Supplementary Material to the Revised Paper: "Decentralized Market Processes to Stable Job Matchings with Competitive Salaries"

Bo Chen, Satoru Fujishige, and Zaifu Yang

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We provide several results that are omitted in the paper due to space limit but might help the reader to have a better understanding of the paper.

The supplementary material contains three parts. The first part presents two auxiliary results demonstrating that it suffices for us to focus on integral payoff vectors. The second part contains an example illustrating non-monotonicity of market values of a labor market along a path of bilateral trades. And the third part shows the major difference between our algorithm and the well-known algorithms in Crawford and Knoer (1981) and Demange, Gale and Sotomayor (1986).

Part I. One Theorem and One Lemma

A state or outcome of the market (F, W, V) consists of a matching μ and a payoff vector $u \in \mathbb{R}^{F \cup W}$ such that $u(x) = V(x)$ for any $x \in I(\mu)$ and $u(x) + u(\mu(x)) = V(x, \mu(x))$ for $x \notin I(\mu)$, where $I(\mu) = \{h \in F \cup W \mid \mu(h) = h\}$ is the set of self-matched agents at μ.

A market state (μ, u) is *stable* or equivalently *a competitive equilibrium* if $u(f) + u(w) \geq V(f, w)$ and $u(x) \geq V(x)$ for all $f \in F$, $w \in W$, $x \in F \cup W$.

Observe that in the definition of market state (μ, u), the payoff $u \in \mathbb{R}^{F \cup W}$ specifies a payoff for every agent $x \in F \cup W$. In fact, this is not necessary, because knowing the payoff vector for either all workers or all firms will automatically specify the payoff for all members on the other side of the market. We can therefore fully characterize each market state by specifying only the payoff vector for workers. For instance, a market state (μ, u) with $u \in \mathbb{R}^W$ specifies a payoff vector $u \in \mathbb{R}^F$ for firms as follows: for each firm $f \in F$, $u(f) = V(f)$ if $\mu(f) = f$, and $u(f) = V(f, u(f)) - u(u(f))$ if $u(f) = w \in W$. Let $CE(W)$ be the set of all competitive equilibrium payoff vectors $u \in \mathbb{R}^W$ for all workers in the market (F, W, V). It is known from Koopmans and Beckmann (1957) and Shapley and Shubik (1971) that the market (F, W, V) has at least one competitive equilibrium and thus $CE(W)$ is not empty. Furthermore, it is well-known from Shapley and Shubik (1971) that the set of competitive equilibrium price vectors is a lattice, i.e., $CE(W)$ is a lattice. The following result shows that the labor market has at least one integral competitive equilibrium payoff vector if all values $V(f, w)$, $V(f)$ and $V(w)$ are integral. This result is not new and is implied by more general results obtained by Ausubel (2006, Corollary to

Proposition 1, p. 625) and Sun and Yang (2009, Theorem 3 (ii), p.939). Below we give a simpler and more direct proof of this special but basic result.

Theorem 1 . *The labor market (F, W, V) has at least one integral competitive equilibrium payoff vector if $V(f, w)$, $V(f)$ and $V(w)$ are integral for all $f \in F$ and $w \in W$.*

Proof. Clearly, it is sufficient to show that $CE(W)$ contains an integral payoff vector in $\mathbb{Z}^W$. Since $CE(W)$ is a nonempty lattice, it contains a unique minimum payoff vector $u^* \in \mathbb{R}^W$, i.e., $u^* \in CE(W)$ and $u^* \leq u$ for all $u \in CE(W)$. We will show that u^* itself is an integral vector. Let (μ^*, u^*) be the competitive equilibrium associated with $u^* \in \mathbb{R}^W$. For every firm $f \in F$, we have $u^*(f) = V(f)$ if $\mu^*(f) = f$, and $u^*(f) = V(f, w) - u^*(w)$ if $\mu^*(f) = w \in W$.

Define $W_1 = \{w \in W \mid u^*(w) \in \mathbb{Z}\}$–the set of workers whose payoffs are integers, and $W_2 = \{w \in W \mid u^*(w) \notin \mathbb{Z}\}$–the set of workers whose payoffs are not integers. Clearly, $W_1 \cup W_2 = W$ and $W_1 \cap W_2 = \emptyset$. We have to prove that W_2 is empty.

Using W_1 and W_2 we define a partition (F_1, F_2, F_3) of F: $F_1 = \{f \in F \mid \mu^*(f) \in W_1\}$, $F_2 = \{f \in F \mid \mu^*(f) \in W_2\}$, and $F_3 = \{f \in F \mid \mu^*(f) = f\}$. Clearly, $F_i \cap F_j = \emptyset$ for any $i \neq j$ and $\cup_{h=1}^3 F_h = F$. $u^*(f)$ is integral for every $f \in F_1$, $u^*(f)$ is integral for every $f \in F_3$, but $u^*(f)$ is not integral for any $f \in F_2$.

Recall that because (μ^*, u^*) is a competitive equilibrium, we have

$$u^*(f) + u^*(w) \geq V(f, w), \ \ \forall f \in F, \forall w \in W \tag{1}$$

$$u^*(x) \geq V(x), \ \ \forall x \in F \cup W \tag{2}$$

Clearly, $u^*(w) \geq V(w)$ for all $w \in W_1$, $u^*(f) \geq V(f)$ for all $f \in F_1$, $u^*(w) > V(w)$ for all $w \in W_2$, $u^*(f) > V(f)$ for all $f \in F_2$, and $u^*(f) \geq V(f)$ for all $f \in F_3$, because $V(x)$ is integral for all $x \in F \cup W$.

It follows from (1) that for any $f \in F_1$ we have

$$u^*(f) + u^*(w) > V(f, w), \ \ \forall w \in W_2 \tag{3}$$

and for any $f \in F_2$ we have

$$u^*(f) + u^*(w) > V(f, w), \ \ \forall w \in W_1 \tag{4}$$

and for any $f \in F_3$ we have

$$u^*(f) + u^*(w) > V(f, w), \ \ \forall w \in W_2 \tag{5}$$

We can then choose a sufficiently small $\epsilon > 0$ for the payoff vector $\bar{u} = u^* - \epsilon \sum_{j \in W_2} e(j) \in \mathbb{R}^W$ so that $\bar{u}(w) > V(w)$ for all $w \in W_2$ and the above inequalities (3), and (5) still hold when u^* is replaced by $\bar{u}$. Here $e(j)$ is the jth unit vector in $\mathbb{R}^W$ for each $j \in W$.

Observe that $\bar{u}(w) = u^*(w) - \epsilon$ for every $w \in W_2$, and $\bar{u}(w) = u^*(w)$ for every $w \in W_1$. For any firm $f \in F_1$, $\bar{u}(f) = u^*(f)$ is integral, and for any firm $f \in F_3$, $\bar{u}(f) = u^*(f) = V(f)$ is integral, and for any firm $f \in F_2$, $\bar{u}(f) = u^*(f) + \epsilon$ is not integral. We have $\bar{u}(x) \geq V(x)$ for all $x \in F \cup W$. Note that (4) trivially holds when u^* is replaced by $\bar{u}$.

We will show that $(\mu^*, \bar{u})$ is also a competitive equilibrium. Observe that for any $f \in F_1$ we have

$$\bar{u}(f) + \bar{u}(w) > V(f, w), \;\; \forall w \in W_2$$

$$\bar{u}(f) + \bar{u}(w) > V(f, w), \;\; \forall w \in W_1$$

and for any $f \in F_2$ we have

$$\bar{u}(f) + \bar{u}(w) > V(f, w), \;\; \forall w \in W_1$$

$$\bar{u}(f) + \bar{u}(w) > V(f, w), \;\; \forall w \in W_2$$

and for any $f \in F_3$ we have

$$\bar{u}(f) + \bar{u}(w) > V(f, w), \;\; \forall w \in W_2$$

$$\bar{u}(f) + \bar{u}(w) \geq V(f, w), \;\; \forall w \in W_1$$

In summary, we have

$$\bar{u}(f) + \bar{u}(w) \geq V(f, w), \;\; \forall f \in F, \; \forall w \in W$$

$$\bar{u}(x) \geq V(x), \;\; \forall x \in F \cup W$$

By definition, $(\mu^*, \bar{u})$ is a competitive equilibrium, which contradicts the fact that u^* is the smallest competitive payoff vector in $CE(W)$, since $\bar{u} < u^*$. Consequently, W_2 must be empty and u^* must be an integral payoff vector. □

A *blocking pair* of a state (μ, u) is a pair (f, w) of firm f and worker w that are not matched under μ but both can improve their well-being by abandoning their partners at μ and matching with each other, i.e., there are $r_f, r_w \in \mathbb{R}$ with $r_f + r_w = V(f, w)$ such that $r_w \geq u(w)$ and $r_f \geq u(f)$ with at least one strict inequality. A state (μ, u) can also be blocked by a single agent $x \in F \cup W$ if x is not self-matched at μ, $x \neq \mu(x)$, but prefers to be single, i.e., $r_x = V(x) > u(x)$.

The following lemma is new and plays an important role in our analysis and is now also included in the paper.

Lemma 1 *Let $V(f,w)$, $V(f)$ and $V(w)$ be integral for all $f \in F$ and $w \in W$. If a state (μ, u) with $u \in \mathbb{Z}^{F\cup W}$ is not blocked by any pair (f,w) with $(r_f, r_w) \in \mathbb{Z}\times\mathbb{Z}$, then it cannot be blocked by any pair (f', w') with $(r_{f'}, r_{w'}) \in \mathbb{R}\times\mathbb{R}$. Consequently, (μ,u) must be a competitive equilibrium.*

Proof. Suppose to the contrary that the statement is not true. Then there would exist a state (μ,u) with $u \in \mathbb{Z}^{F\cup W}$ which is not blocked by any pair (f,w) with $(r_f,r_w) \in \mathbb{Z}\times\mathbb{Z}$, but is blocked by a pair (f',w') with $(r'_f, r'_w) \in \mathbb{R}\times\mathbb{R}$. Because (f',w') blocks (μ,u), $r_{f'} + r_{w'} = V(f',w')$, $r_{f'} \geq u(f')$ and $r_{w'} \geq u(w')$ with at least one strict inequality. Since $V(f',w')$ and $u \in \mathbb{Z}^{F\cup W}$ are integral, we must have that either both $r_{f'}$ and $r_{w'}$ are integral or neither $r_{f'}$ nor $r_{w'}$ is integral. The former case cannot happen by hypothesis. In the latter case, we must have $r_{f'} > u(f')$ and $r_{w'} > u(w')$. Now let $f = f'$ and $w = w'$. We can round up r_f to its next higher integer s_f and round down r_w to its next lower integer s_w. Because u and $V(f,w)$ are integral, clearly we have $s_f + s_w = V(f,w)$, $s_f > u(f)$ and $s_w \geq u(w)$. By definition, (μ,u) is blocked by (f,w) with $(s_f,s_w) \in \mathbb{Z}\times\mathbb{Z}$, contradicting the hypothesis. □

The above two results imply that it is sufficient to focus on integral payoff vectors in $\mathbb{Z}^{F\cup W}$.

Part II. Nonmonotonicity of Market Values Along a Path of Bilateral Trades

The following example shows that an arbitrary sequence of successive pair improvements may yield trading cycles. In particular, the example demonstrates the complexity of finding a deterministic path of pair improvements toward stability in that the choices of both surplus division rules and blocking pairs are important. This example also shows clearly that in the process of pair improvements the overall welfare need not be monotone, some agents may be better off this time but may be worse off another time, and other may have opposing change of welfare.

Example 1: Consider a labor market (F,W,V) with $F = \{a,b\}$, $W = \{x,y\}$, and $V(i,i) = 0$, $\forall i \in F\cup W$, $V(a,i) = 4$ and $V(b,i) = 6$, $\forall i \in W$.

Start with an initial state (μ^0,u^0) with $\mu^0 = \{(a,x),(b,b),(y,y)\}$, i.e., b and y are self-matched, and $u^0(a) = u^0(x) = 2$ and $u^0(i) = 0$ otherwise.

Choose the first blocking pair to be (b,x), resulting in (we list the agents' integral payoffs below the matching)

$$(\mu^1,u^1) = \left\{ \left(\underset{0}{a},\underset{0}{a}\right), \left(\underset{3}{b},\underset{3}{x}\right), \left(\underset{0}{y},\underset{0}{y}\right) \right\}.$$

Now b and y can form the next blocking pair, leading to

$$\left(\mu^2, u^2\right) = \left\{ \left(\underset{0}{a}, \underset{0}{a}\right), \left(\underset{4}{b}, \underset{2}{y}\right), \left(\underset{0}{x}, \underset{0}{x}\right) \right\}.$$

We then choose (a, y) as the next blocking pair, leading to

$$\left(\mu^3, u^3\right) = \left\{ \left(\underset{1}{a}, \underset{5}{y}\right), \left(\underset{0}{b}, \underset{0}{b}\right), \left(\underset{0}{x}, \underset{0}{x}\right) \right\}.$$

Finally, we choose blocking pair (a, x), which when satisfied, gives

$$\left(\mu^4, u^4\right) = \left\{ \left(\underset{2}{a}, \underset{2}{x}\right), \left(\underset{0}{b}, \underset{0}{b}\right), \left(\underset{0}{y}, \underset{0}{y}\right) \right\},$$

completing the cycle. Notice in particular that the market value is not monotonic along the path of pair improvements in the example, namely, the sequence of market values is $(4, 6, 6, 4, 6, 6, 4, \ldots)$.

Part III. The Difference between Our Algorithm and Two Previous Algorithms

The following example illustrates the major difference between our algorithm and the salary adjustment process in Crawford and Knoer (1981) (CK81 in short), and Demange, Gale and Sotomayor (1986) (DGS86).

Example 2: Consider a labor market where $F = \{a, b\}$ and $W = \{x, y, z\}$. Let $V(a, x) = V(a, y) = V(b, x) = V(b, y) = 3$, $V(a, z) = V(b, z) = 2$, and $V(i, i) = 0$ for all $i \in F \cup W$, i.e., all agents have an outside option of 0. We start with an initial market state (μ^0, u^0) as:

$$\left(\mu^0, u^0\right) = \left\{ \left(\underset{1}{a}, \underset{2}{x}\right), \left(\underset{1}{b}, \underset{1}{z}\right), \left(\underset{0}{y}, \underset{0}{y}\right) \right\}.$$

Here the agents' payoffs are listed below each agent and (y, y) indicates that y is matched with itself.

Our Algorithm

We apply our algorithm in Section 4 of our paper to find a path of bilateral trades for (μ^0, u^0) toward a stable outcome. Given (μ^0, u^0), the sets of best firms are $F_x = F_y = F_z = \{a, b\}$ and we choose the lists as $L_x = ab, L_y = ba, L_z = ba$.

First, an execution of steps **S1** and **S2** using lists L_x, L_y and L_z leads to the following successive outcomes:

$$\begin{aligned}
y \text{ applies to } b &: \quad (\mu^1, u^1) = \left\{ \left(\underset{1}{a}, \underset{2}{x}\right), \left(\underset{1}{b}, \underset{2}{y}\right), \left(\underset{0}{z}, \underset{0}{z}\right) \right\}. \\
z \text{ applies to } a &: \quad (\mu^2, u^2) = \left\{ \left(\underset{1}{a}, \underset{1}{z}\right), \left(\underset{1}{b}, \underset{2}{y}\right), \left(\underset{0}{x}, \underset{0}{x}\right) \right\}. \\
x \text{ applies to } b &: \quad (\mu^3, u^3) = \left\{ \left(\underset{1}{a}, \underset{1}{z}\right), \left(\underset{1}{b}, \underset{2}{x}\right), \left(\underset{0}{y}, \underset{0}{y}\right) \right\}. \\
y \text{ applies to } a &: \quad (\mu^4, u^4) = \left\{ \left(\underset{1}{a}, \underset{2}{y}\right), \left(\underset{1}{b}, \underset{2}{x}\right), \left(\underset{0}{z}, \underset{0}{z}\right) \right\}. \\
z \text{ applies to } b &: \quad (\mu^5, u^5) = \left\{ \left(\underset{1}{a}, \underset{2}{y}\right), \left(\underset{1}{b}, \underset{1}{z}\right), \left(\underset{0}{x}, \underset{0}{x}\right) \right\}. \\
x \text{ applies to } a &: \quad (\mu^6, u^6) = \left\{ \left(\underset{1}{a}, \underset{2}{x}\right), \left(\underset{1}{b}, \underset{1}{z}\right), \left(\underset{0}{y}, \underset{0}{y}\right) \right\}.
\end{aligned}$$

Since $(\mu^6, u^6) = (\mu^0, u^0)$, we have found a cycle. Collect the firms involved in the cycle in $F_Q = \{a, b\}$ — these are "over-demanded" firms in the two steps.

Next, we carry out the **Augment** procedure to increase the payoffs of firms a and b by one, using only pair improvements. As described before, we define a bookkeeping set F^*, initially an empty set, i.e., $F^* := \varnothing$. The **Augment** procedure ends when $F^* = F_Q$. Rename μ^6 to be μ and notice that $w^* = y$.

1. Let the first alternating path be $(f_1, w^*) = (b, y)$. We match this pair and increase b's payoff (from that in (μ^6, u^6)) by one to obtain:

$$(\mu, u) = \left\{ \left(\underset{1}{a}, \underset{2}{x}\right), \left(\underset{2}{b}, \underset{1}{y}\right), \left(\underset{0}{z}, \underset{0}{z}\right) \right\}.$$

2. Update F^* so that $F^* = \{b\}$. The new single firm is $w^* = z$. Choose the second alternating path from z to a to be $\underline{(b, z)}, (b, y), \underline{(a, y)}$, where pairs with underscores are currently not matched (notice that the last firm in this alternating path is $a \in F_Q \backslash F^*$). We proceed from the back so that a is matched with y and a's payoff increases to 2, which is a pair improvement. After y breaks up with b, b becomes single and we matched b with z so that b's payoff is back to be 2, which is again a pair improvement. In the end, we have:

$$(\mu', u') = \left\{ \left(\underset{2}{a}, \underset{1}{y}\right), \left(\underset{2}{b}, \underset{0}{z}\right), \left(\underset{0}{x}, \underset{0}{x}\right) \right\}.$$

Since now $F^* = F_Q = \{a, b\}$ after the above two steps, the **Augment** procedure terminates and we are now at step **S4**.

Notice that (μ', u') is not stable ((b, x) is a blocking pair), we rename (μ', u') to be (μ^0, u^0) and let $w^0 = x$. Notice that we still have $F_x = F_y = F_z = \{a, b\}$. Choose $L_x = ba$ (L_y and L_z can be arbitrary). We now carry out step **S1** again. Exactly one execution leads to

$$x \text{ applies to } b: \quad (\mu^1, u^1) = \left\{ \left(\underset{2}{a}, \underset{1}{y} \right), \left(\underset{2}{b}, \underset{1}{x} \right), \left(\underset{0}{z}, \underset{0}{z} \right) \right\}.$$

It is easily verified that this last market state (μ^1, u^1) is indeed a stable outcome–a competitive equilibrium, ending the execution of the algorithm.

Notice that in our algorithm, because of pair improvements there is no monotonicity. It is easy to see that for worker y, the path of his payoffs is $(2, 2, 0, 2, 0, 1)$, and for worker z, the path of his payoffs is $(0, 1, 1, 0, 1, 1, 0)$.

In the following, we will use the same example to illustrate CK81's process and DGS86's. DGS86's process is an important improvement of CK81's in the sense that when valuations are integer, DGS86's process can find a stable outcome in finite time, while CK81's process can (1) only find a discrete core allocation, which is strictly weaker than the concept of competitive equilibrium or stable outcome, and (2) approach an equilibrium only through a limiting argument by letting the sequence of positive increments (e.g., $1, 1/2, 1/4, ...$) converge to zero.

The DGS86's Process

Step 1. The DGS86 process starts with the reservation price of agents on one side of the market, say, $u(a) = u(b) = 0$. The auctioneer announces the price vector $p^0 = (u(a), u(b)) = (0, 0)$.
Step 2. Each worker $w \in W$ reports its demand $D^w(p^0)$ of firms at p^0. We have $D^x(p^0) = \{a, b\}$, $D^y(p^0) = \{a, b\}$, and $D^z(p^0) = \{a, b\}$. The auctioneer checks if there is any over-demanded set of firms. The set $\{a, b\}$ is over-demanded. The auctioneer increases the price of firms a and b each by 1 and announces the current price $p^1 = (u(a), u(b)) = (1, 1)$.
Step 3. Each worker $w \in W$ reports its demand $D^w(p^1)$ of firms at p^1. We have $D^x(p^1) = \{a, b\}$, $D^y(p^1) = \{a, b\}$, and $D^z(p^1) = \{a, b\}$. The auctioneer checks if there is any over-demanded set of firms. The set $\{a, b\}$ is over-demanded. The auctioneer increases the price of firms a and b each by 1 and announces the current price $p^2 = (u(a), u(b)) = (2, 2)$.
Step 4. Each worker $w \in W$ reports its demand $D^w(p^2)$ of firms at p^2. We have $D^x(p^2) = \{a, b\}$, $D^y(p^2) = \{a, b\}$, and $D^z(p^2) = \{z, a, b\}$. The auctioneer checks if there is any over-demanded set of firms. There is no over-demanded set of firms and the auction stops at

one of the two competitive equilibria:

$$(\mu^1, u^1) = \left\{ \left(\underset{2}{a}, \underset{1}{y}\right), \left(\underset{2}{b}, \underset{1}{x}\right), \left(\underset{0}{z}, \underset{0}{z}\right) \right\}.$$

$$(\mu^2, u^2) = \left\{ \left(\underset{2}{a}, \underset{1}{x}\right), \left(\underset{2}{b}, \underset{1}{y}\right), \left(\underset{0}{z}, \underset{0}{z}\right) \right\}.$$

Unlike our algorithm, the DGS86's process uses retention and the path of payoffs of every agent on each side of the market is monotone (increasing or decreasing). For instance, for firm a, the path of his payoffs is $(0, 1, 2)$.

Our algorithm can start from any initial market state, while DGS86's needs to start with the reservation prices of agents on one side of the market. Although DGS86's process can find a competitive equilibrium, it cannot be used to prove our Proposition 1, because DGS86 uses retention, which obviously violates the property of pair improvements. The same thing can be said about CK81's process.

The CK81's Process

The salary adjustment process in CK81 starts with the scenario where every agent is self-matched, and is illustrated with the same example.

Step 1. Both firms start with a salary of zero to each worker. Based on the current salaries, each firm makes an initial offer to their respective favorite worker, y (this corresponds to our definition of "best worker" in Definitions 1). Both firms will make the offer to y.

Step 2. Worker y temporarily accepts firm a's offer and rejects b's offer (since the offers are identical, worker y breaks ties arbitrarily).

Step 3. Firm b increases its salary offer to y by 1. At this point, the offer from firm a remains in force.

Step 4. Worker y accepts b's offer and rejects a's offer. Then firm a increases its salary offer to y by 1 but makes an offer to z. z accepts a's offer. Now there is no rejection and the process stops with a weak core allocation:

$$(\mu^*, u^*) = \left\{ \left(\underset{2}{a}, \underset{0}{z}\right), \left(\underset{2}{b}, \underset{1}{y}\right), \left(\underset{0}{x}, \underset{0}{x}\right) \right\}.$$

Notice that (μ^*, u^*) is a weak core allocation since there is no blocking pair of a firm and a worker that makes **both parties strictly better off**. The outcome (μ^*, u^*) is however not a stable outcome in the sense of our paper, which corresponds to a strict core allocation or a competitive equilibrium. The reason is that there is a blocking pair (a, x) with payoffs $u(a) = 3$ and $u(x) = 0$ that makes a strictly better off while x not worse off. Another blocking pair is (b, x).

Unlike our algorithm, the CK81's process uses retention and the path of payoffs of every agent on each side of the market is monotone (increasing or decreasing). For instance, for worker x, the path of his payoffs is $(0, 1)$.

Our algorithm can start from any initial market state, while CK81's needs to start with the reservation prices of agents on one side of the market.

Both our algorithm and DGS86's can find a competitive equilibrium in finite time, while CK81's can only find a weak core allocation in finite time.

In summary, CK81 has to start with a specific initial market state and can only find a discrete core allocation which is weaker than our notion of stable outcome in finite time. It uses retention and as a result the path of payoffs of every agent on one side of the market is monotone. The use of retention violates the property of pair improvement and thus CK81 cannot be used to prove our Proposition 1.

DGS86 also has to start with a specific initial market state but can find a stable outcome in finite time. However, it uses retention and as a result the path of payoffs of every agent on one side of the market is monotone. The use of retention violates the property of pair improvement and thus DGS86 cannot be used to prove our Proposition 1, either.

Our algorithm employs only bilateral trades or pair improvements and can start from any initial market state and find a stable outcome in finite time. Our algorithm hence enables us to establish Proposition 1.

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