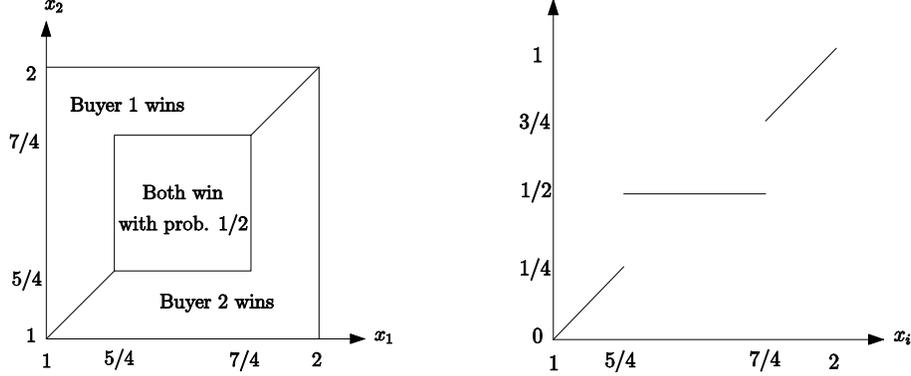


Figure 7. Optimal Allocation  $\mathbf{Q}^{K(t)}$  and Interim Winning Probability  $q_i^{K(t)}$ .

Notice that interestingly, under the optimal allocation rule, the seller randomly allocates the retaliation right to the two importing countries with equal probability if the types are in the middle box of Figure 7. Such randomization, a result of the countervailing positive externalities of the buyers, enables the seller to extract more surplus from extreme types, who have larger incentives to misreport compared to intermediate types. In addition, the optimal allocation rule implies that there is no inefficiency from *auction failure* in this setting: It is always optimal for the seller to sell the retaliation right to an importing country. However, because of such randomization, there *is* inefficiency from assigning the good to a lower-type buyer, rendering the total economic surplus not maximized.

We now derive buyer  $i$ 's payoff and payment and the seller's expected revenue from the optimal selling mechanism. First, using  $q_i^{K(t)}$  and  $\bar{q}_i^{K(t)}$  defined above, we have

$$\begin{aligned}
u_i(\underline{x}) &= u_i(1) \\
&= -\min_x \int_1^x (aq_i^{K(t)}(x) - b\bar{q}_i^{K(t)}(x))dx \\
&= -a \min_x \int_1^x (2q_i^{K(t)}(x) - 1)dx = -\frac{\Delta}{8} \int_1^{5/4} (2(x-1) - 1)dx = \frac{3\Delta}{128}.
\end{aligned}$$

From  $u_i(x_i) = u_i(1) + \int_1^{x_i} (a\bar{q}_i^{K(t)}(y) - b\bar{q}_i^{K(t)}(y))dy$  in Lemma 1, we obtain that

$$u_i(x_i) = \begin{cases} \frac{35}{128}\Delta - \frac{3}{8}\Delta x_i + \frac{\Delta}{8}x_i^2 & \text{if } x_i \notin [\frac{5}{4}, \frac{7}{4}] \\ 0 & \text{otherwise} \end{cases}. \quad (16)$$

It is interesting to observe that  $u'_i(x_i) < 0$  for  $x_i < 5/4$ ,  $u'_i(x_i) = 0$  for  $x_i \in (5/4, 7/4)$  and  $u'_i(x_i) > 0$  for  $x_i > 7/4$ , which shows that  $u_i$  is *not monotonic* as shown in Figure 8. We next solve for the payment function ( $m_i$ ) explicitly. Substituting (16) and (15) into the definition of  $m_i(x_i)$ , defined as

$$m_i(x_i) = q_i^{K(t)}(x_i)\omega(x_i) + \bar{q}_i^{K(t)}(x_i)\lambda(x_i) - u_i(x_i),$$

we derive the payment function for type  $x_i$  as

$$m_i(x_i) = \begin{cases} \frac{29}{128}\Delta + \frac{5}{8}\Delta^2 - (\frac{\Delta}{4} + \frac{\Delta^2}{2})x_i + \frac{\Delta}{8}x_i^2 & \text{for } x_i \notin [\frac{5}{4}, \frac{7}{4}] \\ \frac{\Delta}{8}(1 - \Delta) & \text{otherwise} \end{cases}.$$

Given the above payment function, the seller's expected revenue ( $ER$ ) in the optimal mechanism can be finally calculated as

$$ER = \sum_i \int_1^2 m_i(x_i) f(x_i) dx_i = \frac{55}{196}\Delta - \frac{\Delta^2}{4}.$$

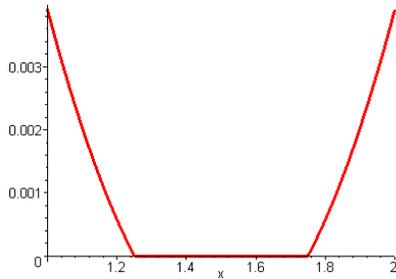


Figure 8. Buyer  $i$ ' Payoff  $u_i(x_i)$ .

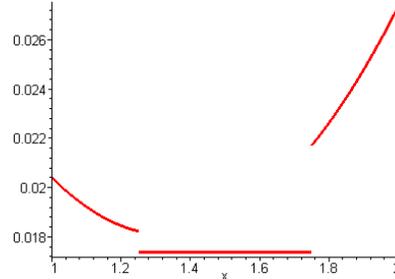


Figure 9. Buyer  $i$ 's Payment  $m_i(x_i)$ .

We now comment on the interim expected payment function  $m_i(x_i)$ , shown in Figure 9. Notice from the figure that very low types, who have small winning probabilities, have to pay more than the intermediate types, whose winning probability is strictly higher. On the other hand, very high types, who have large winning probabilities, again have to pay more than the intermediate types. Such a payment function stands in contrast to commonly observed auctions, where the winning probability is monotonic in the expected payment. In our opinion, this result, together with the other features of the optimal selling mechanism identified above, suggests that it is difficult, if not impossible, to use the commonly observed auctions or their variants to implement the above optimal mechanism.

## 5 Concluding Remarks

In this paper, we study optimal auction design problems with type-dependent and countervailing positive allocative externalities. A distinguishing technical difficulty of such problems is that the type obtaining the reservation utility is endogenously determined by the allocation rule, which is in turn a choice variable for the seller. This difficulty renders the problem not solvable using existing techniques in the characterization of optimal mechanisms.

We provide novel characterization techniques by first modifying Myerson's characterization techniques for the non-regular case to solve the minimax problem for the seller, and then invoking Terkelsen's minimax theorem to show that such solution also solves the seller's original maximin problem. The optimal mechanisms we find mark some interesting features: The optimal allocation rule involves bunching even in the case with monotone hazard rate, as a result of relaxing the bidders' incentive constraints; The type with reservation utility is endogenously determined and is typically not an extreme type; Each buyer's payoff and payment are in general not monotonic in types; And the optimal mechanism may feature several dimensions of ex post inefficiency. As an application, we analyze the problem of selling retaliation rights in the WTO. Our results reveal some potential issues that might arise in implementing the proposal of making retaliation tradeable in the WTO.

It is important to point out that the mechanisms considered in this paper require a great dose of commitment and enforcement power from the seller, excluding the possibility of renegotiation. Although such a commitment assumption is constantly imposed in the optimal mechanism design literature, one should be careful in interpreting our theoretical results in the application of selling retaliation rights in the WTO, as the WTO is a self-enforcing agreement and such commitment and enforcement power is typically unrealistic. Imaginably, ex post, after the governments have made their reports and revealed their information, there are strong incentives to deviate and also to renegotiate, which are indeed commonplace in the WTO.<sup>22</sup> To provide a more complete description of the issues around optimal auction design in selling retaliation rights in the WTO, it is therefore important to also address the effect of potential future renegotiations on the design of optimal mechanisms.<sup>23</sup>

Finally, on the technical side, although our model admits a specific form of allocative positive externalities, our characterization techniques for optimal mechanisms

---

<sup>22</sup>Notice that our optimal mechanisms are in general not ex post efficient, which indeed invites incentives for all parties to renegotiate away ex post inefficiencies.

<sup>23</sup>An immediate problem arises when the mechanism designer cannot perfectly commit to outcomes of the mechanism is that the standard revelation principle is no longer applicable: Once the buyers truthfully reveal their types, the mechanism designer is practically informed and will exploit this information as he is not committed by the mechanism. This is anticipated by the buyers, who would not report truthfully in the first place. The fact that conventional revelation principle fails under imperfect commitment imposes great difficulties in finding an optimal mechanism from all conceivable mechanisms. See Bester and Strausz (2001) for more discussion.

are new and seem to be fairly general. Given the well-known difficulties associated with optimal mechanism design problems with externalities, an extension of our characterization techniques to more general design problems would be both interesting and nontrivial.

## Appendix

**Proof of Lemma 1.** ( $\implies$ ) First, notice that by (IC) in  $P_1$ ,

$$u_i(x_i) = \max_{t \in X} [q_i(t)(A + ax_i) + \bar{q}_i(t)(B - bx_i) - m_i(t)],$$

or  $u_i$  is a maximum of a family of affine functions, hence is a convex function and is differentiable almost everywhere. By the envelope theorem, we have  $u_i'(x_i) = [aq_i(x_i) - b\bar{q}_i(x_i)]$  almost everywhere. As  $u_i(x_i)$  is convex,  $[aq_i(x_i) - b\bar{q}_i(x_i)]$  is a non-decreasing function in  $x_i$ . Finally, the fundamental theorem of calculus implies that (4) is true.

( $\impliedby$ ) Without loss of generality, consider  $z_i \in X$  s.t.  $z_i < x_i$ . Then using (4), we have  $u_i(x_i) - u_i(z_i) = \int_{z_i}^{x_i} (aq_i(t) - b\bar{q}_i(t)) dt$ . By condition (3), we then obtain:

$$u_i(x_i) - u_i(z_i) = \int_{z_i}^{x_i} (aq_i(t) - b\bar{q}_i(t)) dt \geq (x_i - z_i)(aq_i(z_i) - b\bar{q}_i(z_i)),$$

which is equivalent to (IC) as the latter is equivalent to

$$\begin{aligned} u_i(x_i) &= q_i(x_i)(A + ax_i) + \bar{q}_i(x_i)(B - bx_i) - m_i(x_i) \\ &\geq q_i(z_i)(A + ax_i) + \bar{q}_i(z_i)(B - bx_i) - m_i(z_i) \\ &= u_i(z_i) + (aq_i(z_i) - b\bar{q}_i(z_i))(x_i - z_i). \end{aligned}$$

This completes our proof of Lemma 1. ■

**Proof of Lemma 2.** First, set  $\arg \min_z \int_{\underline{x}}^z (aq_i(t) - b\bar{q}_i(t)) dt$  is a non-empty set that belongs to  $X$ , as  $\int_{\underline{x}}^z (aq_i(t) - b\bar{q}_i(t)) dt$  is continuous and  $z$  is chosen from a compact set  $X$ . Now consider the case where  $u_i(\underline{x}) < -\int_{\underline{x}}^{x^*} (aq_i(t) - b\bar{q}_i(t)) dt$ . By (4) in Lemma 1, we have that  $u_i(x^*) = u_i(\underline{x}) + \int_{\underline{x}}^{x^*} (aq_i(t) - b\bar{q}_i(t)) dt < 0$ , which implies that the (IR) constraint is violated for type  $x^*$ . Next, consider the case where  $u_i(\underline{x}) > -\int_{\underline{x}}^{x^*} (aq_i(t) - b\bar{q}_i(t)) dt$ , which yields that  $u_i(x_i) > 0$  for all  $x_i \in X$ . This indicates that the seller can increase the expected revenue by decreasing  $u_i(\underline{x})$ . ■

**Proof of Proposition 1.** First observe that

$$\begin{aligned}
& \int_X \int_X [h_i(F(x_i)) - g_i(F(x_i))] Q_i(x) f(x_1) f(x_2) dx_1 dx_2 \\
&= \int_X [h_i(F(x_i)) - g_i(F(x_i))] q_i(x_i) f(x_i) dx_i \\
&= [H_i(F(x_i)) - G_i(F(x_i))] q_i(x_i) \Big|_{\underline{x}}^{\bar{x}} - \int_X [H_i(F(x_i)) - G_i(F(x_i))] dq_i(x_i) \\
&= - \int_X [H_i(F(x_i)) - G_i(F(x_i))] dq_i(x_i), \tag{A-1}
\end{aligned}$$

where the second equality comes from integration by parts and the third equality is a result of the fact that  $G_i(0) = H_i(0)$  and  $G_i(1) = H_i(1)$ , as  $G_i(\cdot)$  is the convex hull of  $H_i(\cdot)$ , which is by construction continuous as  $F(\cdot)$ ,  $F^{-1}(\cdot)$  and  $V(\cdot; t)$  are all continuous.

Using (A-1), we have

$$\begin{aligned}
& \int_X \int_X V(x_i; t) Q_i(x) f(x_1) f(x_2) dx_1 dx_2 \\
&= \int_X \int_X [h_i(F(x_i)) - g_i(F(x_i))] Q_i(x) f(x_1) f(x_2) dx_1 dx_2 \\
&\quad + \int_X \int_X g_i(F(x_i)) Q_i(x) f(x_1) f(x_2) dx_1 dx_2 \\
&= \int_X \int_X \bar{c}_i(x_i) Q_i(x) f(x_1) f(x_2) dx_1 dx_2 - \int_X [H_i(F(x_i)) - G_i(F(x_i))] dq_i(x_i).
\end{aligned}$$

Consequently, the objective function in  $P_3(t)$  can be alternatively written as

$$\begin{aligned}
& \sum_i \int_X \int_X [V(x_i; t) + J(x_i, x_j; t)] Q_i(x) f(x_i) f(x_j) dx_i dx_j \\
&= \sum_i \int_X \int_X (\bar{c}_i(x_i) + J(x_i, x_j; t)) Q_i(x) f(x_1) f(x_2) dx_1 dx_2 \\
&\quad - \sum_i \int_X [H_i(F(x_i)) - G_i(F(x_i))] dq_i(x_i). \tag{A-2}
\end{aligned}$$

Next, as the object is always transferred — this happens when  $A$  and  $B$  are sufficiently large.<sup>24</sup> Given this, consider  $\mathbf{Q}^{K(t)}$  in the proposition: By definition,  $\mathbf{Q}^{K(t)}$  only puts all weights on the set of bidders whose  $\bar{c}_i(x_i) + J(x_i, x_j; t)$  is non-negative and maximal — observe that by construction,  $J(x_i, x_j; t)$  is symmetric ( $J(x_i, x_j; t) = J(x_j, x_i; t)$ ) and is a common term shared by both buyers. We hence

<sup>24</sup>To see this, notice that  $\bar{c}_i(x_i) = V(x_i; t) \geq V(x_i; \underline{x})$ ,  $\forall t \in [\underline{x}, \bar{x}]$ , as  $V(x_i; t)$  is non-decreasing

have that for any feasible allocation rule  $\mathbf{Q}$ ,

$$\begin{aligned} & \sum_i \int_X \int_X (\bar{c}_i(x_i) + J(x_i, x_j; t)) Q_i^{K(t)}(x) f(x_1) f(x_2) dx_1 dx_2 \\ & \geq \sum_i \int_X \int_X (\bar{c}_i(x_i) + J(x_i, x_j; t)) Q_i(x) f(x_1) f(x_2) dx_1 dx_2, \end{aligned} \quad (A-3)$$

where by construction,  $\mathbf{Q}^{K(t)}$  is feasible.

Moreover, given that there is always sale, we have  $\sum_i Q_i^{K(t)}(x) = 1$ , which implies that  $aq_i^{K(t)}(x_i) - b\bar{q}_i^{K(t)}(x_i) = (a+b)q_i^{K(t)}(x_i) - b$ . To show that the interim probability  $aq_i^{K(t)}(x_i) - b\bar{q}_i^{K(t)}(x_i)$  is non-decreasing in  $x_i$ , notice that by construction,  $\bar{c}_i$  is non-decreasing, as  $F$  and  $g_i$  are non-decreasing. Consequently,  $Q_i^{K(t)}$  is non-decreasing in  $x_i$ , given any  $x_j$ . Therefore,  $q_i^{K(t)}$ , induced from  $Q_i^{K(t)}$ , is also non-decreasing, which implies that  $(a+b)q_i^{K(t)}(x_i) - b$ , hence  $aq_i^{K(t)}(x_i) - b\bar{q}_i^{K(t)}(x_i)$ , is non-decreasing in  $x_i$ , satisfying (3).

Observe next that  $q_i^{K(t)}$  being non-decreasing and  $G_i(\cdot) = \text{conv}H_i(\cdot)$ , hence  $H_i \geq G_i$ , imply that  $\forall i$

$$\int_X [H_i(F(x_i)) - G_i(F(x_i))] dq_i(x_i) \geq 0.$$

In addition, by construction,  $G_i$  is flat whenever  $H_i > G_i$ , hence  $\forall y$  such that  $G_i(y) < H_i(y)$ , we have  $g_i'(y) = G_i''(y) = 0$ . Therefore, if  $H_i(F(x_i)) - G_i(F(x_i)) > 0$ , then  $g_i(F(x_i)) = \bar{c}_i(x_i)$  and hence  $q_i^{K(t)}(x_i)$  are constant over some neighborhood of  $x_i$ . This implies that we always have

$$\int_X [H_i(F(x_i)) - G_i(F(x_i))] dq_i^{K(t)}(x_i) = 0. \quad (A-4)$$

Given the above discussion of  $\mathbf{Q}^{K(t)}$ , (A-2), and properties (A-3) and (A-4), we conclude that  $\mathbf{Q}^{K(t)}$  in Proposition 1 solves  $P_3(t)$ . ■

**Proof of Proposition 2.** The statement is true if a corresponding minimax theorem holds. We invoke the main theorem (Theorem 3) in Terkelsen (1972) and show that the corresponding requirements are all satisfied in our specific setting, which necessarily imply that the pair  $(\mathbf{Q}^{K(t^*)}, t^*)$  also solves  $P_2$ . Theorem 3 in Terkelsen (1972), when solutions exist, states as follows:

---

in  $t$ . We then have

$$\begin{aligned} \bar{c}_i(x_i) + J(x_i, x_j; t) & \geq V(x_i, \underline{x}) + J(x_i, x_j; t) \\ & = v_\omega(x_i) + v_\lambda(x_j) - \frac{bI(x_i < t)}{f(x_i)} - \frac{bI(x_j < t)}{f(x_j)}. \end{aligned}$$

Hence, using our definitions of  $v_\omega(x_i) + v_\lambda(x_j)$ , a sufficient condition for  $\bar{c}_i(x_i) + J(x_i, x_j; t) \geq 0$  is that the buyers' payoffs and the positive externalities are large enough or  $A$  and  $B$  are sufficiently large. Observe that this sufficient condition is by no means necessary.

Let  $X$  be a compact connected set, and  $Y$  be an arbitrary set. If function  $f : X \times Y \rightarrow \mathbb{R}$  satisfies (i) For any  $y_1, y_2 \in Y$ , there exists  $y_0 \in Y$  s.t.  $f(x, y_0) \geq \frac{1}{2}[f(x, y_1) + f(x, y_2)]$  for all  $x \in X$ ; and (ii) Every finite intersection of sets of the form  $\{x \in X : f(x, y) \leq \alpha\}$ , with  $(y, \alpha) \in Y \times \mathbb{R}$ , is closed and connected, then

$$\max_{y \in Y} \min_{x \in X} f(x, y) = \min_{x \in X} \max_{y \in Y} f(x, y).$$

Rewrite the seller's objective function to be  $R(\mathbf{Q}, t) = \pi(\mathbf{Q}) + \phi(\mathbf{Q}, t)$ , where  $\pi(\mathbf{Q})$  and  $\phi(\mathbf{Q}, t)$  are defined in (10). We then have that function  $f$  is  $R$ , variable  $x$  is  $t$ , variable  $y$  is  $\mathbf{Q}$ , set  $X$  is  $[\underline{x}, \bar{x}]$ , and  $Y$  is the set of all  $\mathbf{Q}$ 's that satisfy  $(ND)$  and  $(F)$ , respectively, in our setting.

First, it is easy to show that using the fact that the revenue function is linear in  $\mathbf{Q}$ , we have for any  $\mathbf{Q} = (Q_1, Q_2)$  and  $\mathbf{Q}' = (Q'_1, Q'_2)$  that satisfy  $(F)$  and  $(ND)$ ,

$$R(\eta\mathbf{Q} + (1 - \eta)\mathbf{Q}', t) = \eta R(\mathbf{Q}, t) + (1 - \eta) R(\mathbf{Q}', t), \forall \eta \in [0, 1], t \in X.$$

Setting  $\eta = \frac{1}{2}$  and  $\mathbf{Q}^0 = \frac{\mathbf{Q} + \mathbf{Q}'}{2}$  implies that condition (i) above holds for our setting.

Second, given the non-decreasing condition  $(ND)$ , there are three cases for the function  $\sum_i [aq_i(t) - b\bar{q}_i(t)]$ : (1)  $\sum_i [aq_i(t) - b\bar{q}_i(t)] \geq 0$  for all  $t \in [\underline{x}, \bar{x}] = X$ , (2)  $\sum_i [aq_i(t) - b\bar{q}_i(t)] \leq 0$  for all  $t \in X$ , and (3)  $\sum_i [aq_i(t) - b\bar{q}_i(t)] \leq 0$  for all  $t \in [\underline{x}, \hat{t}]$  and  $\sum_i [aq_i(t) - b\bar{q}_i(t)] > 0$  for all  $t \in [\hat{t}, \bar{x}]$ , where  $\underline{x} < \hat{t} < \bar{x}$ . Let  $(\alpha, \mathbf{Q})$  be any pair s.t.  $\alpha \in \mathbb{R}$  and  $\mathbf{Q}$  satisfies  $(F)$  and  $(ND)$ . Next, because  $\sum_i \int_{\underline{x}}^t (aq_i(z) - b\bar{q}_i(z)) dz$  is (absolutely) continuous and differentiable almost everywhere in  $t$ , with a compact domain  $[\underline{x}, \bar{x}]$ ,<sup>25</sup> the set  $A_\alpha = \{t \in X : R(\mathbf{Q}, t) \leq \alpha\}$ , or equivalently,  $A_\alpha = \{t \in X : \sum_i \int_{\underline{x}}^t (aq_i(z) - b\bar{q}_i(z)) dz \leq \alpha - \pi(\mathbf{Q})\}$  only takes three possible forms:

- a closed interval  $[\underline{x}, t(\alpha, \mathbf{Q})]$  corresponding to case (1), where  $t(\alpha, \mathbf{Q})$  is the value of  $t$  such that  $\sum_i \int_{\underline{x}}^t (aq_i(z) - b\bar{q}_i(z)) dz = \alpha - \pi(\mathbf{Q})$ , or set  $[\underline{x}, \bar{x}]$  if such  $t(\alpha, \mathbf{Q})$  does not exist;
- a closed interval  $[t(\alpha, \mathbf{Q}), \bar{x}]$  corresponding to case (2), where  $t(\alpha, \mathbf{Q})$  is defined similarly as above, or set  $[\underline{x}, \bar{x}]$  if such  $t(\alpha, \mathbf{Q})$  does not exist;
- finally, a closed interval  $[\hat{t}(\alpha, \mathbf{Q}), \bar{t}(\alpha, \mathbf{Q})]$  corresponding to case (3), where we define

$$\begin{aligned} \underline{t}(\alpha, \mathbf{Q}) &= \min \left\{ s : \sum_i \int_{\underline{x}}^s (aq_i(z) - b\bar{q}_i(z)) dz = \alpha - \pi(\mathbf{Q}) \right\} \\ \bar{t}(\alpha, \mathbf{Q}) &= \max \left\{ s : \sum_i \int_{\underline{x}}^s (aq_i(z) - b\bar{q}_i(z)) dz = \alpha - \pi(\mathbf{Q}) \right\}, \end{aligned}$$

or the empty set if  $\left\{ s : \sum_i \int_{\underline{x}}^s (aq_i(z) - b\bar{q}_i(z)) dz = \alpha - \pi(\mathbf{Q}) \right\} = \emptyset$ .

<sup>25</sup>Even if the function  $\sum_i \int_{\underline{x}}^t (aq_i(z) - b\bar{q}_i(z)) dz$  is continuous almost everywhere in  $t$ , we can equivalently focus on  $\mathbf{Q}$  such that  $\sum_i \int_{\underline{x}}^t (aq_i(z) - b\bar{q}_i(z)) dz$  is *always lower semi-continuous* at these discontinuity points. As discontinuity of  $\sum_i \int_{\underline{x}}^t (aq_i(z) - b\bar{q}_i(z)) dz$  only arises at a set with measure zero, such a minor change on  $\mathbf{Q}$  has no effect on the seller's revenue.

It is now immediate to see that any finite intersection of sets  $A_\alpha$ , which are closed intervals or the empty sets, is either the empty set or a closed interval, hence always closed and connected.

We next present a lemma to show rigorously that our solution pair  $(\mathbf{Q}^{K(t^*)}, t^*)$  solves the minimax version of the seller's problem:

**Lemma 4** *The pair  $(\mathbf{Q}^{K(t^*)}, t^*)$  derived from Proposition 1 and (8) is the solution to problem  $(P_2)$ :*

$$(P_2') : \min_{t \in X} \max_{\mathbf{Q}} \{ \pi(\mathbf{Q}) + \phi(\mathbf{Q}, t) \} \text{ subject to } (ND) \text{ and } (F).$$

**Proof.** Define  $R(\mathbf{Q}, t) = \pi(\mathbf{Q}) + \phi(\mathbf{Q}, t)$ . Then for all  $t \in X$ , we have that

$$R(\mathbf{Q}^{K(t^*)}, t^*) \leq R(\mathbf{Q}^{K(t^*)}, t) \leq R(\mathbf{Q}^{K(t)}, t),$$

where the first inequality comes from that  $t^* \in \arg \min_t \int_{\underline{x}}^t (a q_i^{K(t^*)}(z) - b \bar{q}_i^{K(t^*)}(z)) dz$ , which implies that  $t^* \in \arg \min_t R(\mathbf{Q}^{K(t^*)}, t)$ . The second inequality is a result of Proposition 1. ■

We thus conclude that the solution pair  $(\mathbf{Q}^{K(t^*)}, t^*)$  derived from the minimax problem also solves  $P_2$ . ■

**Proof of Lemma 3.** Without loss of generality, we assume that  $x_1 \geq x_2$ . Because  $\bar{c}_1 = \bar{c}_2$ ,  $J(x_1, x_2) = J(x_2, x_1)$  and  $\bar{c}_1$  is non-decreasing,

$$\max(\bar{c}_1(x_1) + J(x_1, x_2), \bar{c}_2(x_2) + J(x_2, x_1)) = \bar{c}_1(x_1) + J(x_1, x_2).$$

First, we consider the case where  $x_1 \leq \frac{3}{2}$ . Because  $J(x_1, x_2)$  is non-increasing in  $x_2$  for  $x_2 \leq \frac{3}{2}$ ,  $J(x_1, x_2) \geq J(x_1, x_1)$  and

$$\bar{c}_1(x_1) + J(x_1, x_2) \geq \bar{c}_1(x_1) + J(x_1, x_1) \quad \text{for } x_1 \leq \frac{3}{2}. \quad (A-5)$$

Similarly, we have

$$\bar{c}_1(x_1) + J(x_1, x_2) \geq \bar{c}_1(x_1) + J(x_1, x_1) \quad \text{for } x_1 > \frac{3}{2}, x_2 > \frac{3}{2}. \quad (A-6)$$

Last, we consider the case where  $x_1 > \frac{3}{2}$  and  $x_2 \leq \frac{3}{2}$ . Because  $\bar{c}_1$  is non-decreasing and  $J(x_1, x_2)$  is non-increasing in  $x_1$  and  $x_2$ , for  $x_1 > \frac{3}{2}$  and  $x_2 \leq \frac{3}{2}$ , we have  $\bar{c}_1(x_1) \geq \bar{c}_1(\frac{3}{2})$  and  $J(x_1, x_2) \geq J(2, \frac{3}{2})$  and

$$\bar{c}_1(x_1) + J(x_1, x_2) \geq \bar{c}_1(\frac{3}{2}) + J(2, \frac{3}{2}) = \frac{\Delta}{8}(1-2\Delta) \geq 0 \quad \text{for } x_1 > \frac{3}{2}, x_2 \leq \frac{3}{2}. \quad (A-7)$$

Inequalities (A – 5) - (A – 6) imply that

$$\begin{aligned} & \max(\bar{c}_1(x_1) + J(x_1, x_2), \bar{c}_2(x_2) + J(x_2, x_1)) \\ & \geq \min_{y \in [1, 2]} (\bar{c}_1(y) + J(y, y)), \forall (x_1, x_2) \in [1, \frac{3}{2}]^2 \cup (\frac{3}{2}, 2]^2. \end{aligned} \quad (A - 8)$$

It remains to show that  $\min_{y \in [1, 2]} (\bar{c}_1(y) + J(y, y)) \geq 0$ . For  $t = \frac{3}{2}$ , simple algebra yields that

$$\bar{c}_1(y) + J(y, y) = \begin{cases} -\Delta^2/4 + \Delta/4 & \text{for } y \in [1, 5/4], \\ \frac{7}{8}\Delta - \frac{\Delta^2}{4} - \frac{\Delta}{2}y & \text{for } y \in (5/4, 3/2], \\ \frac{9}{8}\Delta - \frac{\Delta^2}{4} - \frac{\Delta}{2}y & \text{for } y \in (3/2, 7/4], \\ -\Delta^2/4 + \Delta/4 & \text{for } y \in (7/4, 2]. \end{cases}$$

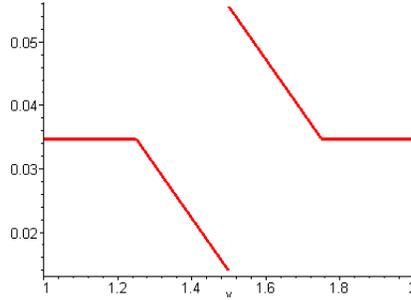


Figure 10. An illustration of function  $\bar{c}_1(y) + J(y, y)$ .

As shown in Figure 10,  $\bar{c}_1(y) + J(y, y)$  is a spline with four linear segments. Because a minimum of a spline must be at a corner of its segments, we check the value of  $\bar{c}_1(y) + J(y, y)$  at each corner. The value of  $\bar{c}_1(y) + J(y, y)$  at each corner is as follows:

$$\begin{aligned} \bar{c}_1(1) + J(1, 1) &= \bar{c}_1(\frac{5}{4}) + J(\frac{5}{4}, \frac{5}{4}) = \bar{c}_1(\frac{7}{4}) + J(\frac{7}{4}, \frac{7}{4}) = \bar{c}_1(2) + J(2, 2) = \frac{\Delta}{4}(1 - \Delta) \geq 0 \\ \bar{c}_1(\frac{3}{2}) + J(\frac{3}{2}, \frac{3}{2}) &= \frac{\Delta}{8}(1 - 2\Delta) \geq 0 \quad \text{and} \quad \lim_{y \rightarrow 3/2^+} \bar{c}_1(y) + J(y, y) = \frac{\Delta}{8}(3 - 2\Delta) \geq 0 \end{aligned}$$

All the inequalities employ the fact that  $\Delta \leq 1/6$ . These inequalities show that  $\bar{c}_1(y) + J(y, y) \geq 0$  at each corner and imply that

$$\min_{y \in [1, 2]} (\bar{c}_1(y) + J(y, y)) \geq 0. \quad (A - 9)$$

Inequalities (A – 8) and (A – 9) yield that

$$\max(\bar{c}_1(x_1) + J(x_1, x_2), \bar{c}_2(x_2) + J(x_2, x_1)) \geq 0, \forall (x_1, x_2) \in [1, \frac{3}{2}]^2 \cup (\frac{3}{2}, 2]^2. \quad (A - 9)$$

Inequalities (A – 7) and (A – 9) imply the lemma. ■

Finally, we present following lemma shows that if the monotone hazard rate condition holds then our claim for (9) holds:

**Lemma 5** *If the monotone hazard rate condition  $\left(\frac{1-F(x_i)}{f(x_i)}\right)$  is non-increasing in  $x_i$  holds, then there exist  $t^*$  such that  $aq_i^{K(t^*)}(t^*) - b\bar{q}_i^{K(t^*)}(t^*) = 0$ .*

**Proof.** We present two claims which together imply the result in the lemma.

*Claim 1:*  $aq_i^{K(t)}(t) - b\bar{q}_i^{K(t)}(t) < 0$  for  $t = \underline{x}$ . For  $t = \bar{x}$ ,  $aq_i^{K(t)}(t) - b\bar{q}_i^{K(t)}(t) > 0$ . For  $t = \underline{x}, \bar{x}$ , with the monotone hazard rate assumption,  $V(x_i; t)$  is strictly increasing in  $x_i$  for all  $x_i \in [\underline{x}, \bar{x}]$ . Following Proposition 1, one can show that for  $t \in \{\underline{x}, \bar{x}\}$ ,

$$Q_i^{K(t)}(x) = 1 \quad \text{if } x_i > x_j$$

$$Q_i^{K(t)}(x) = 0 \quad \text{if } x_i < x_j$$

$$Q_i^{K(t)}(x) = 1/2 \quad \text{if } x_i = x_j$$

It follows that  $q_i^{K(t)}(x_i) = F(x_i)$  and  $\bar{q}_i^{K(t)}(x_i) = 1 - F(x_i)$ . Therefore, for  $t = \underline{x}$ ,  $aq_i^{K(t)}(\underline{x}) - b\bar{q}_i^{K(t)}(\underline{x}) = aF(\underline{x}) - b(1 - F(\underline{x})) = -b < 0$ . Similarly, for  $t = \bar{x}$ ,  $aq_i^{K(t)}(\bar{x}) - b\bar{q}_i^{K(t)}(\bar{x}) = aF(\bar{x}) - b(1 - F(\bar{x})) = a > 0$ . This ends the proof of *Claim 1*.

*Claim 2:*  $aq_i^{K(t)}(t) - b\bar{q}_i^{K(t)}(t)$  is continuous in  $t$ .

Notice that from Proposition 1, we have  $aq_i^{K(t)}(t) - b\bar{q}_i^{K(t)}(t) = (a + b)q_i^{K(t)}(t) - b$ . Therefore, we only need to show that  $q_i^{K(t)}(t)$  is continuous in  $t$ .

We first focus on the case where  $t \in (\underline{x}, \bar{x})$ . Boundedness of  $h_i$  implies that  $H_i(x_i, t) = \int_{\underline{x}}^t h_i(y; t) dy + \int_t^{x_i} h_i(y; t) dy$  is continuous everywhere in  $(x_i, t)$ . Because  $h_i(x_i; t)$  is increasing for  $x_i \in [\underline{x}, t]$ ,  $H_i(x_i; t)$  is convex for  $x_i \in [\underline{x}, t]$ . Similarly,  $H_i(x_i; t)$  is convex for  $x_i \in (t, \bar{x}]$ . Moreover,  $H_i(x_i; t)$  is kinked at  $x_i = t$  and  $H_i'(x_i; t) = h_i(x_i; t)$  for all  $x_i \neq t$ .

Given such properties of  $H_i$ ,  $G_i$  (the convex hull of  $H_i$ ) is defined as the following:

$$\begin{aligned} G_i(x_i; t) &= H_i(x_i; t) \quad \text{for } x_i \notin [x_i^L, x_i^R] \\ G_i(x_i; t) &= H_i(x_i^L; t) + H_i'(x_i^L; t)(x_i - x_i^L) \quad \text{for } x_i \in (x_i^L, x_i^R) \end{aligned} \quad (17)$$

where  $\underline{x} \leq x_i^L < t < x_i^R \leq \bar{x}$  are uniquely defined by the following conditions:

$$\text{Case 1: } x_i^L > \underline{x}, x_i^R < \bar{x}, H_i'(x_i^L; t) = H_i'(x_i^R; t), \frac{H_i(x_i^R; t) - H_i(x_i^L; t)}{x_i^R - x_i^L} = H_i'(x_i^L; t). \quad (18)$$

$$\text{Case 2: } x_i^L = \underline{x}, x_i^R < \bar{x}, H_i'(x_i^L; t) > H_i'(x_i^R; t), \frac{H_i(x_i^R; t) - H_i(x_i^L; t)}{x_i^R - x_i^L} = H_i'(x_i^R; t).$$

$$\text{Case 3 : } x_i^L > \underline{x}, x_i^R = \bar{x}, H_i'(x_i^L; t) > H_i'(x_i^R; t), \frac{H_i(x_i^R; t) - H_i(x_i^L; t)}{x_i^R - x_i^L} = H_i'(x_i^L; t).$$

$$\text{Case 4 : } x_i^L = \underline{x}, x_i^R = \bar{x}, H_i'(x_i^L; t) > H_i'(x_i^R; t).$$

By the implicit function theorem, we can write  $x_i^L$  and  $x_i^R$  as functions of  $t$ ;  $x_i^L(t), x_i^R(t)$ . Because  $H_i(x_i, t)$  and  $h_i(x_i, t)$  are continuous in  $(x_i, t)$  for  $x_i \neq t$  and  $x_i^L \neq t, x_i^R \neq t, x_i^L(t)$  and  $x_i^R(t)$  are both continuous in  $t$ .

With  $G_i$  defined by (17),  $\bar{c}_i(x_i) = G_i'(x_i)$  is strictly increasing for  $x_i \notin [x_i^L, x_i^R]$  and  $\bar{c}_i(x_i)$  is flat for  $x_i \in [x_i^L, x_i^R]$ . Note that  $\bar{c}_1 = \bar{c}_2$ . Hence,  $Q^{K(t)}$  is as followed:

$$\begin{aligned} Q_i^{K(t)}(x) &= 1 \quad \text{for } x_i > x_j \quad \text{and } x \notin [x_i^L, x_i^R]^2 \\ Q_i^{K(t)}(x) &= 0 \quad \text{for } x_i < x_j \quad \text{and } x \notin [x_i^L, x_i^R]^2 \\ Q_i^{K(t)}(x) &= 1/2 \quad \text{otherwise.} \end{aligned}$$

It follows that

$$\begin{aligned} q_i^{K(t)}(x_i) &= F(x_i) \quad \text{for } x_i \notin [x_i^L, x_i^R] \\ q_i^{K(t)}(x_i) &= F(x_i^L) + (F(x_i^R) - F(x_i^L))/2 \quad \text{otherwise.} \end{aligned}$$

From  $x_i^L < t < x_i^R$ , we have  $q_i^{K(t)}(t) = F(x_i^L) + (F(x_i^R) - F(x_i^L))/2$ . Because  $x_i^L$  and  $x_i^R$  is continuous in  $t$ ,  $q_i^{K(t)}(t)$  is continuous in  $t$  for  $t$  in  $(\underline{x}, \bar{x})$ .

Now, we consider the continuity of  $q_i^{K(t)}(t)$  in case that  $t = \underline{x}$ . From the proof of Claim 1, for  $t = \underline{x}$ , we have  $q_i^{K(t)}(t) = F(t)$ . Observe that as  $t \rightarrow \underline{x}$ ,  $x_i^L \rightarrow \underline{x}$ . From (18) and

$$\lim_{x_i^R \rightarrow \underline{x}} \frac{H_i(x_i^R, t) - H_i(\underline{x}; t)}{x_i^R - \underline{x}} = H_i'(\underline{x}; t)$$

as  $t \rightarrow \underline{x}$ ,  $x_i^R > t$  also approaches  $\underline{x}$ . Hence, as  $t \rightarrow \underline{x}$ ,  $x_i^L$  and  $x_i^R$  approaches  $\underline{x}$ . It follows that as  $t \rightarrow \underline{x}$ ,  $q_i^{K(t)}(t) \rightarrow F(\underline{x})$  and  $q_i^{K(t)}(t)$  is right continuous at  $t = \underline{x}$ . Similarly, we can show that  $q_i^{K(t)}(t)$  is left continuous at  $t = \bar{x}$ .

We now establish the continuity of  $q_i^{K(t)}(t)$  for  $t \in [\underline{x}, \bar{x}]$ , which proves *Claim 2*. ■

## References

- [1] Aseff, J. and Chade, H., 2008. "An Optimal Auction with Identity-Dependent Externalities," *Rand Journal of Economics*, 39, 731-746.
- [2] Bagwell, K., Mavroidis P. C. and Staiger, R. W., 2007. "Auctioning Countermeasures in the WTO," *Journal of International Economics* 73, 309-332.
- [3] Bagwell, K. and Staiger, R. W., 1999. "An Economic Theory of GATT," *American Economic Review*, 89, 215-48.
- [4] Baldwin, R., 1987. "Politically Realistic Objective Functions and Trade Policy," *Economics Letters*, 24, 287-90.
- [5] Bester, H. and Strausz, R., 2001. "Contracting with Imperfect Commitment and the Revelation Principle: The Single Agent Case," *Econometrica*, 69, 1077-1098.

- [6] Brocas, I., 2007. "Auctions with Type-dependent and Negative Externalities: the Optimal Mechanism," *mimeo*, USC.
- [7] Figueroa, N. and Skreta, V., 2009a. "The Role of Optimal Threats in Auction Design," *Journal of Economic Theory*, 144, 884-897.
- [8] Figueroa, N. and Skreta, V., 2009b. "Optimal Allocation Mechanisms with Single-Dimensional Private Information," *mimeo*, NYU.
- [9] Grossman, G. M. and Helpman, E., 1994. "Protection for Sale," *American Economic Review*, 84, 833-50.
- [10] Jehiel, P. and Moldovanu, B., 2000. "Auctions with Downstream Interaction among Buyers," *Rand Journal of Economics*, 31, 768-791.
- [11] Jehiel, P., Moldovanu, B. and Stacchetti, E., 1996. "How (Not) to Sell Nuclear Weapons," *American Economic Review*, 86, 814-29.
- [12] Jehiel, P., Moldovanu, B. and Stacchetti, E., 1999. "Multidimensional mechanism design for auctions with externalities," *Journal of Economic Theory* 85, 258-93.
- [13] Jullien, B., 2000. "Participation Constraints in Adverse Selection Models," *Journal of Economic Theory*, 93, 1-47.
- [14] Lawrence, R. Z., 2003. *Crime & Punishments? Retaliation Under the WTO*, Institute for International Economics.
- [15] Lewis, T. R. and Sappington, D.E.M., 1989, "Countervailing Incentives in Agency Problems," *Journal of Economic Theory*, 49, 294-313.
- [16] Maggi, G., 1999. "The Role of Multilateral Institutions in International Trade Cooperation," *American Economic Review*, 89, 190-214.
- [17] Maggi, G. and Rodriguez, A., 1995. "On Countervailing Incentives," *Journal of Economic Theory*, 66, 238-263.
- [18] Myerson, R.B., 1981. "Optimal Auction Design," *Mathematics of Operation Research*, 6, 58-73.
- [19] Terkelsen, F., 1972. "Some Minimax Theorems," *Math. Scand.* 31, 405-413.
- [20] WTO 2002. "Dispute Settlement Body - Special Session - Negotiations on Improvements and Clarifications of the Dispute Settlement Understanding - Proposal by Mexico," November 4, catalogue record TN/DS/W/23.