The Role of Aggregate Information in a Binary Threshold Game*

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Abstract

We analyze the problem of coordination failure in the presence of imperfect information in the context of a binary-action sequential game with a tipping point. An information structure summarizes what each agent can observe before making her decision. Focusing on information structures where only "aggregate information" from past history can be observed, we characterize information structures that can lead to various (efficient and inefficient) Nash equilibria. When individual decision making can be rationalized using a process of iterative dominance (Moulin 1979), we derive a necessary and sufficient condition on information structures under which a unique and efficient dominance solvable equilibrium outcome is obtained. Our results suggest that if sufficient (and not necessarily perfect) information is available, coordination failure can be overcome without centralized intervention.

Keywords: Dominance Solvability, Imperfect Information, Pareto Efficiency, Tipping Point, Threshold, Weak Dominance.

JEL Classification: C72, C73, D80.

1 Introduction

A threshold or tipping point is a boundary which if crossed leads to an *irrevocable* change of state. Strategic complementaries in games in which the net return from one's actions depends positively on how many others have taken the same action often lead to payoff structures characterized by a threshold. Sequential binary threshold games in which agents move sequentially to choose one of two actions are common in the economics literature. Herding behavior, agglomeration and cascading, evolution of social norms, network formation, formation of clubs and customs unions as well as studies of insurance

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and of diffusion of technology, represent some examples of problems that can be fruitfully modeled and studied using such games.

We consider a class of sequential binary threshold games where a group of agents move in a predetermined sequence to decide whether to participate in a cause, and each agent prefers to participate if and only if total participation exceeds a tipping point. Specific examples include sequential decisions of whether to participate in a revolt, whether to consume a product with network externalities, and whether to adopt an innovation or a production standard. Our objective is to investigate how the ability to coordinate on certain socially preferable outcomes corresponds to the (imperfect) information the agents have about the participation behavior of their predecessors.

It is known that strategic complementarities in games typically result in the existence of multiple equilibria with various levels of participation. For example, the classic paper by Milgrom and Roberts (1990) shows that for supermodular games, where the strategy sets have a lattice structure and where there are strategic complementaries, the set of serially undominated strategy profiles has a maximal and minimal element (pure-strategy Nash equilibria), corresponding to upper and lower bounds respectively on the collective behavior of players. Two characteristics distinguish our study from this earlier work. Firstly, our focus is *not* on strategic complementarities and positive externalities but rather on the theoretical implications of a tipping point—payoff functions with a *single-crossing* property. In particular, our specification allows for some degree of congestion (negative spillovers or strategic substitutability) in our model. Secondly, and more importantly, the key aspect of our paper is *the consideration of and the modeling of information and its role* in the determination and efficiency of outcomes of the game.

We model information as a signal received by each player before she moves. The information is *anonymous* in that it is about how many people rather than who has participated, and the information is possibly only an imperfect signal on the *aggregate* participation level in that an agent's signal, while correct, may only reveal the aggregate participation level of a subset of the previous players. We identify key properties of the information structure and their relationship to the occurrence of various types of Nash equilibria, as well as to our game being dominance solvable using a process of iterative elimination of dominated strategies.[1]

Our first result is concerned with the relationship between information structures and pure-strategy Nash equilibria. While the existence of the maximal (full participation) and the minimal (no participation) Nash equilibrium has been discussed in the literature, the existence of these outcomes are independent of the information structure of our game. We extend the literature by considering *intermediate* pure-strategy Nash equilibria involving partial participation. We show that the existence of such (efficient or inefficient) intermediate Nash equilibria depends critically on the information structure of the game. We

[1]We provide a detailed discussion on the rationale and justification of the solution concept of iterated weak dominance in Section 4.1. In particular, we also discuss there the relationship between our dominance-solvable solution concept and the more standard solution concept of perfect Bayesian equilibrium for our game.

provide a necessary and sufficient condition on the information structure for the existence of *every* possible type of *intermediate* pure-strategy Nash equilibrium. We find that a defining feature of such information structures is the presence of a pair of agents who get information signals that are unaffected by each other's act of participation.

Our second and **main** result characterizes the relationship between information structures, plausibility of strategies and dominance solvability. We follow Moulin (1979), Marx and Swinkels (1997), and others by assuming that rational agents iteratively delete (weakly) dominated strategies to obtain a set of iteratively undominated (admissible) strategies. We define the game to be dominance solvable when every such iteratively undominated strategy profile leads to the same (unique) outcome, which turns out to be the efficient one with full participation. We provide a sufficient condition on the information structure for the game to be dominance solvable. Our main contribution, however, lies in showing that the same condition is also *necessary* for dominance solvability.

We define an information chain of length m as a set of m agents who are informationally linked in that the first agent's participation impacts the information signal the second agent receives, and the second agent's participation impacts the information signal the third agent receives, and so on. The tipping point λ represents the degree of participation that is needed to improve the payoffs of the participating agents over the *status quo* of non-participation. We show that dominance solvability obtains and maximal participation occurs *if and only if* the information structure admits an information chain of length at least λ. Here, the length of the information chain can be interpreted as the degree of transparency of the information structure, i.e., the degree to which the information about the *aggregate* action of earlier players filters through to agents moving later in the sequence. This result implies that the greater the need for coordination (the larger the λ), the greater the need for informational transparency that is necessary for dominance solvability. Our analysis hence suggests that a non-coersive decentralized policy tool for preventing coordination failure would be to increase the flow of information between agents and to allow them to act in their rational self interest.

The problem studied in our paper belongs to a class of binary choice problems with externalities, first studied in the classic paper of Schelling (1973). Such binary choice problems, while appearing simple, have found various applications in the social sciences.[2] In this broad literature, our paper is more closely related to several studies on threshold games with imperfect information. In an influential study, Watts (2002) considers binary threshold games in random networks to investigate how global cascades are triggered by small shocks. While both Watts (2002) and our paper study how observability of the other agents' behavior (interpreted as connectivity of networks in Watts (2002)) affects global cascades, our model is deterministic and the agents in our (more specific) model are forward looking and more strategic, in that each agent takes full account of how her action

[2] Granovetter (1978) studies a binary threshold model where a group of agents with different thresholds decides whether to participate in a riot. Binary threshold models have also been applied to segregation (Schelling 1969), public good games (Dawes et al. 1986), crime (Glaeser, Sacerdote, and Scheinkman 1996), etc. Chwe (2001) contains an array of social phenomena involving coordination with thresholds.

affects later adopters when making an optimal decision. Chwe (2000) characterizes the minimal sufficient communication networks for coordination in binary threshold games with incomplete information and heterogeneous thresholds. Chwe (2000) and our paper are related and are both concerned with how coordination can arise through some type of communication. While our model and characterization bear some resemblance to those in Chwe (2000), our results are not subsumed or implied by Chwe's results. We will relate our results to Chwe (2000) in more detail in Section 4. Finally, binary threshold games can be modeled as special network games where neighborhood structure affects payoffs (see e.g. Galeotti et al. 2010). Compared to this literature, while our setting is specific, our analysis of imperfect observability in a sequential binary threshold game, to our knowledge, has not been studied in this literature.

The rest of the paper is organized as follows: Section 2 introduces the model. Section 3 presents our results on pure-strategy Nash equilibria, while Section 4 provides our main result on dominance solvability, together with some discussion on our solution concept and various assumptions in our model. Omitted proofs are collected in an Appendix.

2 The Model

Consider a set of agents $N = \{1, \ldots, n\}$ with $n \geq 3$ who move sequentially. We assume that for all $j \in \{1, \ldots, n-1\}$ agent j moves before agent $(j+1)$ according to the exogenously given order $(1, \ldots, n)$. For $j \in N$, agent j's action $b_j \in \{1, 0\}$ represents the choice between participating ("joining"), $b_j = 1$, and not participating ("not joining"), $b_j = 0$. Possible interpretations of action b_j include whether to buy an excludable public good or a club good, whether to follow a product standard for firms, and whether to adopt a social norm. We denote $b = (b_j, b_{-j}) = (b_1, \ldots, b_n)$ as an action profile or an outcome.

We assume that each agent has preference orderings $\succeq_j$ on the set of outcomes that depend on the agent's decision (to participate or not) and on the total *number* of participants (the participation level). As in Schelling (1973), such a specification is reasonable when either the individual characteristics of the other agents are unimportant for the underlying problem (such as with equal cost sharing for a public project) or when the information about the relevant characteristics of the other agents is just not available (such as with purchasing health insurance when privacy concerns dictate that personal information not be available even though how many other people have joined may be public information).[3] Given the finiteness of the set of outcomes, each agent's preferences $\succeq_j$ can be represented by an (ordinal) utility function $u_j(b) = u_j(b_j, \sum_{i \in N} b_i)$.

[3]Schelling (1973) considers a similar but *static* model of binary choices with externalities where the agents' preferences are given by a utility function $g : \{1, 0\} \times \{0, \ldots, n-1\} \to \mathbb{R}$, and for $\alpha \in \{0, \ldots, n-1\}$, $g(1, \alpha)$ (resp., $g(0, \alpha)$) is the value of participating (resp., not participating) when α other individuals participate. Our model admits a tipping point (where Schelling's model may or may not have one) and thus our model is not more general than Schelling's model. Nor is our model a special case as it allows for types of preferences not modeled by Schelling (see Section 4.1).

We impose that the agents' preferences are characterized by a *common* threshold λ given by the participation level required for the participation of agent j to be an improvement over *non-participation by* j.

Assumption 1 (Tipping Point) *There exists* $\lambda \in \{2, \ldots, n\}$ *such that for all* j *and action profiles* b, b' *such that* $b_j = 1$ *and* $b'_j = 0$, $u_j(b) > u_j(b')$ *if* $\sum_{i \in N} b_i \geq \lambda$ *and* $u_j(b) < u_j(b')$ *if* $\sum_{i \in N} b_i < \lambda$.

In what follows, we also make the following simplifying assumption:

Assumption 2 (Identical Utility from Status Quo) *For all* j *and action profiles* b, b' *with* $b_j = b'_j = 0, u_j(b) = u_j(b')$. *We normalize* the status quo utility *to be zero (i.e.,* $u_j(b) = u_j(b') = 0$*).*

While Assumption 1 is essential for the results in our model, Assumption 2 is less important and not needed for the proof of our main theorems.[4]

Hence, letting $f_j\left(\sum_{i \in N} b_i\right) = u_j(1, \sum_{i \in N} b_i)$, we can write

$$u_j(b) = b_j f_j\left(\sum\nolimits_{i \in N} b_i\right), \tag{1}$$

where f_j has a *single-crossing* property—f being positive for values greater than λ and negative for values less than λ.

In examples below, preferences are identical and the subscript j will be dropped from f_j with $u_j(b) = b_j f(\sum_N b_i)$. Example 1 shows how the tipping point assumption (Assumption 1) can arise from economies of scale in production or from positive spillovers in consumption (network externalities) even in the presence of congestion. Clearly, it can arise through some combination of these two effects.

Example 1 (i) (**Network Externalities with Congestion**) *Suppose the per capita cost of servicing a communication network is a constant* $c > 0$ *and the benefit received by anyone joining the network is* $v(\sum_N b_i)$. *Thus, the net benefit for an agent joining the network is* $f(\sum_N b_i) = v(\sum_N b_i) - c$. *We can allow for some congestion arising when* $\sum_N b_i$ *is large leading to declining benefits—for some* $\lambda \in \{2, ..., n\}$, *Assumption 1 will be satisfied as long as* $v(j) < c$ *for all* $j \leq \lambda - 1$ *and* $v(k) > c$ *for all* $k \geq \lambda$.

(ii) (**Economies of Scale**) *Consider a health insurance service provided to the agents at average cost. The industry's technology is characterized by economies of scale so that the average cost,* $C(\sum_N b_i)/(\sum_N b_i)$, *strictly decreases as more agents enroll for the service (i.e., the service is provided by a "natural monopoly"). If the "gross" value of the service to each agent is a constant* $v > 0$, *then the net benefit from joining* $f(\sum_N b_i) = v - C(\sum_N b_i)/(\sum_N b_i)$ *captures the positive spillovers associated with the economies of scale.*

[4] It is only needed for Proposition 1 and its associated footnote establishing the connection of our results to the Rawlsian maximum and utilitarian maximum.

As in (i), Assumption 1 holds if $C(1) > v$ and $v > C(n)/n$. Once again one can modify this example to allow for congestion with Assumption 1 being satisfied as long as the congestion effects are not too strong.

Example 2 illustrates a *bang-bang* type of utility function that can arise in the context of a political economy/voting model.

Example 2 (**Voting**) *A group of agents decide whether to join a political party. Agents who join receive benefits from supporting the party if and only if the party wins. If the party loses the supporters of the party are penalized. The party wins if it receives the support of at least $\lambda \in \{2, \ldots, n\}$ agents, otherwise it loses. If the party wins, the (fixed total) spoils of victory (taken here to be 100) are divided equally among the supporters of the party. Letting $b_i = 1$ represent support for the party, the following function f satisfies Assumption 1*:

$$f\left(\sum_N b_i\right) = \begin{cases} \frac{100}{\sum_N b_i} \text{ if } \sum_N b_i \geq \lambda, \\ -1, \text{ otherwise.} \end{cases}$$

2.1 Information Structure and Information Chain

We now introduce several crucial concepts (and the associated assumptions) which summarize the information each agent receives before making her move. Let $\kappa(\cdot)$ be an operator that associates with each $j \in N$ a unique number $\kappa(j) \in \{0, \ldots, j-1\}$. The **information structure** $\mathcal{I}$ is defined as an n-vector of non-negative integers $\mathcal{I} = (\kappa(1), \ldots, \kappa(n))$. If $\kappa(j) = 0$ then j receives no information before making her decision. If $\kappa(j) > 0$, then $\kappa(j)$ represents a specific agent moving before j and the report that j receives is an integer from the set $\{0, \ldots, \kappa(j)\}$ indicating exactly how many (though not which) of the agents from the set of agents $\{1, \ldots, \kappa(j)\}$ have chosen to participate.[5] Notice that agent j only observes the actual *aggregate* number of participations from the *first* $\kappa(j)$ agents. Whenever $\kappa(j) \geq j'$, any alteration in j''s action, *ceteris paribus*, is reflected in the aggregate report that j receives. So for agents $j, j' \in N$, if $\kappa(j) \geq j'$ (resp., $\kappa(j) < j'$), then j's information *covers* (resp. does not cover) j''s action. We will abbreviate these and say that 'j covers j'' and 'j does not cover j''.

We will assume that the information structure described above is monotone:

Assumption 3 ($\mathcal{I}$-Monotonicity) *For all $j, j' \in N$, $j \geq j'$ implies that $\kappa(j) \geq \kappa(j')$.*

Assumption 3 requires that the sequence of numerical reports that agents $1, \ldots, n$ receive is a *nested* and (weakly) *monotonically increasing* set of integers. An intuitive interpretation of Assumption 3 is that agents moving later have at least as much (payoff

[5] Consequently, agent j knows how many agents from $\{1, \ldots, \kappa(j)\}$ have *not* participated. In some cases, for instance if $\kappa(j) = 1$ or if the report received is equal to $\kappa(j)$, j is able to deduce not only how many but also who has joined the group.

relevant) information as those who move earlier, which is indeed a natural and reasonable assumption in many applications.[6] For instance, think of a scenario where before taking any action, each agent checks a common website that reports the number of "hits" and the total number of people who have joined. If collecting this information takes time and the information is updated at fixed time intervals, this can result in data collection lags and data publication lags.[7] Example 3 provides an illustration.

Example 3 *Let* $N = \{1, 2, 3\}$*. There are five possible monotone information structures:*

$$\mathcal{I}_1 = (0,0,0)\ ;\ \mathcal{I}_2 = (0,0,1)\ ;\ \mathcal{I}_3 = (0,0,2)\ ;\ \mathcal{I}_4 = (0,1,1)\ ;\ \mathcal{I}_5 = (0,1,2)\,.$$

Information structure $\mathcal{I}_1$ *is the extreme case where no one has any information about previous agents' moves, while* $\mathcal{I}_5$ *represents the polar opposite case where before they move each agent knows the complete aggregate history of previous moves, i.e., each agent knows exactly how many of the previous agents have joined. For instance, if there is no collection lag (the website operators collect and process the information instantaneously) but suppose the website is updated every two periods, then we obtain the information structure* $\mathcal{I}_3 = (0,0,2)$*. Here the two-period publication lag implies that before they move, agents 1 and 2 get no information while* 3 *sees the entire aggregate history. Suppose that in addition to the publication lag, there is a collection lag and it takes one period to collect and process the data. Then,* 3 *will receive a report that only includes the information about 1's move, leading to the information structure* $\mathcal{I}_2 = (0,0,1)$*.*

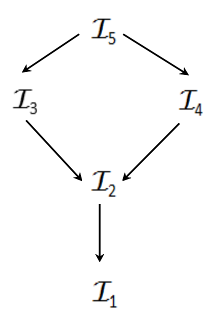

Figure 3. The information structures in Example 3.

The information structures $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ and $\mathcal{I}_5$ can be ranked in that there is a greater availability of (payoff relevant) information as we move from $\mathcal{I}_1$ to $\mathcal{I}_5$.[8] However, the information contents of $\mathcal{I}_3$ and $\mathcal{I}_4$ cannot be ranked. The lattice showing the quasi-

[6]If $\kappa(j) = 1$ and $\kappa(j') = 2$, then even though by looking at her signal j knows whether 1 has joined or not joined and j' may not, j' will receive at least as much *payoff relevant* information as j does, since all that matters for payoffs is how many individuals rather than which individuals have joined.

[7]We discuss an alternative modeling of information in Section 4.1.

[8]Agent 3 in the information structure $\mathcal{I}_2$ knows whether 1 has moved but does not in the structure $\mathcal{I}_3$. Nevertheless agent 3 has more payoff relevant information in $\mathcal{I}_3$ because the payoffs only depend on the total number of individuals participating.

ordering of the information structures in Example 3 is presented in Figure 3, where each arrow represents the relation of "contains more payoff relevant information than."

In general, we can compare information structures by comparing the associated vectors (in the vector sense) with larger ones representing smaller lags and hence a greater availability of information. While some such comparisons may not be possible, the vector comparisons of information structures will nevertheless always be a *partial order*, a reflexive, antisymmetric and transitive (possibly incomplete) binary relation on the set of all information structures with the *maximal* and *minimal* information structures being $\mathcal{I}^{\max} = (0, 1, \ldots, n-1)$ and $\mathcal{I}^{\min} = (0, 0, \ldots, 0)$, respectively.

Recall that when agent j's information covers agent j', the marginal impact of j''s move on the total number of participations can be observed by j. Thus, j''s knowledge about the aggregate history of the play *and* the impact of j''s action on this aggregate history filter through to agent j. An important property of an information structure is given by the concept of an *information chain* which measures how *easily* the information flows within the model.

An ordered set of agents $(i_1, \ldots, i_m) \subseteq N$ forms an **information chain** of length m if and only if for all $s \in \{2, \ldots, m\}$, agent i_{s-1}'s information covers agent i_s. In other words, an information chain of length m is a set of m informationally linked agents who can be ordered such that i_1's information covers all the others in the chain, i_2's information covers all agents in the chain except i_1, and so on. Clearly, the existence of an information chain of length m implies the existence of an information chain of length m' with $m' \leq m$. From finiteness, it follows that an information chain of maximal length m^* always exists. We interpret m^* as a summary measure of the extent to which available information filters through from agents moving earlier to those moving later and as an indicator of the *transparency* of the information structure. Note, however, that m^* is a property of the information structure that does not measure how much information is available in an information structure. In particular, while information structures with strictly more information will have chains of maximal length that are at least as large, information structures with less information may have maximal information chains that are just as large as information structures with more information. In Example 3, the maximal lengths of the information chains for $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$ and $\mathcal{I}_5$ in N are respectively $0, 2, 2, 2$ and 3. Though $\mathcal{I}_4 = (0, 1, 1)$ has "more (payoff relevant) information" than $\mathcal{I}_2 = (0, 0, 1)$, $\mathcal{I}_4$ is *not* any more transparent (as measured by m^*) than $\mathcal{I}_2$.

2.2 Normal Form Game

We now describe the normal form representation of the threshold game. A pure strategy of agent $j \in N$ is a $(\kappa(j)+1)$-dimensional binary vector of conditional moves (or actions) $a_j = (a_j(1), \ldots, a_j(\kappa(j)+1))$. For all $l \in \{1, \ldots, \kappa(j)+1\}$ we interpret the *conditional move* $a_j(l) \in \{1, 0\}$ as agent j's decision on whether to participate *if* given the information that $(l-1)$ agents out of the the first $\kappa(j)$ agents are going to participate. The conditional action $a_j(l)$ becomes j's move b_j if and only if j receives the report that exactly $(l-1)$

out of $\kappa(j)$ previous agents have chosen to participate. In this case we will say that the *pre-requisite* of the l^{th} coordinate of j's strategy has been satisfied. The set of all possible pure strategies of j is denoted by A_j and a pure strategy profile is $a = (a_j, a_{-j}) = (a_1, \ldots, a_n) \in A = \Pi_{j\in N} A_j$.

For a strategy profile a, the path of play is a set of coordinates $a_j(\ell)$ such that for all $j \in \{1, ..., n\}$, the pre-requisite of $a_j(\ell)$ is met and hence $a_j(\ell) = b_j(a)$. In this case, we will say that the ℓ^{th}-coordinate of j's strategy is *on the path of play.* Notice that one and only one coordinate of each individual is on the path of play and this *unique* path of play yields the action profile $b(a)$. For any other coordinate l whose pre-requisite is not satisfied, the l^{th}-coordinate of j's strategy is *off the path of play.* Finally, abusing notation we will denote agent j's payoff from a as being given by $u_j(a) = u_j(b(a))$.

Gathering together the notation developed so far we define the normal form game $\mathcal{G}_0$ as the quadruple $\langle N, A, \mathcal{I}, \{u_j\}\rangle$. Example 4 illustrates the above concepts for $\mathcal{G}_0$.

Example 4 *Let* $N = \{1, 2, 3\}$, $u_j(b) = b_j f\left(\sum_N b_i\right)$ *and* $f(1) = -3$, $f(2) = f(3) = 4$ *and hence* $\lambda = 2$. *Consider the information structure* $\mathcal{I} = (0, 1, 1)$ *and the strategy profile* a*:*

Agent j	*1*	*2*	*3*
Strategy a_j	$(\mathbf{0})$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$

Given the strategy profile a*, agents 2 and 3 join if and only if they observe that agent 1 has not joined. The pre-requisites of* $a_2(1)$ *and* $a_3(1)$ *are satisfied and the path of play is represented by the sequence of conditional actions* $a_1(1)$, $a_2(1)$ *and* $a_3(1)$ (*coordinates in* ***boldface***), *resulting in action profile* $b(a) = (0, 1, 1)$ *and a payoff vector* $(0, 4, 4)$.

3 Equilibrium and Efficiency

We now analyze the relationship between information structures and the occurrences of various pure-strategy Nash equilibria in our setting.

A pure-strategy Nash equilibrium (PSNE) of $\mathcal{G}_0$ is a strategy profile (a_j, a_{-j}) such that for all $j \in N$ and all $a'_j \in A_j$, $u_j(a_j, a_{-j}) \geq u_j(a'_j, a_{-j})$. The action profile $b(a)$ associated with a PSNE a is called a pure-strategy Nash equilibrium outcome (PSNEO). Checking whether a strategy profile is a PSNE amounts to verifying that unilateral deviations are not beneficial and it suffices to consider unilateral deviations that change the coordinate (of a deviating agent's strategy) along the path of play. Given (a_j, a_{-j}), we will refer to the coordinates of an agent's strategy off the path of play as (payoff) *irrelevant coordinates* for (a_j, a_{-j}). Furthermore, since a unilateral change in the coordinate of an agent's strategy along the path of play *necessarily* changes that agent's utility, the coordinates on the path of play will be called as (payoff) *relevant coordinates* for (a_j, a_{-j}).

Since every agent can always play a strategy in which all the coordinates are zero, the *non-participation* payoff is the reservation payoff. It follows that *all* agents receive non-negative payoffs in a PSNE and we must either have $\sum_N b_i(a) \geq \lambda$ or $\sum_N b_i(a) = 0$ for a

PSNE a. This observation leads us to consider three mutually exclusive (and exhaustive) types of Nash equilibria:

(i) A *maximal* PSNE is a PSNE where all *agents* join in equilibrium. A maximal PSNE results in a unique PSNEO with payoffs $(f_1(n), \ldots, f_n(n))$. This outcome is necessarily Pareto efficient since any other outcome will reduce the utility of some individual.

(ii) A *minimal* PSNE is a PSNE where *nobody* joins in equilibrium. The associated unique PSNEO with payoffs $(0, \ldots, 0)$ is strongly inefficient in that all achievable benefits from cooperation are lost. Every other PSNEO (weakly) Pareto dominates the minimal PSNEO in that some agents are better off and no one is worse off.

(iii) An *intermediate* PSNE is a PSNE that is neither maximal nor minimal. In particular, an intermediate PSNE has *exactly* $\lambda + \tau$ agents join in equilibrium for some $\tau \in \{0, \ldots, (n - \lambda - 1)\}$.

While an intermediate PSNEO represents a Pareto improvement over the *status quo*, an intermediate PSNEO may be either (Pareto) efficient or inefficient. For instance, if f_j is non-decreasing for all j, all the intermediate PSNEOs are inefficient and are Pareto dominated by the maximal PSNEO. However, it is possible that these intermediate PSNEO are *all* efficient as well. This will happen for instance when f_j reaches a maximum at the tipping point λ and is strictly decreasing for all participation levels greater than λ (see Example 2). The following remark summarizes Pareto efficient PSNEOs in our setting (the proof is straightforward and is hence omitted):[9]

Remark 1 In the game $\mathcal{G}_0$, a PSNEO, denoted as b, is Pareto efficient if and only if all outcomes with a larger participation level have a strictly lower utility from participating for at least one of the participants in b.

From a welfare point of view, the maximal PSNEO is attractive: Given that in a PSNE, a participant receives positive utility while a non-participant receives the *status quo* utility level of zero, the maximal PSNEO is not only Pareto efficient, but also corresponds to a Rawlsian maximum:[10]

Proposition 1 *All social welfare functions reach a Rawlsian maximum at the maximal PSNEO.*

We now discuss how the existence of various PSNEOs of the game $\mathcal{G}_0$ is related to the information structure of the game, starting with a simple proposition, which is an immediate consequence of our tipping point assumption (Assumption 1):

[9] If $f_j = f$ is strictly quasi-concave (strictly single peaked), then the intermediate PSNEOs can be partitioned such that PSNEOs with participation levels above λ and below "$\arg\max f$" will be inefficient and those with participation levels of "$\arg\max f$" and above will be efficient.

[10] The Rawlsian welfare function is defined as $W = \max_a \min_j \{u_j(a)\}$. See Sen (1977) and Hammond (1976). In addition, notice that the "utilitarian" welfare function given by the sum of utilitites may or may not be maximized at the maximal PSNEO. Adding an individual adds to her utility but because of congestion it may reduce the utility of other participants. Clearly, the preferences being monotone (non-decreasing) in $\sum b_j$ is sufficient for such a maximum.

Proposition 2 *The game $\mathcal{G}_0$ always admits maximal and minimal PSNEOs regardless of the information structure $\mathcal{I}$.*

Unlike maximal and minimal PSNEs, whether an intermediate PSNE exists depends on the information structure of $\mathcal{G}_0$. One can check that with $n = 3$, $\lambda = 2$ and $\mathcal{I}^{\min} = (0, 0, 0)$, no intermediate PSNE exists since the best response to any contingency when two other agents are participating is to participate. On the other hand, one can verify that the strategy profile a in Example 4 illustrates an intermediate PSNE with outcome $b(a) = (0, 1, 1)$ where two out of the three agents participate. The main result in this section, Theorem 1, provides a characterization of the information structure of $\mathcal{G}_0$ that can give rise to an intermediate PSNE with **exactly** $(\lambda + \tau)$ participants for $\tau \in \{0, \ldots, (n - \lambda - 1)\}$.

Theorem 1 *In the game $\mathcal{G}_0$, let $\tau \in \{0, \ldots, n - \lambda - 1\}$. Then, there exists a PSNE $\overline{a}$ such that $\sum_N b_i(\overline{a}) = (\lambda + \tau)$ if and only if there exists $j^* \in \{n - \lambda - \tau + 1, \ldots, n - \tau - 1\}$ such that $\kappa(j^*) \geq n - \lambda - \tau$, $\kappa(j^* + 1) < j^*$, and $n - j^* - \tau - 1 \geq |\{j : \kappa(j) = j^*|$.*

Corollary 1 *In the game $\mathcal{G}_0$, if either $\mathcal{I} = \mathcal{I}^{\max} = (0, 1, \ldots, n - 1)$ or $\mathcal{I} = \mathcal{I}^{\min} = (0, \ldots, 0)$ then there does not exist an intermediate PSNE.*

Corollary 2 *In the game $\mathcal{G}_0$, there is a PSNE $\overline{a}$ such that $\sum_N b_i(\overline{a}) = \lambda$ if and only if there exists $j^* \in \{n - \lambda + 1, \ldots, n - 1\}$ such that $\kappa(j^*) \geq n - \lambda$ and $\kappa(j^* + 1) < j^*$.*

Corollary 3 *In the game $\mathcal{G}_0$, let $\tau \in \{0, \ldots, n - \lambda - 1\}$. If $\mathcal{I} = (\kappa(1), \ldots, \kappa(n))$ where $\kappa(j) = 0$ for $j \in \{1, \ldots, n - \lambda - \tau\}$ and $\kappa(j) = n - \lambda - \tau$ for $j \in \{n - \lambda - \tau + 1, \ldots, n\}$, then $\mathcal{G}_0$ has a PSNE with exactly $\lambda + \tau$ participants.*

Theorem 1 completely characterizes the information structures that can induce intermediate PSNEs. The rough intuition behind Theorem 1 is that in an intermediate PSNE, some early agents do not join because a unilateral deviation from such an agent from not joining to joining triggers a chain reaction of defections from among the later participants starting with j^*. It is only if the scale of defections is large enough to make the total number of participants fall below the critical number λ will such a deviation be rendered undesirable thus supporting a PSNE.

According to Theorem 1, two conditions on the information structure are important for the existence of intermediate PSNEs: The first is imperfect observability. In particular, there should be an agent j^* whose move is not covered by the next agent $j^* + 1$. Second, there should be enough agents after j^* whose defections are able to bring the total number of participants below λ—this requires $j^* \leq n - \tau - 1$ and an upper limit "$n - j^* - \tau - 1$" on the number of agents j in $\{j^* + 2, ..., n\}$ who have $\kappa(j) = j^*$. Notice that for a relevant deviation to be undesirable for a non-participant, the cascade of defections needs to involve at least τ other agents *after* j^* *and* $j^* + 1$ to defect after the deviation, and hence $j^* \leq n - \tau - 1$. The condition "$n - j^* - \tau - 1 \geq |\{j : \kappa(j) = j^*\}|$" on the other hand requires that *not too many* agents $\hat{\jmath}$ moving after j^* have $\kappa(\hat{\jmath}) = j^*$—such agent

$\hat{\jmath}$ **cannot** be a part of this chain of defections since agent $\hat{\jmath}$'s report remains unchanged when j^* switches to not participate in reaction to an extra participant in j^*'s report.

The corollaries follow directly from Theorem 1. Corollary 1 concludes that under the maximal and minimal information structures, an intermediate PSNE cannot occur. Corollaries 2 identifies the information structures that are consistent with intermediate PSNEs with *exactly* λ participations. Corollary 3 shows the existence of information structures for which every feasible intermediate participation level is a possible PSNEO.

Finally, observe that some agents use unintuitive (non-monotone) strategies in an intermediate PSNE—they switch to not participating upon seeing more previous participations. The use of such non-monotone strategies may not, *prima facie*, be unreasonable since agents may gang up to prevent others from joining when additional participation reduce the utility of existing participants. This being said, such strategies may lead to the credibility problems associated with intermediate PSNEs, an issue we turn to next.

4 Dominance Solvability and Efficiency

As mentioned earlier, while some of the intermediate PSNEs are 'sensible' from an efficiency point of view, a potential problem for intermediate and minimal PSNEs is that such equilibria can turn out not to be credible, i.e., such a PSNE can involve players committing to conditional moves off the path of play that are against their self interest. For instance, the intermediate PSNE in Example 4 is not plausible since it involves agent 3 committing to a conditional move off the path of play that can negatively affect her payoff: After observing agent 1 participating agent 3 has a strict incentive to participate. Arguing along exactly similar lines one can see that Example 5 below presents an implausible minimal PSNE:

Example 5 *Let* $N = \{1,2,3\}$, $\mathcal{I} = (0,0,1), u_j(b) = b_j f\left(\sum_{i\in N} b_i\right)$ *and* $f(1) < 0 < f(2) < f(3)$. *A minimal PSNE that involves a non-credible "threat" is:*

Agent j	*1*	*2*	*3*
Strategy a_j	$(\mathbf{0})$	$(\mathbf{0})$	$(\mathbf{0},0)$

Given such an issue for intermediate and minimal PSNEs, the rest of this section aims to answer the following questions: When will the elimination of such implausible strategies result in eliminating the coordination failure represented by the minimal PSNEO? And more importantly, when can the only credible PSNEO attain the Rawlsian maximum in which all individuals participate?

A standard refinement to weed out PSNEs involving non-credible strategies is subgame perfection. However, it is easily verified that the game in Example 5 admits *no* proper subgames and hence all its PSNEs, including the minimal PSNE in that example, are subgame perfect. While stronger refinements satisfying the requirement of sequential rationality (Kreps and Wilson 1982) such as perfect Bayesian equilibrium and sequential equilibrium have been used to filter out non-credible equilibria, we adopt an alternative approach. It is well recognized that in games with *perfect* information that an alternative approach of *dominance solvability* (Moulin 1979) based on iterative elimination of weakly dominated strategies has a similar intuitive basis to and is closely related to subgame perfection. In our game with *imperfect* information this alternative approach is used to obtain a compact and succinct characterization of information structures that leave the maximal equilibrium as the only "plausible" PSNEO, thus eliminating the possibility of coordination failure and giving rise to the Rawlsian maximal efficient outcome.[11]

To formalize this process of iterative elimination, we introduce the following notation:

In $\mathcal{G}_0 = \langle N, A, \mathcal{I}, \{u_j\}\rangle$, a strategy a'_j is (*weakly*) *dominated* by a strategy a_j if $u_j(a_j, a_{-j}) \geq u_j(a'_j, a_{-j})$ for all a_{-j} and for some a'_{-j}, $u_j(a_j, a'_{-j}) > u_j(a'_j, a'_{-j})$. Let $\mathcal{R}$ be a function such that the game $\mathcal{G}_1 = \mathcal{R}(\mathcal{G}_0)$ is obtained by eliminating *all dominated strategies of all the agents* in $\mathcal{G}_0$.[12] Applying the operator $\mathcal{R}$ successively generates a sequence of games $\mathcal{G}_1$, $\mathcal{G}_2$, ..., where $\mathcal{G}_{h+1} = \mathcal{R}(\mathcal{G}_h)$ for $h \geq 0$. Since only strategies are eliminated (and none added), as one goes from one game in the sequence to the next, the strategy profiles in $\mathcal{G}_{h+1}$ is a subset of the set of strategy profiles in games $\mathcal{G}_1, \ldots, \mathcal{G}_h$ with the games in $\{\mathcal{G}_1, \mathcal{G}_2, \ldots\}$ otherwise having the same players and the same information structure as $\mathcal{G}_0$. In addition, the payoff functions in $\mathcal{G}_h \in \{\mathcal{G}_1, \mathcal{G}_2, \ldots\}$ are the same as those in $\mathcal{G}_0$ except for the fact that these functions are restricted to a smaller domain in $\mathcal{G}_h$. If $\mathcal{G}_s = \mathcal{R}(\mathcal{G}_s)$ for some $\mathcal{G}_s \in \{\mathcal{G}_1, \mathcal{G}_2, \ldots\}$, then $\mathcal{G}_s$ is said to be *irreducible.* Since $\mathcal{G}_0$ has a finite set of players each with a finite set of pure strategies, an irreducible game always exists. We will analyze the sequence $\{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_M\}$ where $\mathcal{G}_M$ is the *first* irreducible game in the sequence. Observe that our concepts of strategy profile, action profile, PSNE, PSNEO, contingency, and the concepts of coordinates of individual strategies being on and off the path of play (i.e., relevant and irrelevant coordinates) defined for $\mathcal{G}_0$ extend naturally and unambiguously from $\mathcal{G}_0$ to $\mathcal{G}_h$ for all $h \in \{1, \ldots, M\}$. The game $\mathcal{G}_0 = \langle N, A, \mathcal{I}, \{u_j\}\rangle$ is *dominance solvable* if $\mathcal{G}_M$ has only one outcome.

The intuitive argument for this iterative elimination process and its relationship to the credibility of strategies derives from the sequential structure of our game and the standard argument used to justify subgame perfection that at any "stage" of the game the action being proposed under a strategy must be credible in the sense that it should be consistent with the agents' incentives at that stage of the game. In our context, this implies that when it is individual j's turn to move if individual j can conclude (from the information he receives and based on the assumption that other players use credible strategies) that $(\lambda - 1)$ other individuals will participate then all of j's credible strategies involve a move in which individual j participates. All other strategies are not credible. To see how this process works, consider Example 5. The first round of elimination removes agent 3's

[11] As mentioned previously, our dominance solvable outcome is also closely related to the perfect Bayesian equilibrium outcome of the game. A detailed comparison of the two solution concepts in our game is provided in Section 4.1.

[12] We will discuss the issue of the order of elimination in the reduction process in Section 4.1. In addition, we only use weak domination in our arguments and we will drop the adjective 'weak' hereafter.

non-credible dominated strategies "$(0,0)$" and "$(1,0)$", both prescribing 3 to "not join" upon seeing a previous participation. Thus, $\mathcal{G}_1$ incorporates credible (rational) behavior of individual 3. Thus, from $\mathcal{G}_1$, we can conclude that "(1)" is the only credible strategy of agent 1 based on the information that he has (i.e., that agent 3 covers him) and the assumption of agent 3 behaving rationally. Thus $\mathcal{G}_2$ incorporates credible behavior of both agents 1 and 3. Arguing similarly $\mathcal{G}_M = \mathcal{G}_3$ which contains the only credible strategies of the three players (the strategy profiles a' and a'' below).

Agent j	1	2	3
a'	$(\mathbf{1})$	$(\mathbf{1})$	$(0,\mathbf{1})$
a''	$(\mathbf{1})$	$(\mathbf{1})$	$(1,\mathbf{1})$

Both a' and a'' have outcome $(1,1,1)$. Dominance solvability hence filters out the non-credible equilibrium, resulting in the maximal PSNEO being the only credible PSNEO.

The next example shows a game $\mathcal{G}_0$ that is not dominance solvable and the resulting $\mathcal{G}_M$ admits both a minimal and an intermediate PSNEs.

Example 6 *Consider $\mathcal{G}_0$ with $N = \{1,2,3,4\}$, $\mathcal{I} = (0,1,1,1)$ and payoffs $u_j(b) = b_j f\left(\sum_{i\in N} b_i\right)$ where $f(1) = f(2) = -3$, $f(3) = 5$ and $f(4) = 2$, and hence $\lambda = 3$. Here, no strategy in $\mathcal{G}_0$ is dominated (hence $\mathcal{G}_0 = \mathcal{G}_M$) and the intermediate PSNE (a) and minimal PSNE (a') shown below survive the iterated elimination process:*

Agent j	*1*	*2*	*3*	*4*
a	$(\mathbf{0})$	$(\mathbf{1},0)$	$(\mathbf{1},0)$	$(\mathbf{1},0)$
a'	$(\mathbf{0})$	$(\mathbf{0},0)$	$(\mathbf{0},0)$	$(\mathbf{0},0)$

In general, whether the game $\mathcal{G}_0$ is dominance solvable and whether $\mathcal{G}_M$ necessarily has a maximal PSNE as its unique outcome after the iterated elimination process depends on the tipping point λ and m^*, the maximal length of information chains in $\mathcal{G}_0$. Recall that m^* is a *property* of the information structure that provides a summary measure of information transfer possibilities within the game. In particular, m^* measures how easily the information is filtered through from early players to later players in the game, allowing an earlier player to predict how a later player would behave if she acted rationally. Whether m^* weakly exceeds the coordination threshold λ becomes key to determining whether the game $\mathcal{G}_0$ is dominance solvable. Associating rational decentralized decision making with individuals playing strategies that survive iterated dominance, the condition $m^* \geq \lambda$ determines whether decentralized rational decision making leads to the maximal PSNE which is efficient and gives us the Rawlsian maximum welfare (Theorem 2).[13]

[13] The games in Examples 4 and 5 both have $\lambda = m^* = 2$ and are hence dominance solvable. The additional information that agent 2 has in Example 4 (as compared to Example 5) is of no importance in predicting the outcome of the game. In contrast, the game in Example 6 has $\lambda = 3 > m^* = 2$, and is hence not dominance solvable.

Theorem 2 *Let $\mathcal{G}_0$ be a game with tipping point λ and let m^* be the maximal length of an information chain in $\mathcal{G}_0$. Then $\mathcal{G}_0$ is dominance solvable if and only if $m^* \geq \lambda$.*

Corollary 4 *(i) If $\mathcal{G}_0$ is dominance solvable, then $\mathcal{G}_M$ has a unique efficient outcome given by the maximal PSNEO. (ii) If $\mathcal{G}_0$ is not dominance solvable, then $\mathcal{G}_M$ has a minimal PSNE in which no individual participates.*

Corollary 5 *(i) If $\lambda = 2$, then $\mathcal{G}_0$ is dominance solvable iff the information structure of $\mathcal{G}_0$ is not minimal. (ii) If $\lambda = n$, then $\mathcal{G}_0$ is dominance solvable iff the information structure of $\mathcal{G}_0$ is maximal.*

Remark 2 As noted earlier, Chwe (2000) studies a binary threshold (coordination) game with incomplete information and a communication network where agents can communicate with their neighbors about their willingness to participate. Chwe characterizes the minimal sufficient communication networks (in terms of a chain of cliques) for coordination to occur regardless of the prior. While Chwe's characterization result appears to share some similarity with Theorem 2, the two sets of results are qualitatively different: The communication in Chwe's model is direct and resolves the agents' uncertainty about the others' types (willingness to participate) and a minimal sufficient network guarantees that everyone participating arises as **an** equilibrium in the state where everyone is willing—other equilibria with coordination failure still exist. The communication in our model however takes place indirectly through the agents' actions and is to resolve strategic uncertainty about the others' choices. In addition, the existence of an information chain with length m^* guarantees that everyone participating is the **only** "equilibrium" outcome in the game. Our results, weaved around imperfect observability and akin to equilibrium selection, are hence not subsumed or implied by Chwe's characterization.

One implication of Theorem 2 is that policies promoting an improved flow of the information between agents can be used as a policy tool to promote coordination. The intuitive reason as to why the condition "$m^* \geq \lambda$" leads to dominance solvability can be seen from an argument similar to backward induction (in our proof of the sufficiency of "$m^* \geq \lambda$"). This is not surprising since dominance solvability has a similar intuitive basis to subgame perfection for games with perfect information. Why "$m^* \geq \lambda$" is necessary for dominance solvability is however more opaque and represents a deeper and more important result. It identifies for us the property of the information structure *without* which we *cannot* rule out coordination failure by using dominance solvability. The proof of necessity consists of showing that the iterated elimination process breaks down if $m^* < \lambda$ and that this necessarily leads to at least two equilibrium outcomes the maximal PSNEO and the minimal PSNEO surviving in $\mathcal{G}_M$. We present a detailed example (Example 7) in the proof of Theorem 2 to illustrate the breakdown of the iterated elimination process when $m^* < \lambda$.

An Algorithm for Finding m^*. Since an information structure $\mathcal{I}$ typically admits information chains of various lengths, it is important to identify a maximal information chain of length m^* from any given information structure. To this end, we define the *canonical information chain* of $\mathcal{G}_0$ as the information chain of maximum length m^* constructed using the following algorithm: Construct an ordered set of agents $(i_1, i_2, \ldots, i_m)$ such that $i_1 = n$ and $i_{s+1} = \kappa(i_s)$ whenever $\kappa(i_s) > 0$ and $\kappa(i_m) = 0$. Since the set of agents is finite and the covering relation is both asymmetric ($i_s \neq \kappa(i_{s+1})$ for all s) and transitive, the sequence is finite and well defined and involves the repeated application of the operator $\kappa(\cdot)$. Thus, $i_2 = \kappa(n)$, $i_3 = \kappa^2(n) = \kappa(\kappa(n))$,..., $i_m = \kappa^{m-1}(n)$ and $\kappa(i_m) = \kappa^m(n) = 0$. Figure 3 provides a graphical illustration.

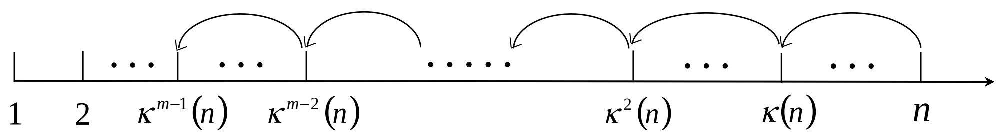

Figure 3. Constructing the Canonical Information Chain of Maximal Length.

We now argue that $m = m^*$ and that this canonical information chain is of maximal length. It suffices to show that if there is an information chain in $\mathcal{G}_0$ given by $(j_1, \ldots, j_t)$ then $m \geq t$. To this end, we establish a one-to-one function from the set $\{j_1, \ldots, j_t\}$ to a subset $\{i_1, \ldots, i_m\}$ of the canonical sequence. Notice that by the definition of an information chain, "j_1 covers j_2" implies (by $\mathcal{I}$-Monotonicity) "$i_1 = n$ covers j_2" and in particular $\kappa(n) = i_2 \geq j_2$. Similarly, "j_2 covers j_3", using $i_2 \geq j_2$, implies "i_2 covers j_3 and $i_3 = \kappa(i_2) = \kappa^2(n) \geq j_3$." Repeating this argument we have that for each $j_k \in \{j_1, \ldots, j_t\}$ there exists a distinct i_k such that $i_k \geq j_k$. This establishes a one to one function from the set $\{j_1, \ldots, j_t\}$ to $\{i_1, \ldots, i_m\}$, proving that $m \geq t$. Our next proposition summarizes this property of the canonical information chain.

Proposition 3 *For the game $\mathcal{G}_0$ the ordered set of agents given by $(i_1, \ldots, i_m)$, where $i_1 = n$, $i_2 = \kappa(i_1)$,..., and $i_m = \kappa^{m-1}(i_1)$ and $\kappa^m(i_1) = \kappa(i_m) = 0$, is an information chain with maximal length $m = m^*$.*

4.1 Extensions and Robustness

We now discuss some issues associated with our solution concept of dominance solvability, as well as the implications of relaxing our assumptions on our modeling of preferences and the information structure for Theorem 2.

Order Independence

Though the concept of using weak dominance to eliminate implausible equilibria has strong support in the literature, the process of iterated weak dominance has faced a particular criticism.[14] While our operator $\mathcal{R}$ specifies that in each stage of the iteration all dominated strategies are removed, one can propose alternative operators which remove some but not necessarily all dominated strategies in each stage of the iteration. It has been argued that this change in the order of elimination of dominated strategies can matter and that the different sequences of games generated by alternative operators may lead to different irreducible games. In some of these cases it has been shown that depending on the order of iterative elimination process the irreducible game may not have a unique outcome, creating ambiguity about the outcome of the game. In our case, if the game is dominance solvable, then no matter what alternative operator $\overline{\mathcal{R}} \neq \mathcal{R}$ is used (as long as some dominated strategies are eliminated in each stage of the iteration) the order of elimination will not matter and the irreducible game obtained from alternative operators will have exactly the same unique PSNEO as $\mathcal{G}_M$. The intuition for this being true is that for our particular game, if under our specified operator $\mathcal{R}$, a coordinate of all strategies of some agent becomes 1 at some stage of the reduction process then either this will also be true under any alternative operator or that coordinate will have become irrelevant in the irreducible game. This order independence result is not surprising in our model since our game satisfies the "non-bossiness" condition in Satterthwaite and Sonnenschein (1981) and it has been well understood from various discussions in the literature that this condition leads to order independence (see e.g. Rochet 1980, Mailath, Samuelson and Swinkels 1993, and Marx and Swinkels 1997).

Information Structure

In our basic model, to ease exposition, we have imposed the condition that the information received by all the agents about previous participations is nested or monotone (Assumption 3). We interpreted this assumption as implying that those playing later observe no less information than those playing earlier. It is important to point out that Assumption 3 is *stronger* than necessary: Our main result still goes through by assuming that the underlying information structure is monotone **only** among the agents in a largest information chain.[15] In particular, what the agents out of a maximal information chain observe (or what they know other than the existence of the key agents in the chain) is not essential for Theorem 2 to hold. In this sense, we can also interpret Theorem 2 as identifying a key group of agents in an information chain whose information signals are nested in achieving an overall coordination in the entire group.

On the other hand, the requirement of a monotone information structure among the agents in a largest information chain—in particular, the signals received by the agents in the chain should be *nested*—remains critical for our results. Suppose we interpret "how much information an agent gets" instead by the dimensionality of her strategy space and

[14] A different concern about iterated weak dominance is that iterated weak dominance is not equivalent to and cannot be grounded by assuming it is common knowledge that players do not play weakly dominated strategies. See Samuelson (1992).

[15] We thank the referees of our paper for raising our attention on this issue.

impose a weaker assumption that later players in the sequence have strategy spaces which are dimensionally no smaller than those of players playing earlier. The following example of a 7-player game shows that our results fail under this weaker assumption. The set of individuals in the cells below the players shows which individuals are covered by that player (e.g. 6 covers the individuals in the set $\{4, 3, 1\}$).

Agent i	1	2	3	4	5	6	7
Set $\varkappa(i)$	$\varnothing$	$\varnothing$	$\{1\}$	$\{1, 2, 3\}$	$\{4, 2, 1\}$	$\{4, 3, 1\}$	$\{4, 3, 2\}$

Let $\lambda = 4$. One can verify that information chains of length 4 exist (i.e., $(7, 4, 3, 1)$, $(6, 4, 3, 1)$, and $(5, 4, 3, 1)$) and the game is dominance solvable. However, if we remove agent 7, while a maximal chain still has length 4, the game is no longer dominance solvable. When agent 4 gets a report that two agents out of $\{1, 2, 3\}$ have joined, by joining, agent 4 is not sure that agent 5 or agent 6 will receive a report that three agents have joined since the two participations agent 4 sees may have come from agents 2 and 3 and neither 5 nor 6 covers *both* these agents.

Preferences

There are two aspects of our assumptions about preferences (Assumptions 1 and 2) that can easily be extended without affecting Theorem 2: (a) Anonymity (dependence of f only on the *number* of participants), and (b) the *status quo* payoff being constant and independent of the number of participants and thus allowing us to normalize the *status quo* utility to being zero for all individuals (see Footnote 3).

We start with a general formulation of the "Tipping Point" Assumption. For each $j \in N$, partition the set of outcome profiles $\mathcal{B}$ into (i) $\mathcal{B}_j^+(\lambda) = \{b \in \mathcal{B} : b_j = 1, \sum_N b_i \geq \lambda\}$, a set of outcomes where j joins together with at least $(\lambda - 1)$ other individuals, (ii) $\mathcal{B}_j^-(\lambda) = \{b \in \mathcal{B} : b_j = 1, \sum_N b_i < \lambda\}$, a set of outcomes where j joins and the total number of participants is less than λ, and (iii) $\mathcal{B}_j^0 = \{b \in \mathcal{B} : b_j = 0\}$, outcomes where j does not join. Endow each agent j with a reflexive and binary weak preference relation $\succsim_j$ on the set $\mathcal{B}$, with an asymmetric component $\succ_j$ (representing strict preferences) and a symmetric component $\backsim_j$ (representing indifference). In the context of this general definition of preferences, consider the following assumption:

Assumption 4 *There exists $\lambda \in \{2, ..., n\}$ such that for all $j \in N$, (if $b \in \mathcal{B}_j^+(\lambda)$ and $b' \in \mathcal{B}_j^0$ then $b \succ_j b'$) and (if $b \in \mathcal{B}_j^-(\lambda)$ and $b' \in \mathcal{B}_j^0$ then $b' \succ_j b$).*

Notice that unlike the preferences in our base model the preferences $\succsim_j$ can be be non-anonymous and intransitive. Since *no* restrictions are imposed on each agent's preference relation *within the sets* $\mathcal{B}_j^+(\lambda)$, $\mathcal{B}_j^-(\lambda)$ and $\mathcal{B}_j^0$, the preferences can be non-anonymous: The dependence of preferences on "who" is participating (rather than just on how many agents are participating) is not necessarily ruled out. Moreover, the condition is consistent with the *status quo* values depending on the set of participants. Also, the lack of any restriction on the binary comparisons using $\succsim_j$ *within* the sets $\mathcal{B}_j^+(\lambda)$, $\mathcal{B}_j^-(\lambda)$ and $\mathcal{B}_j^0$ implies that $\succsim_j$ need *not* even be an ordering on $\mathcal{B}$ (indeed, $\succsim_j$ need neither be transitive nor complete). This model of preferences can thus accommodate a variety of non-traditional preferences, including those not representable by a utility function.[16] The formal definitions of PSNE and Dominance Solvability do not depend on the traditional model of the agents' preferences (as being a weak order) and these concepts can easily be defined in terms of the binary relations $\succsim_j$ and $\succ_j$. As mentioned earlier, for our model, only the changes in strategies along the path of play have an impact on the outcome profile. Thus any such unilateral change along the path of play by any agent j necessarily involves a comparison either between an outcome profile in $\mathcal{B}_j^+(\lambda)$ and an outcome profile in $\mathcal{B}_j^0$ or one between an outcome profile in $\mathcal{B}_j^-(\lambda)$ with one in $\mathcal{B}_j^0$. Since Assumption 4 is sufficient to inform us on both these types of comparisons and since these comparisons are *identical* to those implied by the preferences in our base model, this type of extension leaves our principal result Theorem 2 and its proof intact.

Perfect Bayesian Equilibrium

While it has its critics (as shown above), dominance solvability via iterated weak dominance has been widely used in the literature.[17] In particular, it is known that for generic (finite) extensive games with perfect information, iterated weak dominance is closely related to subgame perfect equilibrium, in that there is an order of elimination, parallel to the procedure of backward induction, that leads to the same outcome of the (unique) subgame perfect equilibrium of the game.[18] With imperfect information in our extensive game, the relationship between iterated weak dominance and subgame perfect equilibrium is however more obscure: In Example 5, while all Nash equilibria including the minimal PSNE are also subgame perfect equilibria, dominance solvability selects a unique outcome of full participation, regardless of the elimination order.

That being said, one remaining issue is whether for our binary threshold game, a standard solution concept, perfect Bayesian equilibrium (PBE), will lead to similar outcomes as dominance solvability does. As we have argued above, for our results to hold we do not need representable preferences. However, to facilitate a comparison to PBE, we now specialize to the case where the agents' preferences can be rationalized as maximizing ex-

[16] For a justification of the use of such a non-traditional ("quasitransitive") preference in games, see Basu and Pattanaik (2014).

[17] Moulin (1986) and Marx and Swinkels (1997) provide justification for using this concept. Also see Gretlein (1982) for a discussion of this procedure in voting games. The procedure of iterative elimination of weakly dominated strategies has also been applied to chess-like games and two-player strictly competitive games (Ewerhart 2000, 2002), signaling future actions by burning money (Ben-Porath and Dekel 1992), finite dynamic bargaining games (Tyson 2010), and auctions (Azrieli and Levin 2011). Epistemic conditions for the procedure of iterated elimination of weakly dominated strategies are provided in Brandenburger et al. (2008).

[18] However, since iterated weak dominance can be applied using different orders of elimination, the two procedures (iterated weak dominance and backward induction) can lead to different outcomes and are hence different in general. See Chapter 6.6 of Osborne and Rubinstein (1994).

pected utility. In particular, we will show that when our game is dominance solvable, it is possible to find a unique PBE which gives us the same outcome as that under dominance solvability.

Recall the canonical information chain of maximal length m (Figure 3), which is an ordered set of agents $\{i_1, \ldots, i_m\}$ with $i_1 = n$, $i_2 = \kappa(n)$, $\ldots$, $i_m = \kappa^{m-1}(n)$. We denote agent i's set of information sets as $H_i = \{H_i(1), \ldots, H_i(\kappa(i)+1)\}$—i.e., at information set $H_i(l)$, agent i observes an aggregate signal of '$l-1$' previous participations. Agent i's belief is a function assigning to all $H_i(l) \in H_i$ a probability measure on the set of histories in $H_i(l)$. Agent i's (behavior) strategy is a sequence of functions mapping from H_i to participation probabilities $a_i = (a_i(1), \ldots, a_i(\kappa(i)+1))$ where $a_i(l) : H_i(l) \to [0,1]$. We denote the extensive form of the binary threshold game as Γ. Finally, a strategy profile $(a_i)_{i \in N}$ is a *perfect Bayesian equilibrium* of Γ if for each $i \in N$, $a_i(l)$ is a best response at information set $H_i(l)$ for all $l \in \{1, \ldots, \kappa(i)+1\}$, given a_{-i}.

Notice that we have suppressed the belief profile in defining PBE for ease of exposition: The agents' beliefs about the histories of participations are derived via Bayes' rule along the equilibrium path, while beliefs off the equilibrium path can be arbitrary—indeed, for our purpose here, no further restrictions (other than that only payoff relevant information is relevant for simplicity) are needed for beliefs off the equilibrium path.

Proposition 4 *Consider the binary threshold extensive game Γ with a canonical information chain with length m. Then Γ admits a unique perfect Bayesian equilibrium outcome in which all agents participate whenever $m \geq \lambda$.*

Therefore, exactly the same condition (i.e., $m \geq \lambda$ as in Theorem 2) implies that full participation among all the agents still arises as the unique PBE outcome. Proposition 4 hence provides partial justification of our solution concept of dominance solvability in that it predicts in our setting the same unique outcome as the standard PBE solution.[19]

However, compared to perfect Bayesian equilibrium, one useful advantage from adopting the solution concept of dominance solvability in our characterization of information structures is that the characterization can be done in an 'ordinal' fashion—involved discussions on the agents' beliefs of the past participation histories can be avoided and we can simply impose a *single-crossing* condition (Assumption 1) without specifying further details of the agents' payoffs for various aggregate participation levels.

[19] It is worth pointing out that a recent study (Koriyama and Núñez 2015) shows that for any finite normal-form game satisfying the TDI condition in Marx and Swinkels (1997) that is dominance solvable by weak dominance, the unique dominance-solvable outcome must coincide with the payoff of a *proper* equilibrium. This result justifies our dominance solvable outcome—since our game satisfies the TDI condition, our dominance solvable outcome hence also coincides with the outcome of a proper equilibrium of our threshold game. We thank Sean Horan for bringing this to our attention.

5 Conclusion

We have considered a sequential binary tipping point game with the objective of analyzing the collective behavior of the agents under various information structures. We have found that an information structure is important in determining the number and type of pure-strategy Nash equilibria of the game. While both maximal coordination and coordination failure arise as possible pure-strategy Nash equilibrium outcomes in our game irrespective of its information structure, an adequate transmission of and availability of information leads rational agents to achieve a maximal pure strategy Nash equilibrium as the *unique* admissible outcome. Hence, an important implication of our results is that improving the flow of the information among agents acting independently can be used as a policy tool to promote decentralized coordination, avoid inefficiency, and achieve an egalitarian maximal welfare in binary threshold models. In addition, an alternative interpretation of our main result is that it identifies a key group of agents who are informationally linked—those in a longest information chain—in achieving an overall coordination for environments similar to our setting.[20]

Appendix

Proof of Theorem 1. ($\Longrightarrow$) Let $\tau \in \{0, \ldots, (n-\lambda-1)\}$. Suppose there is a PSNE a with exactly $(\lambda+\tau)$ agents participating: $\sum_{i\in N} b_i(a) = \lambda+\tau$. Since the set $N_1 = \{(n-\lambda-\tau), \ldots, n\}$ has $(\lambda+\tau+1)$ agents, $\sum_{i\in N} b_i(a) = \lambda+\tau$ implies that there exists $j \in N_1$ such that $b_j(a) = 0$. Let $\tilde{j}$ be the "last" agent in N_1 such that $b_j(a) = 0$, i.e., let $\tilde{j} = \max\{j : j \in N_1 \text{ and } b_j(a) = 0\}$. Given such $\tilde{j}$, we have

$$b_j(a) = 1 \text{ for all } j > \tilde{j}, \text{ and} \tag{2}$$
$$\tilde{j} \geq (n-\lambda-\tau). \tag{3}$$

Consider a profile $\widetilde{a} = (\widetilde{a}_{\tilde{j}}, a_{-\tilde{j}})$ that differs from a *only* in $\tilde{j}$'s strategy such that the relevant coordinate of $\tilde{j}$ under a is changed from $b_{\tilde{j}}(a) = 0$ to $b_{\tilde{j}}(\widetilde{a}) = 1$. Since a is a PSNE, this unilateral deviation must decrease $\tilde{j}$'s utility to less than zero. In other words at least $(\tau+2)$ agents in $\left\{(\tilde{j}+1), \ldots, n\right\}$ who join under a (with $b_j(a) = 1$) must no longer join under $\widetilde{a}$ (with $b_j(\widetilde{a}) = 0$). Define $J = \{j_1, \ldots, j_{\tau+2}\}$ with $j_i < j_{i+1}$ for all $i \leq \tau+1$ be the set of the *first* $(\tau+2)$ agents such that $b_j(a) = 1$ and $b_j(\widetilde{a}) = 0$ for $j \in J$ (i.e., the *first* $(\tau+2)$ agents switching from joining to not joining after $\tilde{j}$'s deviation). This implies that there are at least $\tau+2$ agents moving after $\tilde{j}$ and hence

$$\tilde{j} \leq n-\tau-2. \tag{4}$$

[20] See our discussion of Information Structure in Section 4.1.

Inequalities (3) and (4) together imply that

$$(n - \lambda - \tau + 1) \leq j_1 \leq (n - \tau - 1). \tag{5}$$

Since $\tilde{j}$ has made a unilateral deviation and $j_1 \in J$ is such that $b_{j_1}(a) = 1$ and $b_{j_1}(\widetilde{a}) = 0$, it must be that j_1's information covers $\tilde{j}$, i.e., $\kappa(j_1) \geq \tilde{j}$. By the choice of J, since j_1 and j_2 are the first two agents switching from joining under a to not joining under $\widetilde{a}$, and that all agents j with $j_1 < j < j_2$ (if any) have $b_j(a) = b_j(\widetilde{a}) = 1$ by definition of j_1 and j_2, it follows that j_2's information does not cover j_1, i.e., $\kappa(j_2) < j_1$.[21] Since $j_2 \geq j_1 + 1$, Assumption 3 and $\kappa(j_2) < j_1$ then jointly imply that

$$(j_1 + 1)\text{'s information does not cover } j_1, \text{ i.e., } \kappa(j_1 + 1) < j_1. \tag{6}$$

Finally, by our choice of J, since j_1 is the first agent switching after $\tilde{j}$'s deviation, the number of agents who move after agent $(j_1 + 1)$ and who cover j_1 but not j_2 can be at most $(n - (j_1 + 1)) - \tau = n - j_1 - \tau - 1$, i.e., $|\{j : \kappa(j) = j_1\}| \leq n - j_1 - \tau - 1$. The reason is that the agents in $\{j : \kappa(j) = j_1\}$, when this set is non-empty, receive the same information signal under a and $\widetilde{a}$ and will have the same move under both profiles and cannot belong to J. Since $\lambda \geq 2$, the number of agents in $\{j : \kappa(j) = j_1\}$ cannot be too large for there to be at least $(\tau + 2)$ agents in J. It is easy to check that if $|\{j : \kappa(j) = j_1\}| > n - j_1 - \tau - 1$ then the number of agents who remain in the set $\{j_1, j_1 + 1, j_1 + 2, \ldots, n\} \backslash \{j : \kappa(j) = j_1\}$ is strictly less than $n - j_1 + 1 - (n - j_1 - \tau - 1) = \tau + 2$. Thus,

$$|\{j : \kappa(j) = j_1\}| \leq n - j_1 - \tau - 1. \tag{7}$$

Using (5), (6), (7) and letting j_1 be the j^* in Theorem 1 completes the proof.[22]

($\Longleftarrow$) Let $\tau \in \{0, \ldots, n - \lambda - 1\}$. By our hypothesis we have $j^* \geq n - \lambda - \tau + 1$ with $\kappa(j^*) \geq n - \lambda - \tau$, $\kappa(j^* + 1) < j^*$, and $n - j^* - \tau - 1 \geq |\{j : \kappa(j) = j^*\}|$. We will construct a PSNE a such that exactly the last $(\lambda + \tau)$ agents participate on the path of play.

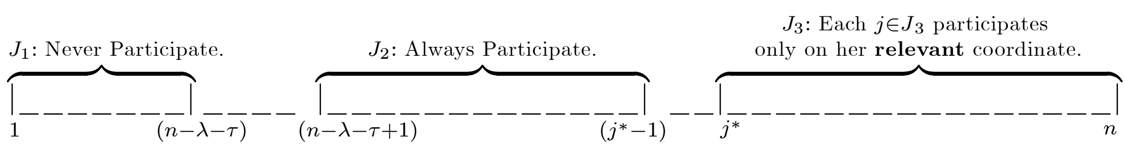

Figure 4. Equilibrium strategy profile a with exactly $(\lambda + \tau)$ participations.

Consider a strategy profile a (see Figure 4) such that:

[21] If $\kappa(j_2) \geq j_1$, j_2 would receive the same report under both a and $\widetilde{a}$, contradicting $b_{j_2}(a) \neq b_{j_2}(\widetilde{a})$.

[22] An alternative and simpler argument for (7) is the following: Given that $\kappa(j_1) \geq \tilde{j} \geq n - \lambda - \tau$ and $\kappa(j_1 + 1) \leq \kappa(j_2) < j_1$, we have that $\{j : \kappa(j) = j_1\} \subset \{j > j_2 : j \notin J\}$, i.e., the agents whose information **exactly covers** j_1 (hence such agents are not in J) belong to those who move after j_2. The fact that the set $\{j > j_2 : j \notin J\}$ has at most "$n - (j_1 + 1) + 1 - (\tau - 1)$" agents directly implies (7). We thank a referee for pointing this out.

- For all $j \in J_1 = \{1, \ldots, n-\lambda-\tau\}$, $a_j(l) = 0$ for all $l \in \{1, \ldots, \kappa(j)+1\}$, i.e., all of the first $(n-\lambda-\tau)$ agents never participate regardless of what they observe.

- For all $j \in J_2 = \{n-\lambda-\tau+1, \ldots, j^*-1\}$, $a_j(l) = 1$ for all $l \in \{1, \ldots, \kappa(j)+1\}$, i.e., all agents moving after the agents in J_1 but before agent j^* choose to participate regardless of what they observe.

- For all $j \in J_3 = \{j^*, \ldots, n\}$, $a_j(l) = 1$ iff the l^{th} coordinate of j is on the path of play, i.e., agents moving after agent (j^*-1) choose to join *only* in those coordinates that are relevant under a (i.e., such an agent has "1" in the single coordinate of her strategy that is on the path of play and has "0" elsewhere).

By construction, on the path of play only the last $(\lambda+\tau) = |J_2 \cup J_3|$ agents choose to join under a and hence we have $\sum_{i \in N} b_i(a) = \lambda + \tau$. Given the payoff in (1) and Assumption 1, the strategies for agents in J_1, J_2 and J_3 then imply

$$u_j(a) = \begin{cases} 0, \text{ if } j \leq n-\lambda-\tau, \\ f_j(\lambda+\tau) > 0, \text{ if } j \geq n-\lambda-\tau+1. \end{cases} \tag{8}$$

The remainder of the proof establishes that the above profile a is a PSNE.

If $j \in J_2 \cup J_3$, then $j \geq n-\lambda-\tau+1$ and using (8), such a j's unilateral deviation either gives exactly $f_j(\lambda+\tau)$ or decreases j's payoff from $f_j(\lambda+\tau)$ to 0. Hence j is playing a best response at a.

Next, consider a unilateral deviation of a relevant coordinate by $j \in J_1$ with a resulting profile $a' = (a'_j, -a'_j) = (a'_j, -a_j)$. Given $\kappa(j^*+1) < j^*$, agents j^* and (j^*+1) now both observe a signal at a' indicating that one more agent (than at a) has joined. According to a', since $a_{-j} = a'_{-j}$, both j^* and j^*+1 now switch from "join" ($b_{j^*}(a) = b_{j^*+1}(a) = 1$) to "not join" ($b_{j^*}(a') = b_{j^*+1}(a') = 0$) after j's deviation.[23] Now consider agents $j' \geq j^*+2$ (i.e., $j' \in J_3 \setminus \{j^*, j^*+1\}$). There are three possible cases:

- If $\kappa(j') < j^*$, j' now (like agents j^* and (j^*+1)) gets a signal under a' indicating that one more agent has joined than under a. Hence, by construction j' will switch from joining to not joining.

- If $\kappa(j') = j^*$, j''s signal will not change and agent j' will, as before, participate.[24]

- If $\kappa(j') > j^*$, then j' gets a signal indicating *at least* one less agent has joined under a' than under a.[25] According to $a_{j'} = a'_{j'}$, j' will switch from joining to not joining.

[23] To be specific, agents j^* and (j^*+1) both observe $(j^*-n+\lambda+\tau-1)$ previous agents joining before j's deviation and $(j^*-n+\lambda+\tau)$ agents joining after j's deviation.

[24] Since $n-j^*-\tau-1 \geq |\{j : \kappa(j) = j^*\}|$, there are at most $(n-j^*-\tau-1)$ such agents.

[25] Since $\kappa(j') \geq j^*+1$, one is added to j''s information report because $j \in J_1$ joins, while two is subtracted from j''s report because j^* and j^*+1 have switched to "not join."

To summarize, after the unilateral deviation from $j \in J_1$, the set of agents who choose to join on the path of play is given by $\{j\} \cup J_2 \cup \{j : \kappa(j) = j^*\}$. It follows that

$$\sum_{i \in N} b_i(a') = 1 + (j^* - (n - \lambda - \tau + 1)) + |\{j : \kappa(j) = j^*\}| \leq \lambda - 1.$$

This implies that j's deviation is not profitable: j's payoff from her deviation on her relevant coordinate in a is at most $f_j(\lambda - 1)$ which is less than 0 (Assumption 1). ■

Proof of Theorem 2

We will introduce some notation, define some key concepts, and prove preliminary lemmas that will be used to prove Theorem 2. The first set of lemmas (Lemma 1 to Lemma 4) holds for all $\mathcal{G}_0$ whether or not $\mathcal{G}_0$ is dominance solvable and is unrelated to the particular information structure of $\mathcal{G}_0$. These lemmas provide useful insights into the iterative reduction process of dominated strategies. The second set of lemmas (Lemma 5 and Lemma 6) relates the existence of information chains in $\mathcal{G}_0$ to this reduction process and provides results that represent crucial parts of the proof of Theorem 2.

For convenience, we will abuse notation and use $a_j \in \mathcal{G}_h$ to indicate that a_j is *possible* in the game $\mathcal{G}_h$, i.e., a_j has not been eliminated and is a strategy of j in game $\mathcal{G}_h$. Similarly $a \in \mathcal{G}_h$, $a_{-j} \in \mathcal{G}_h$, and $a_j(l) \in \mathcal{G}_h$ will indicate respectively that the profile a, the contingency a_{-j} and the conditional action $a_j(l)$ are possible in $\mathcal{G}_h$.

Definition 1 *For $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_M\}$, a conditional action $a_j(l) \in \mathcal{G}_h$ is a **best response conditional action** (BRCA) in $\mathcal{G}_h$ iff there is $a_{-j} \in \mathcal{G}_h$ such that the l^{th}-coordinate of j's strategy is on the path of play and a_j is a best response to the contingency a_{-j} in $\mathcal{G}_h$.*

For the games in $\{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$, as dominated strategies are iteratively eliminated, certain paths of play occurring in earlier games may not appear in later games. As this happens, some coordinates of an agent's strategy become irrelevant, i.e., no path of play that is possible in the game passes through these coordinates. It is important to identify which coordinates and which paths persist. The next lemma shows that an undominated strategy is closely related to BRCAs and these coordinates of individual strategies persist from game to game. Lemma 1 (i) shows that if some coordinates of a strategy in a game consist of BRCAs then these BRCAs survive in that some undominated strategy in that game has these BRCAs in its coordinates, while Lemma 1 (ii) shows that for a strategy to be undominated in a game, **all** coordinates of that strategy must either be a BRCA or irrelevant in that game.[26]

Lemma 1 (Persistence) *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_{M-1}\}$ and $j \in N$. (i) For $a_j \in \mathcal{G}_h$ and $L \subseteq \{1, \ldots, \kappa(j) + 1\}$, if for all $l \in L$, $a_j(l)$ is a BRCA, then there exists $a'_j \in \mathcal{G}_{h+1}$ such*

[26] The results in Lemma 1(i) are stated in the form of a set of coordinates rather than an individual coordinate. We choose this somewhat cumbersome formulation to simplify our proof for Lemma 1(ii).

that $a'_j(l) = a_j(l)$ for all $l \in L$. (ii) If a^*_j is undominated in $\mathcal{G}_h$, then a^*_j is such that for all $l \in \{1, \ldots, \kappa(j)+1\}$, either $a^*_j(l)$ is a BRCA in $\mathcal{G}_h$ or the l^{th} coordinate of j's strategy is irrelevant in $\mathcal{G}_h$.

Proof. (i) Let $a_j \in \mathcal{G}_h$ be such that $a_j(l)$ is a BRCA in $\mathcal{G}_h$ for all $l \in L$. By the definition of a BRCA, for each such l, there is a strategy profile $a^l = \left(a_j, a^l_{-j}\right) \in \mathcal{G}_h$ such that the l^{th} coordinate of j is on the path of play and a_j is a best response for a^l_{-j} in $\mathcal{G}_h$. If a_j itself is undominated, then $a_j \in \mathcal{G}_{h+1}$ and we are done. If a_j is dominated, then by finiteness of A_j and transitivity of the dominance relation, there is an undominated strategy $a'_j \in \mathcal{G}_h$ that dominates a_j. This implies that for each of the contingencies $a^l_{-j} \in \mathcal{G}_h$, the payoff of j from a'_j is at least as large as that from a_j. But, since a_j is a best response to $a^l_{-j} \in \mathcal{G}_h$, we must have $a'_j(l) = a_j(l)$ for all $l \in L$. Finally, as a'_j is undominated in $\mathcal{G}_h$, $a'_j \in \mathcal{G}_{h+1}$.

(ii) Assume to the contrary that there exists an undominated strategy $a^*_j \in \mathcal{G}_h$ and a non-empty set of coordinates $\widetilde{L}$ of a^*_j where $\widetilde{L} = \{l : l \in \{1, \ldots, \kappa(j)+1\}, a^*_j(l)$ is relevant and $a^*_j(l)$ is not a BRCA in $\mathcal{G}_h\}$. Replace of all the coordinates of a^*_j in $\widetilde{L}$ with the corresponding BRCAs in $\mathcal{G}_h$ to construct a new strategy a'_j which agrees with a^*_j in all coordinates other than those in $\widetilde{L}$. By construction, the coordinates of a'_j are either irrelevant or are BRCAs in $\mathcal{G}_h$ and *if* this a'_j exists in $\mathcal{G}_h$, it would dominate a^*_j. Since all j's contingencies in $\mathcal{G}_h$ are also contingencies for j in $\mathcal{G}_{h-1}$ and since every relevant coordinate of a'_j is a BRCA in $\mathcal{G}_h$, it must be the case that these coordinates must also be BRCAs in $\mathcal{G}_{h-1}$. Denoting all these relevant coordinates of a'_j which are BRCAs by L, Lemma 1 (i) (applied to $\mathcal{G}_{h-1}$) implies that there exists $a''_j \in \mathcal{G}_h$ such that each of a''_j's coordinates either coincides with that of a'_j or the coordinate is irrelevant in $\mathcal{G}_h$. Thus, since a''_j "generates exactly the same outcomes as" a'_j in $\mathcal{G}_h$, it dominates a^*_j contradicting our hypothesis that a^*_j is undominated in $\mathcal{G}_h$. ■

In our setting, another consequence of eliminating dominated strategies is that certain relevant coordinates of an agent's strategies become "fixed" in that for these coordinates all the agent's strategies take on the same value. Once this has happened in a game, these coordinates remain fixed in subsequent games. We will be particularly interested in cases where the possibility of non-participation is eliminated by dominance and some coordinate of an agent's strategy is *reduced to 1* in a game and is fixed at 1 in all subsequent games. Moreover, if all relevant coordinates greater than or equal to some coordinate are also fixed at one, we will say that this particular coordinate of the agent's strategy has been *strictly reduced to* 1 and the coordinate is *strictly fixed at 1*.

Definition 2 *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_{M-1}\}$, $j \in N$ and $l \in \{1, \ldots, \kappa(j)+1\}$. The conditional action $a_j(l)$ is **fixed at 1** in $\mathcal{G}_{h+1}$ iff for all $a_j \in \mathcal{G}_{h+1}$, $a_j(l) = 1$. The conditional action $a_j(l)$ is **strictly fixed at 1** in $\mathcal{G}_{h+1}$ iff (i) $a_j(l) = 1$ is **fixed at 1** in $\mathcal{G}_{h+1}$ and (ii) for all $s \in \{1, \ldots, \kappa(j)+1-l\}$ either $a_j(l+s) = 1$ or the $(l+s)^{th}$ coordinate of j's strategy is irrelevant in $\mathcal{G}_{h+1}$. If $a_j(l)$ is fixed at 1 in $\mathcal{G}_{h+1}$ and $a_j(l)$ is not fixed at 1 in $\mathcal{G}_h$ we will say that the l^{th} coordinate of j's strategy is **reduced to 1** in game $\mathcal{G}_h$. If $a_j(l)$ is* *strictly fixed at 1 in $\mathcal{G}_{h+1}$ and $a_j(l)$ is not strictly fixed at 1 in $\mathcal{G}_h$ we will say that the l^{th} coordinate of j's strategy is* ***strictly reduced to 1*** *in $\mathcal{G}_h$.*

Remark 3 As per the terminology we have adopted in Definition 2, a coordinate is "not fixed" in the game in which it is "reduced" and only becomes fixed in subsequent games and $a_j(l) = 1$ being strictly fixed at 1 in $\mathcal{G}_{h+1}$ implies that $a_j(l) = 1$ is fixed at 1 in $\mathcal{G}_{h+1}$. It is also clear that $a_j(l) = 1$ being fixed at 1 in $\mathcal{G}_h$ implies that $a_j(l)$ is fixed at 1 in $\mathcal{G}_{h'}$ for all $h' \geq h$. Moreover, since an irrelevant coordinate of a strategy in a game is irrelevant in all subsequent games, it also follows that if $a_j(l)$ is strictly fixed at 1 in $\mathcal{G}_h$ then $a_j(l)$ is strictly fixed at 1 in $\mathcal{G}_{h'}$ for all $h' \geq h$.

We next introduce notation for a particular set of "participating" strategies ($\mathcal{P}$) and a set of "non-participating" strategies ($\mathcal{NP}$). These are simply book-keeping devices that we shall use to describe the process of coordinates being "reduced" to 1. Here, $\mathcal{P}$ is the set of strategy profiles where all the agents choose to participate whenever their information signal shows maximal previous participations. The largest coordinate of each agent's strategy hence takes the value 1 in $\mathcal{P}$.

$$\mathcal{P} = \{a \in \mathcal{G}_0 : a_j(\kappa(j) + 1) = 1 \text{ for all } j \in N\}.$$

On the other hand, $\mathcal{NP}(r)$ is the set of strategy profiles where agents decide not to participate upon observing less than $(r - 1)$ participations. Here, *all the coordinates of all the agents' strategies* less than or equal to the r^{th}-coordinate are zero.

$$\mathcal{NP}(r) = \{a \in \mathcal{G}_0 : \text{For all } j \in N,\ a_j(l) = 0 \text{ for all } l \leq \min\{\kappa(j) + 1, r\}, r \in \mathbb{N}\}.$$

By varying r, we obtain a nested set of subsets of $\mathcal{NP}(r)$ with $\mathcal{NP}(r-1) \supseteq \mathcal{NP}(r)$. We will use the terminology $\mathcal{P}$ in $\mathcal{G}_h$ ($\mathcal{NP}(r)$ in $\mathcal{G}_h$) to indicate the strategy profiles in $\mathcal{P}$ (the strategy profiles in $\mathcal{NP}(r)$) that survive the elimination process from $\mathcal{G}_0$ to $\mathcal{G}_h$.[27] We will also use $a_j \in \mathcal{P}$ ($a_j \in \mathcal{NP}(r)$) to indicate a strategy of j with $a_j(\kappa(j) + 1) = 1$ (with $a_j(l) = 0$ for all $l \leq \min\{\kappa(j) + 1, r\}$).

Lemma 2 (Unanimous Participation) *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_M\}$. Then $\mathcal{P} \neq \varnothing$ in $\mathcal{G}_h$ and every strategy profile $a \in \mathcal{P}$ is a PSNE in $\mathcal{G}_h$ with $\sum_N b_j(a) = n$.*

Proof. In $\mathcal{G}_0$ since no strategies have been eliminated, $\mathcal{P} \neq \varnothing$. In addition, for all $j \in N$, a_j is a best response to a_{-j} for $(a_j, a_{-j}) \in \mathcal{P}$ in $\mathcal{G}_0$ since any unilateral deviation on the path of play with $a_j(\kappa(j) + 1) = 0$ will reduce agent j's payoff from positive to zero. Thus, by the persistence Lemma 1, there is then an undominated strategy a_j with $a_j(\kappa(j) + 1) = 1$ in $\mathcal{G}_0$. Since this is true for all j, we have $\mathcal{P} \neq \varnothing$ in $\mathcal{G}_1$. A similar argument establishes the result inductively. ■

[27] For all games $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_M\}$, $\mathcal{P} \cap \mathcal{NP}(r) = \varnothing$ for all $r \geq 1$ and both $\mathcal{P}$ and $\mathcal{NP}(r)$ are non-empty in $\mathcal{G}_0$ for all possible values of $r \geq 1$.

Lemma 2 shows that in all games $\mathcal{G}_h$ there always exists an (efficient) PSNEO in which each agent receives a payoff of $f_j(n)$—hence the efficient PSNEO will never be eliminated in the reduction process.[28] An immediate consequence of Lemma 2 is Corollary 6, which shows that for dominance solvability it is **necessary** that after the process of iterative dominance, the irreducible game $\mathcal{G}_M$ should satisfy $\mathcal{NP}(1) = \varnothing$.

Corollary 6 *If $\mathcal{NP}(1) \neq \varnothing$ in $\mathcal{G}_M$ then $\mathcal{G}_M$ has at least two outcomes, one in which $\sum_N b_j = n$ and another in which $\sum_N b_j = 0$ and hence $\mathcal{G}_0$ is not dominance solvable.*

The following special type of a pre-requisite will play an important role in the sequel:

Definition 3 *The pre-requisite of the l^{th} coordinate of j's strategy is satisfied **exactly** for a contingency a_{-j} in the profile (a_j, a_{-j}) if $\sum_{i=1}^{l-1} b_i(a_j, a_{-j}) = l-1$, where $b(a_j, a_{-j})$ is the action profile induced by (a_j, a_{-j}). The contingency a_{-j} is called an **exact contingency** for the l^{th} coordinate of j's strategy.*

Under an exact contingency a_{-j}, the information that j receives represents a full and complete aggregate report of what *actually* occurs and this report is generated by exactly the *first* $(l-1)$ individuals participating. Our next lemma shows that for all $r \leq \kappa(n)+1$, if $\mathcal{NP}(r) \neq \varnothing$ in $\mathcal{G}_h$, then for each agent, for every (possible) coordinate in the agent's strategy that is no larger than r, there exists an exact contingency in $\mathcal{G}_h$ for such a coordinate of the agent to be on the path of play.

Lemma 3 (Exact Contingency) *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_M\}$ be such that $\mathcal{NP}(r) \neq \varnothing$ in $\mathcal{G}_h$ for some $r \in \{1, \ldots, \kappa(n)+1\}$. Then for $j \in N, l \in \{1, \ldots, \min\{\kappa(j)+1, r\}\}$, there is a strategy profile $a^* \in \mathcal{G}_h$, a^* depending on l, such that the l^{th} coordinate of j's strategy is on the path of play under a^*, and $\sum_{i=1}^{l-1} b_i(a^*) = \sum_N b_i(a^*) = l-1$.*

Proof. The proof is done by construction. First, $\mathcal{P} \neq \varnothing$ in $\mathcal{G}_h$ (Lemma 2) implies that for each $i \in N$, there is a strategy $\hat{a}_i \in \mathcal{G}_h$ with $\hat{a}_i(\kappa(i)+1) = 1$. Since by the hypothesis $\mathcal{NP}(r) \neq \varnothing$ in $\mathcal{G}_h$, there is also a strategy $\check{a}_i \in \mathcal{NP}(r)$ for all $i \in N$ such that $\check{a}_i(1) = \cdots = \check{a}_i(\min\{\kappa(i)+1, r\}) = 0$. Let $j \in N$ and $l \in \{1, \ldots, \kappa(j)+1\}$ with $1 \leq l \leq r$ and consider a strategy profile a^* where (see Figure 5)

$$\begin{aligned} \text{for } i &\in \{1, \ldots, l-1\}, \, a_i^* = \hat{a}_i, \\ \text{for } i &\in \{l, \ldots, n\}, \, a_i^* = \check{a}_i. \end{aligned} \tag{9}$$

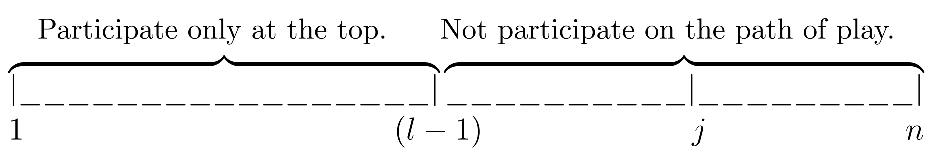

Figure 5. The Strategy Profile a^* for the l^{th} Coordinate of j's Strategy (An agent $i \leq l-1$ chooses $a_i^* \in \mathcal{P}$, while an agent $i \geq l$ chooses $a_i^* \in \mathcal{NP}(r)$).

[28] A similar argument *cannot* be applied to the PSNE with $\sum_N b_j(a) = 0$ as under certain information structure, $a_1 = a_1(1) = 0$ may no longer be a BRCA at some stage of the reduction process.

Hence, under the profile $a^* \in \mathcal{G}_h$, $a_j(l)$ is on the path of play, and $\sum_1^{l-1} b_i(a^*) = \sum_N b(a^*) = l-1$, where $a_i^* \in \mathcal{NP}(r)$ for all $i \geq l$, establishing the result. ■

The construction in (9) yields two key implications: First, since all strategies are possible in $\mathcal{G}_0$, it follows that $\mathcal{G}_0$ satisfies $\mathcal{NP}(\lambda) \neq \varnothing$. Second, for the case where $2 \leq r \leq \lambda - 1$ as one proceeds along the sequence $\mathcal{G}_1, \ldots, \mathcal{G}_{M-1}$, *except* possibly for the first round (from $\mathcal{G}_0$ to $\mathcal{G}_1$), the maximum possible 'reduction' per round of elimination is 'one' in the following sense: If $r \leq \lambda - 1$ and one has $\mathcal{NP}(r) \neq \varnothing$ in $\mathcal{G}_h$ (i.e., every agent has a strategy in $\mathcal{G}_h$ with zeros in all coordinates no larger than the r^{th} coordinate), then it holds that $\mathcal{NP}(r-1) \neq \varnothing$ in $\mathcal{G}_{h+1}$ (i.e., every agent has a strategy in $\mathcal{G}_{h+1}$ with zeros in all coordinates no larger than the $(r-1)^{\text{th}}$ coordinate). We summarize the above in Lemma 4. In particular, the maximal reduction Lemma 4 implies that $\mathcal{G}_1$ satisfies $\mathcal{NP}(\lambda - 1) \neq \varnothing$ whether or not the game $\mathcal{G}_0$ is dominance solvable.

Lemma 4 (Maximal Reduction) *(i) $\mathcal{G}_0$ satisfies $\mathcal{NP}(\lambda) \neq \varnothing$. (ii) Let $j \in N$ and $\mathcal{G}_h \in \{\mathcal{G}_1, \ldots, \mathcal{G}_{M-1}\}$ satisfy $\mathcal{NP}(r) \neq \varnothing$ in $\mathcal{G}_h$ for some $r \in \{2, \ldots, \lambda - 1\}$. Then for all $l \in \{1, \ldots, \min\{r-1, \kappa(j)+1\}\}$, $a_j(l) = 0$ is BRCA in $\mathcal{G}_h$ and $\mathcal{NP}(r-1) \neq \varnothing$ in $\mathcal{G}_{h+1}$.*

We now present two lemmas that represent the key steps in the proof of Theorem 2.

Lemma 5 (Sufficiency) *Consider the canonical sequence of agents given by the ordered set $(i_1, i_2, \ldots, i_{m^*})$ where $i_1 = n$ and $i_{s+1} = \kappa(i_s)$ and $\kappa(i_{m^*}) = \kappa^{m^*-1}(n) = 0$. If $m^* \geq \lambda$ then $\mathcal{G}_0$ is dominance solvable.*

Proof. Consider the game $\mathcal{G}_0$. From $m^* \geq \lambda$ and Assumption 3 ($\mathcal{I}$-Monotonicity) we know that i_1 covers at least (m^*-1) agents, i.e., agents $i_2, \ldots, i_{m^*}$, and hence $\kappa(i_1) = \kappa(n) \geq m^* - 1 \geq \lambda - 1$. Observe that there exists a contingency such that the path of play passes through the λ^{th} coordinate of i_1's strategy in $\mathcal{G}_0$ where all strategies are possible. Moreover, for **every** contingency in $\mathcal{G}_0$ with a path of play passing through the λ^{th} or higher coordinate of i_1, i_1 knows that at least $(\lambda-1)$ individuals have participated before she moves and hence $a_n(\lambda) = a_{i_1}(\lambda + s) = 1$ is BRCA in $\mathcal{G}_0$.[29] Thus, using the persistence Lemma 1, we can conclude that for $i_1 = n$ the λ^{th} coordinate is reduced to 1 in $\mathcal{G}_1$ and all coordinates $a_{i_1}(\lambda)$ and $a_{i_1}(\lambda + s)$ are strictly fixed at 1 for all $s \in \{1, \ldots, \kappa(n) + 1 - \lambda\}$ in $\mathcal{G}_{1+t}$ for all $t \in \{0, \ldots, M-1\}$. This implies $\mathcal{G}_1$ satisfies $\mathcal{NP}(\lambda) = \varnothing$. By the maximal reduction Lemma 4, we know that $\mathcal{G}_1$ satisfies $\mathcal{NP}(\lambda - 1) \neq \varnothing$.

Similarly, the existence of a chain of length λ implies that $\kappa(i_2) \geq \lambda - 2$. By the exact contingency Lemma 3, there is an exact contingency in $\mathcal{G}_1$ passing through the $(\lambda-1)^{\text{th}}$ coordinate of i_2. Since $\kappa(i_1) = i_2$, if $a_{i_2}(\lambda - 1) = 1$, this path of play must pass through the λ^{th} coordinate of i_1's strategy. Since $a_{i_1}(\lambda)$ is strictly fixed at 1 in $\mathcal{G}_{1+t}$ for all $t \in \{0, \ldots, M-1\}$, for **any** contingency in $\mathcal{G}_1$ where the path of play passes through the $(\lambda-1)^{\text{th}}$ coordinate or higher of i_2, we have that $a_{i_2}(\lambda - 1) = 1$ is a BRCA. By the

[29] In $\mathcal{G}_0$, since all strategies are possible, the set of strategies for which this is true is non-empty.

persistence Lemma 1, $a_{i_2}(\lambda - 1)$ is strictly fixed at 1 in $\mathcal{G}_{2+t}$ for all $t \in \{0, \ldots, M-2\}$. In addition, by the maximal reduction Lemma 4, $\mathcal{NP}(\lambda - 2) \neq \varnothing$ in $\mathcal{G}_2$.

Using a similar argument repeatedly for $\mathcal{G}_3, \ldots, \mathcal{G}_{\lambda-1}$ and $i_3, \ldots, i_\lambda$, we have that $a_{i_\lambda}(1)$ is strictly fixed at 1 in $\mathcal{G}_\lambda$ and that for all $a \in \mathcal{G}_\lambda$, $\sum_N b_i(a) \geq \lambda$. This implies that in $\mathcal{G}_{\lambda+1}$ for every $a \in \mathcal{G}_{\lambda+1}$, for every agent j, and for every relevant coordinate l of j's strategies on the path of play, $a_j(l) = 1$ is a BRCA. Using the persistence Lemma 1, it follows that for all $a \in \mathcal{G}_M$, $\sum_N b_i(a) = n$. Thus, $\mathcal{G}_0$ is dominance solvable. ■

Recall that dominance solvability of $\mathcal{G}_0$ implies $\mathcal{G}_M$ must satisfy $\mathcal{NP}(1) = \varnothing$ (Corollary 6), i.e., in $\mathcal{G}_M$ it should not be possible for every individual to have some strategy with zero in the first coordinate. Furthermore, with dominance solvability, using the maximal reduction Lemma 4, we know that $\mathcal{G}_1$ satisfies $\mathcal{NP}(\lambda - 1) \neq \varnothing$. Thus, using Lemma 4 repeatedly we conclude that as one proceeds along the sequence $\{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$ we will encounter (sequentially) games which satisfy "$\mathcal{NP}(\lambda - 1) = \varnothing$ and $\mathcal{NP}(\lambda - 2) \neq \varnothing$" followed by games satisfying "$\mathcal{NP}(\lambda - 2) = \varnothing$ and $\mathcal{NP}(\lambda - 3) \neq \varnothing$" and so on until we will get to the set of games satisfying "$\mathcal{NP}(2) = \varnothing$ and $\mathcal{NP}(1) \neq \varnothing$" and then to games satisfying $\mathcal{NP}(1) = \varnothing$, which, as we have noted, is a condition necessary for dominance solvability. The above process with coordinates successively becoming fixed at 1 *breaks down* when λ is strictly less than the length of the longest information chain m^*. Before offering a proof, we illustrate the underlying intuition of such *break-down* using the example below:

Example 7 *Let $N = \{1, \ldots, 6\}$, $\lambda = 4$, $\kappa(6) = \kappa(5) = \kappa(4) = 3$, $\kappa(3) = 2$, and $\kappa(2) = \kappa(1) = 0$ (and hence $m^* = 3$). In $\mathcal{G}_1$, the 4^{th} coordinate of agents 4,5, and 6 get reduced to 1 and by Lemma 4 all coordinates of all agents less than or equal to the 3^{rd} coordinate have 0 as a BRCA. In particular, the 3^{rd} coordinate of agent 3 is not reduced to 1. In $\mathcal{G}_2$, the 3^{rd} coordinate of agent 3 gets reduced to 1 and again by Lemma 4 all coordinates of all agents less than or equal to the 2^{nd} coordinate have 0 as a BRCA. Now by Lemma 4, the 2^{nd} coordinate of some agent would have to be reduced to 1 in some game in the subsequent sequence $\mathcal{G}_3, \mathcal{G}_4, \ldots$ in order for the game to be dominance solvable. However, this cannot happen: Neither 1 or 2 has a 2^{nd} coordinate and 0 remains a BRCA in the 2^{nd} coordinate of the remaining agents.*

Note that the reduction process described above follows the canonical sequence $n = 6$ and $\kappa(6) = 3$ followed by $\kappa(3) = 2$, and the process 'fails' in the 3^{rd} step (recall that $\lambda = 4$) because $\kappa(2) = 0 \neq 1$—agent 2 has no 2^{nd} coordinate that reduces to 1. Our necessity proof hinges on this crucial link between reduction of single coordinates to 1 in each step of the reduction process through dominance and the next element in the canonical sequence having the next lowest coordinate that can then be reduced.

To establish this relationship, we will develop a notion which partitions the sequence $\{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$ into *blocks* such that the single coordinate reductions occur in the *last* element of each block. Thus, for $\lambda - 1 \geq k \geq 1$, we denote the set of games satisfying "$\mathcal{NP}(\lambda - k + 1) = \varnothing$ and $\mathcal{NP}(\lambda - k) \neq \varnothing$" by the **block** of games $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ with the first and last games in $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ (as ranked in the sequence $\mathcal{G}_1, \ldots, \mathcal{G}_M$) being denoted by $\mathcal{G}_{\min}^{\lambda-k}$ and $\mathcal{G}_{\max}^{\lambda-k}$, respectively.[30] Note that in all games in the block $\left\{\mathcal{G}_i^{\lambda-k}\right\}$, all agents have strategies in which all the coordinates less than or equal to the $(\lambda-k)^{\text{th}}$ coordinate is zero, while in the game $\mathcal{G}_{\max}^{\lambda-k}$ the $(\lambda-k)^{\text{th}}$ coordinate reduces to 1 in all strategies for some individual and that this coordinate is fixed at 1 in all subsequent games starting with $\mathcal{G}_{\min}^{\lambda-k-1}$. Using the sets $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ with $1 \le k \le \lambda-1$ we will establish a property of the canonical sequence of agents $i_1, i_2, \ldots, i_{m^*}$ that will be critical to demonstrate that the process with successive coordinates becoming fixed at 1 *breaks down* when λ is strictly less than the length of the longest information chain (m^*).

In Lemma 6, we show that in each round of reduction of zeros to ones (in the sense of Lemma 4) that occurs between the blocks partitioning $\{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$, the reduction of the coordinate always has to start with an agent less than or equal to a distinguished agent from the canonical sequence $i_1, i_2, \ldots, i_{m^*}$ (where the selection of this distinguished agent depends on the block $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ under consideration).

Lemma 6 (Non-Reduction) *Let $s \in \{1, \ldots, \lambda-1\}$. If $\mathcal{G}_h \in \left\{\mathcal{G}_i^{\lambda-s}\right\}$ then for all agents j such that $j > i_{s+1}$ and $\kappa(j) \ge (\lambda-s-1)$, $a_j(\lambda-s) = 0$ is a BRCA in $\mathcal{G}_h$.*

Proof. We will provide a proof by induction. In each step, we explicitly construct a strategy profile to establish the result.

Basis Step. $s = 1$.

We first consider the block $\{\mathcal{G}_i^{\lambda-1}\}$. The analysis here leads to the reduction that takes place between $\mathcal{G}_{\max}^{\lambda-1}$ and $\mathcal{G}_{\min}^{\lambda-2}$, where the $(\lambda-1)^{\text{th}}$ coordinate of some agent becomes fixed at 1. Recall that all games in the block $\{\mathcal{G}_i^{\lambda-1}\}$ satisfy $\mathcal{NP}(\lambda-1) \neq \varnothing$ and $\mathcal{G}_{\min}^{\lambda-1} = \mathcal{G}_1$.

Consider an agent j with $j > i_2$ and $\kappa(j) \ge (\lambda-2)$ in a game $\mathcal{G}_h \in \left\{\mathcal{G}_i^{\lambda-1}\right\}$. Since $\mathcal{NP}(\lambda-1) \neq \varnothing$ in $\mathcal{G}_h$, there exists $a_{j'} \in \mathcal{NP}(\lambda-1)$ in $\mathcal{G}_h$ for all $j' \ge j$. Moreover, since $\kappa(j) \ge (\lambda-2)$ and $\mathcal{NP}(\lambda-1) \neq \varnothing$, the exact contingency Lemma 3 implies that there exists an **exact** contingency a_{-j}^* for the $(\lambda-1)^{\text{th}}$ coordinate of j in $\mathcal{G}_h$.

Construct a contingency $a_{-j} \in \mathcal{G}_h$ as follows (see Figure 6):

- Let all agents $j'' < j$ use their strategies from the exact contingency a_{-j}^*.
- Let all agents $j' > j$ use the strategies $a_{j'} \in \mathcal{NP}(\lambda-1)$.

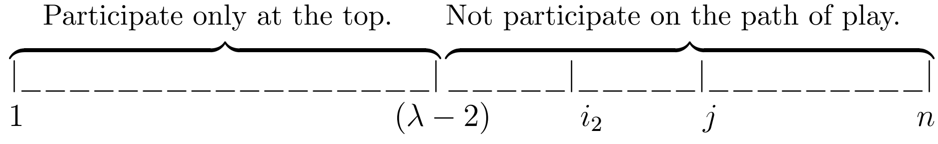

Figure 6. An Illustration of the Contingency a_{-j} for **Basis Step**.

[30] It is possible that $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ is a singleton with $\mathcal{G}_{\min}^{\lambda-k} = \mathcal{G}_{\max}^{\lambda-k}$.

By the construction of a_{-j}, the first $(\lambda-2)$ agents participate on the path of play, while the other agents playing before j choose to not participate. Since, $j > i_2 = \kappa(n) = \kappa(i_1)$, j is not covered by any individual. It follows that all agents moving after j do not participate along the path of play since all such agents after j are using their strategies from $\mathcal{NP}(\lambda-1)$. Hence, irrespective of whether $a_j(\lambda-1) = 1$ is possible or not in $\mathcal{G}_h$, $a_j(\lambda-1) = 0$ is a BRCA for j in $\mathcal{G}_h$, establishing **Basis Step**.[31]

Applying the above to the game $\mathcal{G}_{\max}^{\lambda-1} \in \left\{\mathcal{G}_i^{\lambda-1}\right\}$ and using the persistence Lemma 1, if $\mathcal{G}_{\min}^{\lambda-2}$, the next game after $\mathcal{G}_{\max}^{\lambda-1}$ in the sequence $\mathcal{G}_1, \ldots, \mathcal{G}_M$, exists, we then have:

$$\text{If } j > i_2 \text{ and } \kappa(j) \geq (\lambda-2) \text{, then there is } a_j \in \mathcal{G}_{\min}^{\lambda-2} \text{ with } a_j(\lambda-1) = 0. \quad (10)$$

In other words, a reduction of the $(\lambda-1)^{\text{th}}$ coordinate to 1 of any agent moving after i_2 cannot take place in the block $\left\{\mathcal{G}_i^{\lambda-1}\right\}$ and therefore any such reduction (if any) has to necessarily take place for an agent moving before i_2.

Inductive Step. $s \geq 2$.

Consider the inductive hypothesis: If $\mathcal{G}_h \in \left\{\mathcal{G}_i^{\lambda-s+1}\right\}$ then for all agents j' such that $j' > i_s$ and $\kappa(j') \geq (\lambda-s)$, $a_{j'}(\lambda-s+1) = 0$ is a BRCA in $\mathcal{G}_h$.

Using this inductive hypothesis, we need to show that if $\mathcal{G}_h \in \left\{\mathcal{G}_i^{\lambda-s}\right\}$ then for all agents j such that $j > i_{s+1}$ and $\kappa(j) \geq (\lambda-s-1)$, $a_j(\lambda-s) = 0$ is a BRCA in $\mathcal{G}_h$.

Analogous to (10), use the inductive hypothesis and Lemma 1 to obtain:

$$\text{If } j' > i_s \text{ and } \kappa(j') \geq (\lambda-s) \text{, then there is } a_{j'} \in \mathcal{G}_{\min}^{\lambda-s} \text{ with } a_{j'}(\lambda-s+1) = 0. \quad (11)$$

Consider the first game in the block $\{\mathcal{G}_i^{\lambda-s}\}$: $\mathcal{G}_{\min}^{\lambda-s}$.

Let j be such that $j > i_{s+1}$ and $\kappa(j) \geq (\lambda-s-1)$. We will consider two cases: Case 1. $a_j(\lambda-s) = 0$ for all $a_j \in \mathcal{G}_{\min}^{\lambda-s}$. Case 2. There exists $a_j \in \mathcal{G}_{\min}^{\lambda-2}$ with $a_j(\lambda-s) = 1$.

Case 1. Since $\mathcal{NP}(\lambda-s) \neq \varnothing$ in $\mathcal{G}_{\min}^{\lambda-s}$, $a_j(\lambda-s) = 0$ is possible and the $(\lambda-s)^{\text{th}}$ coordinate of j is relevant (Lemma 3) in $\mathcal{G}_{\min}^{\lambda-s}$. In addition, as $a_j(\lambda-s) = 0$ for all $a_j \in \mathcal{G}_{\min}^{\lambda-s}$, $a_j(\lambda-s) = 0$ is a BRCA in $\mathcal{G}_h$, completing the proof in this case.

Case 2. In this case we have

$$\text{there exists } a_j \in \mathcal{G}_{\min}^{\lambda-s} \text{ with } a_j(\lambda-s) = 1. \quad (12)$$

Let j' be any agent that covers j, i.e., $\kappa(j') \geq j$. Since $\kappa(j) \geq (\lambda-s-1)$ and $\kappa(j') \geq j$, we have $\kappa(j') \geq (\lambda-s)$. In addition, we know that since j' covers j and $j > i_s$ it must be the case that agent j' moves after i_s.[32] Statement (11) shows that for any such j', there is $a_{j'} \in \mathcal{G}_{\min}^{\lambda-s}$ with $a_{j'}(\lambda-s+1) = 0$. This allows us to construct the following strategy

[31] It is possible for some $j > i_2$ to have $a_j(\lambda-1) = 0$ for all $a_j \in \mathcal{G}_h$. For example, consider agent n.

[32] The key difference between **Basis Step** and **Inductive Step** is that an agent j (with $j > i_{s+1}$) may be covered by another agent j' in **Inductive Step** while an agent j (with $j > i_2$) *cannot* be covered by any agent in **Basis Step**. This creates additional complications in constructing the contingency a_{-j} toward the result of $a_j(\lambda-s) = 0$ being a BRCA in **Inductive Step.**

profile $a \in \mathcal{G}_{\min}^{\lambda-s}$ (and hence a contingency $a_{-j} \in \mathcal{G}_{\min}^{\lambda-s}$) to show that $a_j(\lambda-s)=0$ is a BRCA for j in $\mathcal{G}_{\min}^{\lambda-s}$ (see Figure 7):

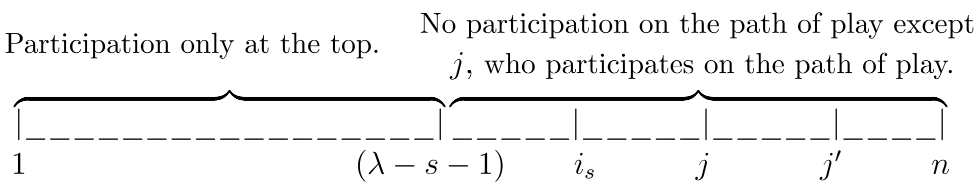

Figure 7. An Illustration of the Strategy Profile a for **Inductive Step**.

- Let agent j use $a_j \in \mathcal{G}_{\min}^{\lambda-s}$ with $a_j(\lambda-s)=1$ (see (12)).

- Let all agents j'', who do not cover j (i.e., $\kappa(j'')<j$), use the strategies in the corresponding exact contingency a^*_{-j} so that the $(\lambda-s)^{\text{th}}$ coordinate of j's strategy is on the path of play. (This includes all agents moving before j and possibly some agents moving after j. Notice that $\mathcal{NP}(\lambda-s)\neq\varnothing$ in $\mathcal{G}_{\min}^{\lambda-s}$ and the exact contingency Lemma 3 imply that this is possible.)

- Let all agents j' with $j'>j$ and $\kappa(j')\geq j$ use $a_{j'} \in \mathcal{G}_{\min}^{\lambda-s}$ such that $a_{j'}(\lambda-s+1)=0$. (Recall that our above arguments for j' imply that this is possible.)

Notice that by construction, there are (exactly) $(\lambda-s)$ agents participating on the path of play (i.e., $\sum_N b_i(a)=\lambda-s$). In particular, only agent j and the first $(\lambda-s-1)$ agents moving before j participate and no agent moving after j participates. In addition, for each agent j' that covers j, the $(\lambda-s+1)^{\text{th}}$ coordinate of j' is on the path of play. Since $\sum_N b_i(a)=\lambda-s\leq\lambda-2$, we have that $a_j(\lambda-s)=0$ is a BRCA for j in $\mathcal{G}_{\min}^{\lambda-s}$. In addition, under exactly the same strategy profile a, $a_{j'}(\lambda-s+1)=0$ is a BRCA for all j' with $j'>j$ and $\kappa(j')\geq j$. If $\mathcal{G}_{\min}^{\lambda-s}=\mathcal{G}_{\max}^{\lambda-s}$, our proof is complete. If not, the persistence Lemma 1 implies that $a_{j'}(\lambda-s+1)=0$ persists and is possible in the next game after $\mathcal{G}_{\min}^{\lambda-s}$ which we denote by $\mathcal{G}_{\min+1}^{\lambda-s}$. Hence, if $\mathcal{G}_{\min+1}^{\lambda-s}$ exists, we have the following:

$$\text{if } j'>i_s \text{ and } \kappa(j')\geq(\lambda-s), \text{ then there is } a_{j'}\in\mathcal{G}_{\min+1}^{\lambda-s} \text{ with } a_{j'}(\lambda-s+1)=0. \quad (13)$$

Using an argument similar to the above used for $\mathcal{G}_{\min}^{\lambda-s}$, we can show that $a_j(\lambda-s)=0$ is a BRCA in $\mathcal{G}_{\min+1}^{\lambda-s}$ for any agent $j>i_{s+1}$. The persistence Lemma 1 and the repeated use of this argument establish **Inductive Step** for all games in $\{\mathcal{G}_i^{\lambda-s}\}$. ■

Proof of Theorem 2. Consider the canonical sequence of agents given by the ordered set $(i_1,i_2,\ldots,i_{m^*})$ where $i_1=n$, $i_2=\kappa(i_1)=\kappa^1(n)$, $\cdots$, and $\kappa(i_{m^*})=\kappa^{m^*-1}(n)=0$. By Proposition 3, m^* is the maximum length of an information chain in $\mathcal{G}_0$. The sufficiency Lemma 5 already establishes that $m^*\geq\lambda$ implies that $\mathcal{G}_0$ is dominance solvable. Hence we only need to show that if $\mathcal{G}_0$ is dominance solvable then we have $m^*\geq\lambda$.

In $\mathcal{G}_0$, choose any $j\in N$ and any coordinate $l\leq\kappa(j)+1$ in j's strategy with $l\leq\lambda-1$. Since all strategies are possible in $\mathcal{G}_0$, we can construct an exact contingency such that the path of play passes through the l^{th} coordinate of j's strategy. Hence, $a_j(l) = 0$ is a BRCA. Using the persistence Lemma 1 (i) it follows that $\mathcal{G}_1$ satisfies $\mathcal{NP}(\lambda - 1) \neq \varnothing$.

By the maximal reduction Lemma 4, as we proceed along the sequence $\mathcal{G}_1, \ldots, \mathcal{G}_M$, we will encounter (sequentially) the $(\lambda - 1)$ non-empty blocks $\{\mathcal{G}_i^{\lambda-k}\}_{\lambda-1 \geq k \geq 1}$ where in the block $\{\mathcal{G}_i^{\lambda-k}\}$ all agents have strategies in which all the coordinates less than or equal to the $(\lambda - k)^{\text{th}}$ coordinate are zero and in the game $\mathcal{G}_{\max}^{\lambda-k}$ the $(\lambda - k)^{\text{th}}$ coordinate of some individual is reduced to 1 and becomes fixed at 1 in all subsequent games.

Consider the first block $\{\mathcal{G}_i^{\lambda-k}\}$ with $k = \lambda - 1$.[33] By definition, some agent's $(\lambda - 1)^{\text{th}}$ coordinate is reduced to 1 in game $\mathcal{G}_{\max}^{1}$. There must be some individual j' such that $\kappa(j') \geq \lambda - 1$. Thus, noticing that the hypothesis of Lemma 6 is non-vacuously satisfied, using the persistence Lemma 1, we can conclude that $j' \leq i_2$.

Notice that the existence of these $(\lambda - 2)$ more of such blocks implies that the hypothesis of the non-reduction Lemma 6 is non-vacuously satisfied in each step of the reduction process and that in the game $\mathcal{G}_{\max}^{\lambda-s}$ for $\lambda - 1 \geq s \geq 2$, the $(\lambda - s)^{\text{th}}$ coordinate can be on the path of play and reduces to 1 in $\mathcal{G}_{\max}^{\lambda-h}$ **only** for some individual $j \leq i_s$ where $\kappa(j) \geq \lambda - s - 1$.[34] Hence, for $\mathcal{NP}(1) = \varnothing$ to be true (i.e., the $(\lambda - \lambda + 1)^{\text{th}} = 1^{\text{st}}$ coordinate of some agent's strategy to be reduced to 1), we must have an agent $j \leq i_\lambda$ with $\kappa(j) \geq 0$. Accordingly, we have that $\mathcal{G}_0$ being dominance solvable implies the existence of the canonical sequence $i_1, \ldots, i_{m^*}$ of length at least equal to λ. ■

Proof of Proposition 4. We proceed with our analysis using a procedure similar to backward induction, focusing mainly on the agents in the canonical information chain $\{i_1, \ldots, i_m\}$ with $i_1 = n, \ldots, i_m = \kappa^{m-1}(n)$.

Given that $m \geq \lambda$, we have $\kappa(n) \geq \lambda - 1$, i.e., $i_1 = n$ has at least λ information sets. Agent n's beliefs in each of her information sets are on the participation history of all agents i such that $i_1 > i > i_2$.[35] For n's information sets $H_n(l)$ with $l \geq \lambda - 1$, it is a dominant strategy for n to participate regardless of n's beliefs in $H_n(l)$. For any of n's other information sets, n's optimal strategy depends on n's beliefs of the participation history of i such that $i_1 > i > i_2$.

Next consider agent $i_2 = \kappa(n)$. The canonical information chain implies that $\kappa(i_2) \geq \lambda - 2$. Given agent n's optimal play, for i_2's information sets $H_{i_2}(l)$ with $l \geq \lambda - 2$, it is a dominant strategy for i_2 to participate regardless of i_2's beliefs. And i_2's optimal strategy in her other information sets again depend on her beliefs of the participation history of all agents i such that $i_2 > i > i_3$.

[33] Since $\mathcal{G}_1$ belongs to this block, it is nonempty.

[34] Observe that for the l^{th} coordinate of an agent j to be reduced to 1, it must be the case that $\kappa(j) \geq l - 1$. This implies that when we have dominance solvability, the hypothesis of the non-reduction Lemma 6 is true for each of the blocks $\{\mathcal{G}_i^{\lambda-k}\}_{\lambda-1 \geq k \geq 1}$.

[35] Notice that agent i_1 observes the aggregate participation level among all agents in $\{1, \ldots, i_2\}$, but not the participation outcome of i for $i_1 > i > i_2$. Given that we only focus on payoff-relevant information, there is no uncertainty about the participation history of agents in $\{1, \ldots, i_2 - 1\}$—recall the $\mathcal{I}$-monotonicity Assumption 3. Hence, agent i_1's beliefs in her information sets should only be on the unobservable choices of i for $i_1 > i > i_2$.

Applying a similar arguments as those for i_1 and i_2 repeatedly for the remaining agents in $\{i_1, \ldots, i_m\}$, we have that agent i_m who has only one information set—by definition, $\{i_1, \ldots, i_m\}$ is the longest canonical information chain and hence $\kappa(i_m) = 0$—participates with probability 1 in any perfect Bayesian equilibrium.

Given the optimal strategies of agents in $\{i_1, \ldots, i_m\}$, we have that all the m agents in the chain participate on the equilibrium path in every perfect Bayesian equilibrium. Given that $m \geq \lambda$, this further implies that in every perfect Bayesian equilibrium, there are at least λ agents participating on the equilibrium path (regardless of what the other agents' strategies are). Given the payoff specifications, it follows that all the other agents participate in every perfect Bayesian equilibrium as well, i.e., there is a unique perfect Bayesian equilibrium outcome where everyone participates. ■

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