Elimination Contests with Collusive Team Players*

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Abstract

We consider a standard two-stage elimination (Tullock) contest where multiple (team) players can perfectly and publicly collude with each other throughout. We analyze and compare equilibrium outcomes under various seedings where the collusive players meet or are separated in the group stage. We identify the impact of collusion on the contest organizer and non-collusive players, as well as the organizer's optimal seeding. We find that collusion, while always undermining fairness of the competition, can hurt or benefit the organizer, depending on the discriminatory powers of the two stages. We also discuss issues such as sequential group-stage competitions, comparison between the elimination contest and the corresponding one-shot contest, secret collusion, and large discriminatory powers.

Keywords: Collusion, Elimination Contest, Seeding, Team, Tullock Contest.

JEL Classification: C72, D44, D72, D74.

1 Introduction

Collusion in sports, or match fixing to restrict competition, has been a long-standing and widespread problem in professional sports. Indeed, match fixing events, which undermine the integrity and fairness of sports, have occurred in various professional sports in the past.[1] A famous incident, called the "disgrace of Gijón" in the 1982 FIFA World Cup, occurred in a group-stage football match in Gijón, Spain, where West Germany and

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[1]A Wikipedia page https://en.wikipedia.org/wiki/List_of_match-fixing_incidents lists numerous match-fixing incidents, ranging (alphabetically) from American football to volleyball.

Austria were accused of fixing the match so that both would progress to and face more ideal opponents in the next round, at the expense of Algeria.[2,3] Electronic sports (eSports), which are organized multi-player video game competitions, offer another setting where match-fixing problems are "incredibly widespread." Unlike traditional sports, eSports and its research (on proper training of eSports athletes and composition of teams) are still at an infant stage and professional eSports players are much less known or recognized, according to a 2018 Guardian article.[4] One particular phenomenon of eSports though, according to the Guardian article, is that its hyper-digital nature makes the athletes more integrated and enables them to easily communicate and coordinate their actions.

In this paper, we analyze collusion among players using a standard elimination contest with multiple stages. Elimination contests, while perhaps arising more prominently in traditional sports and eSports contests, have also been widely used in other real-life competitions such as internal job promotions in organizations, political competitions where candidates compete and exert efforts and resources in multiple stages, science or research competitions with a preliminary stage and a final stage, as illustrated in Moldovanu and Sela (2006)[20].

To be concrete, we consider a stylized four-player elimination contest, where the (ex ante identical) players are arranged into pairwise competitions in a group stage, and the two resulting group-stage winners then compete to win a prize in a final stage. The two stages are modeled as Tullock contests. While such two-stage elimination contests are familiar in the contest literature, our setting differs from previous research in two respects. First, we allow for the discriminatory powers of the two stages to be *different*. Such a flexible feature captures real-life elimination contests where competition can vary from one round to another, such as the format of the competition, the winning criterion of each round, the judging panel of each round, etc.[5] Second, there is explicit collusion

[2] After 1982, FIFA changed the World Cup group stage format so that all the teams in a group would play final matches simultaneously. We discuss the impact of making the group stage competitions sequential rather than simultaneous in Section 4.1.

[3] Match fixing incidents have also occurred in badminton competitions in the Summer Olympics, where players from the same country colluded in semi-finals to increase the probability of winning the gold medal in finals. See https://en.wikipedia.org/wiki/List_of_match-fixing_incidents#Badminton.

[4] See https://www.theguardian.com/games/2018/jul/31/its-incredibly-widespread-why-esports-has-a-match-fixing-problem.

[5] For example, the format of the postseason of a sports league (such as NBA, NFL, NCAA) is typically different from its previous regular season. In addition, popular TV shows such as American Idol and Golden Balls also exhibit different rules or formats in different rounds. We **emphasize**, however, that for our analysis to be applicable, such changes in competition or discriminatory powers should affect symmetric players **symmetrically**, rather than reflect bias against some player, so as to be consistent with the interpretation of discriminatory power in the literature.

among a set of players in our contest. Formally, it is publicly known that two players out of the four, called *collusive agents*, are team players who can coordinate their play and aim to maximize their joint payoff throughout the contest.

The existence of public collusion introduces (hidden) heterogeneity into our otherwise symmetric setting and also prompts the remaining players to strategically respond to it. Consequently, how to seed the four players in the group stage has direct implications on their subsequent interactions. The majority of our analysis focuses on investigating equilibrium outcomes and the impact of collusion under two group-stage seedings, one where the collusive agents meet in the group stage, and the other where the collusive agents are separated in the group stage. More explicitly, our analysis answers the following questions: If the seeding of the group stage can be arranged, should the organizer separate the collusive agents or let them compete against each other in the group stage? How does each arrangement affect the organizer's objective (i.e., total expected effort or bid from the players)? How does each arrangement affect the non-collusive players?

The equilibrium for the seeding where the two collusive agents meet in the group stage is relatively simple. The agents collude in the group stage by exerting zero effort and the final features exactly one agent and one non-collusive player. Under this seeding, while the outcome is unfair in that the agents exert no effort in the group stage and the organizer suffers with a revenue loss, the non-collusive players' ex ante winning probability and expected payoffs are not affected by collusion.

For the seeding where the agents are separated in the group stage, however, collusion will affect both the organizer's revenue and the non-collusive players' winning probability and welfare. Intuitively, optimal collusion between the agents entails that the agents bid aggressively in the group stage, in hopes of both entering the final to secure winning *and* to save final-stage bidding cost. This incentive discourages the non-collusive players, who then bid less in the group stage compared to the corresponding benchmark with no collusion, distorting their winning probability and their expected payoffs downwards. The impact of collusion on the organizer is however less clear: while collusion may boost revenue in the group stage, the organizer suffers with a revenue loss when the two collusive agents meet in the final.

A detailed comparison of the two seedings reveals that the organizer's optimal seeding hinges crucially on the discriminatory powers of the two Tullock contests. Roughly, if the discriminatory powers are both relatively small (i.e., if luck plays significant roles in determining winning), then separating the agents in the group stage is optimal as the revenue increase in the group stage dominates the revenue loss in the final. If however the discriminatory powers, especially that of the final stage, are large, then the revenue loss in the final stage is too significant, making the seeding of the agents meeting in the group stage optimal. Indeed, if the discriminatory power of the final stage is sufficiently large, the organizer's expected revenue can go to zero, i.e., the non-collusive players are so discouraged (by the agents' collusive advantage from coordinated play in the group stage and hefty bidding cost in the final) that they essentially bid zero in the group stage, enabling the collusive agents to win almost with certainty at minimal cost.

Another interesting implication of the above discouragement effect is that under the seeding where the agents are separated, if the group-stage discriminatory power is sufficiently large and that of the final-stage is sufficiently small, the aggressive bidding by the collusive agents actually enables the organizer to receive an expected revenue strictly higher than that from the non-collusion benchmark. In this sense, the organizer can, somewhat counter-intuitively, benefit from collusion, if the group-stage bid is much more important than the final for the organizer.

Finally, to further explore the implications of collusion, we consider several related issues in our extensions in Section 4. First, when the group-stage competitions are sequential rather than simultaneous, we show that under the seeding where the agents are separated, the additional information generated from the first group competition enables the collusive agents to better coordinate their group-stage bids and benefit *further* from collusion. Second, we also compare the performance of the elimination contest with that of a corresponding one-shot contest in the presence of collusion, which is a standard exercise in the literature (e.g., Gradstein and Konrad 1999[13] and Fu and Lu 2012[10]). Noteworthy results here are that collusion may actually hurt the collusive agents due to strategic responses from the other players and that seeding and heterogeneity generated by collusion actually provide the organizer with flexibility, which modifies the threshold of the discriminatory power in the comparison between the two contest formats. Third, we consider a setting where the collusion of the agents is secret rather than public, which benefits the organizer but further impairs the non-collusive players' welfare. Finally, we also discuss equilibrium characterization in our setting when the discriminatory powers of the two stages can be large.

1.1 Literature

Our paper is connected to several strands of contest literature. First, our paper relates to the extensive literature on elimination contests with various hierarchical structures. Rosen (1986)[24], one of the earliest in this strand, analyzes the optimal prize structure in a fixed multi-stage elimination tournament, and finds it optimal to allocate disproportionate weight to top-ranking prizes to maintain incentives. More recent studies have also endogenized the hierarchical structure of elimination contests, making it a contest organizer's choice variable. Gradstein and Konrad (1999)[13] considers a multi-stage (Tullock) contest where in each stage, identical players are divided into groups, only the winner of each group advances to the next stage, and the organizer chooses group sizes and the number of stages. It is shown that the effort-maximizing structure is either the pairwise multi-stage structure (if the contest rule is not so discriminatory) or otherwise the one-shot simultaneous structure. Fu and Lu (2012)[10] considers a more general multi-stage Tullock elimination contest in which the organizer chooses both the hierarchical structure and the prize allocation. They show that when the contest technology is sufficiently noisy, the optimal contest features a winner-takes-all prize and has as many stages as possible (i.e., exactly one contestant is eliminated in each stage). Compared to this literature, our contest is simpler, but we consider the novel issue of collusion in elimination contests and analyze its impact. Our setting also features asymmetric players due to collusion.[6] We will relate to this literature further in Section 4.2.

Similar to our setting, several studies have used two-stage elimination contests to analyze specific issues. Amegashie (1999)[2] compares a one-stage contest with a two-stage contest where multiple finalists are selected in the first stage, followed by a final stage selecting a winner, and finds that the one-stage contest leads to a higher total effort (or more wasteful rent-seeking expenditure). Cohen, Maor and Sela (2018)[5] studies a two-stage elimination contest where two finalists are first selected from an n-player Tullock contest and then compete in a final Tullock contest, and finds that favoring the finalist who is top ranked in the first stage in the final always increases expected total effort. Closer to our analysis on optimal seeding is Groh et al. (2012)[14], which considers a two-stage all-pay elimination contest with four asymmetric players. The authors investigate optimal seeding rules (according to various criteria), which sort the players into two groups for elimination in the first stage, and find that seedings that delay encounter of strong players are optimal for various criteria.[7]

[6] There is also an extensive literature on Tullock contests with asymmetric valuations or endogenous network structure (see, e.g., Nti (1999)[23], Feng and Lu (2017)[12], Huremovic (2021)[16], and references therein). Asymmetry in our setting however is different and is due to collusion, which leads to different payoff functions and strategy sets among the players.

[7] There are also studies on two-stage elimination all-pay contests. Moldovanu and Sela (2006)[20] provides a comparison between a two-stage all-pay contest and a static contest with incomplete information. Zhang and Wang (2009)[28] studies optimal disclosure policies in a two-stage elimination all-pay contest with private contestants' prize valuations.

Finally, our paper is also related to the literature of (static) contests with team players or alliances, such as Nitzan (1991)[22], Skaperdas (1998)[26], Esteban and Ray (2001)[6], Münster (2007)[21], Konrad and Kovenock (2009)[18], Fu, Lu and Pan (2015)[11], and Barbieri and Topolyan (2021)[3] among many others.[8] Our setting differs in that our collusive players are exogenously specified and without individual incentives. While endogenous team formation and individual players' incentives can be readily incorporated in our study, we have chosen to omit such generalizations so as to focus on the designing issues for the contest organizer when facing such team players.

2 The Model

Consider a contest where four (risk-neutral) players, $i = 1, \ldots, 4$, compete for a single prize. The contest is organized as an elimination tournament with two stages: In the group stage, the players are divided into two pairs to compete, producing two finalists, one from each pair, with the losing players in the group stage being eliminated. The two finalists then compete in the final stage and the winner wins the prize.

The players all have an identical and publicly known valuation of the prize, $v > 0$. It is publicly known that player 1 and player 2 are team players (e.g., they are players from the same group/country) and can perfectly coordinate their play if desired. To differentiate the players (except in general discussions of a generic player), we call the colluding players 1 and 2 as *agents*, and players 3 and 4 as (non-colluding) *players* hereafter.

Each player's payoff is defined as the expected payoff from winning the prize net of his bid. Specifically, player i's expected payoff $u_i^\ell(b_i, b_j)$ when playing against player j in stage $\ell \in \{1, 2\}$, given a bid profile (b_i, b_j), is:

$$u_i^\ell(b_i, b_j) = \frac{b_i^{r_\ell}}{b_i^{r_\ell} + b_j^{r_\ell}} K_\ell - b_i, \tag{1}$$

where K_ℓ is the (expected if $\ell = 1$) prize awarded in the stage and r_ℓ is the stage-ℓ discriminatory power, which measures the effectiveness of players' efforts (or bids here) in affecting their winning probabilities in stage ℓ. If both players bid zero, the winner is then chosen uniformly at random. Hence, we have adopted the celebrated Tullock contest success function throughout, which has been widely employed in the literature and has

[8] Boudreau et al. (2019)[4] studies alliance formation and collusion in conflicts where collusion is achieved in repeated competitions. Our paper also shares some similarity with Huck et al. (2002)[15], which studies cooperation of a subgroup of contestants and its profitability, though our focus is on cooperation in a multi-stage elimination contest.

been put on a firm axiomatic footing by Skaperdas (1996)[25].[9]

Finally, to focus on the analysis of collusion and its effects, we assume that the hierarchical structure of the elimination contest is fixed and is not part of the organizer's decision, perhaps due to historical or logistic considerations (e.g., as in various sports events).[10] The organizer however can choose a particular seeding on how the four players are arranged into pairwise matches and whether the two pairwise matches in the first stage should take place simultaneously or sequentially, which we analyze in Section 4.1. The organizer's objective is to use such instruments to maximize the total expected revenue induced from all four players. A non-collusive player maximizes his *individual* expected winning prize net of his bid(s), while the two colluding agents coordinate their bids perfectly throughout to maximize their *joint* expected payoffs. We adopt the natural solution concept of subgame perfect equilibrium (SPE) in the sequel.

Remark 1 *In our four-player elimination contest, there are only* ***three*** *decision makers, the two non-collusive players and the team. Subgame perfection here implies that the choices of relevant players among these three decision makers form a Nash equilibrium in every subgame. Notice that while each non-collusive player makes a one-dimensional choice in a relevant subgame, the team makes a* ***two-dimensional choice*** *in each subgame where both agents are present. The main technical difficulty in our analysis lies in characterizing the team's equilibrium strategies in the elimination contest. As we will see, while it is still manageable to characterize pure-strategy subgame perfect equilibria of the game, constructing a mixed-strategy equilibrium where one or both collusive agents randomize is complicated in that the mixed-strategy space of the team is too big to manage. In particular, the collusive agents' randomizations can be* ***correlated arbitrarily****. We will come back to this complication when we discuss the elimination contest with large discriminatory powers* (r_1, r_2) *in Section 4.4.*

[9] As is common in Tullock contests, the marginal cost of each player's effort here is 1 and our results will be qualitatively the same for any constant (positive) marginal cost.

[10] Our elimination contest certainly differs from the optimal sequential elimination contest characterized in Fu and Lu (2012)[10] based on the instruments of contest sequence and prize allocation. Observe, however, that our setting is also different in that (1) there are colluding agents here and (2) we allow for heterogeneous discriminatory powers across the two stages ($r_1 \neq r_2$).

3 Results

3.1 The Non-Collusive Benchmark

As a starting point, we consider here the scenario where no player colludes with each other in the competition. The scenario will serve as a useful benchmark in comparison to the collusive setting in later sections.

First, as is common in the literature, we impose the following restriction on r_ℓ:

Assumption 1 $r_1, r_2 \in (0, 2)$.

Assumption 1 guarantees the existence of a pure-strategy equilibrium in the non-collusive benchmark. Assumption 1 differs from the standard requirement in a one-shot n-player Tullock contest where the discriminatory power is less than $n/(n-1)$ (Konrad 2009[17]). The difference is driven by that we have a dynamic elimination contest where each player faces only one opponent in each game, abating competition to some degree, and we also allow for different discriminatory powers across stages.

Proposition 1 characterizes the symmetric equilibrium in the non-collusive benchmark. We adopt the convention hereafter that "b" denotes a group-stage bid while "B" denotes a final-stage bid.

Proposition 1 (Non-Collusive Elimination Benchmark) *Suppose Assumption 1 holds. In the elimination contest with non-colluding players, there is a unique symmetric SPE with the following bids b in the group stage and B in the final*

$$b = \frac{r_1(2-r_2)}{16}v, \quad B = \frac{r_2}{4}v. \tag{2}$$

The expected payoffs of each player (U_N) and the organizer (Π_N) are:

$$U_N = \frac{(2-r_1)(2-r_2)}{16}v, \quad \Pi_N = \frac{r_1(2-r_2)}{4}v + \frac{r_2}{2}v. \tag{3}$$

Hence, all players bid identically in each stage in equilibrium, and each player hence has an (ex ante) equal chance (1/4) of winning the prize v. In addition, since r_ℓ measures the competitiveness of stage ℓ, each player's equilibrium payoff is decreasing in r_1 and r_2, while the organizer's payoff is increasing in r_1 and r_2. Notice that Assumption 1 ensures each player's equilibrium payoff to be non-negative.

We also present a standard lemma on the corresponding one-shot Tullock contest:[11]

[11]Since Lemma 1 is standard, its proof is omitted.

Lemma 1 (Non-Collusive One-shot Benchmark) *In a corresponding one-shot Tullock contest with four identical non-colluding players and a discriminatory power* $r \leq \frac{4}{3}$*, there is a unique symmetric Nash equilibrium with bid* $b = \frac{3r}{16}v$ *and expected payoffs of* $\Pi_O = \frac{3r}{4}v$ *for the organizer and* $U_O = \frac{4-3r}{16}v$ *for each player.*

If $r_1 = r_2 = r \in \left(0, \frac{4}{3}\right]$, Proposition 1 and Lemma 1 imply that if both contests are logistically feasible, the organizer strictly prefers the one-shot Tullock contest (resp., the elimination contest) if and only if $r > 1$ (resp., $r < 1$), and is indifferent between the two when $r = 1$. This is consistent with the previous literature (see, e.g., Gradstein and Konrad (1999)[13]).

Notice that for an elimination contest with identical and non-colluding players, how to arrange the four players in the group stage (and whether the group stage is conducted sequentially or not) has no effect on the players' equilibrium bidding choices. With colluding agents (and in general with heterogeneous players), however, how to seed the players in the group stage will affect bidding incentives of all the players, as we now show.

3.2 Collusive Equilibrium under Various Seedings

We now analyze how to optimally organize competitions in the group stage when there are collusive team players for the organizer.[12] Recall that the non-colluding players are "players" and the colluding players are "agents," who can coordinate their strategies and maximize their joint payoff in the elimination contest.

To begin with, collusion in our context introduces two technical complications in our analysis. First, although the players are ex ante identical, collusion introduces hidden heterogeneity among the players. Such heterogeneity is reflected in the collusive agents' strategies, which are not chosen independently by the two agents, and in the collusive agents' payoff functions, which is the agents' joint or team payoff. And it is well known in the Tullock literature that heterogeneity can substantially complicate equilibrium analysis. The second technical complication comes from the fact that the agents will optimally choose their bidding strategies in coordination, not just in a particular competition but throughout the entire elimination contest. As a result, one has to consider a strictly larger set of strategies, which further complicates the equilibrium analysis.

[12]Since our setting only has two stages, there is nothing the organizer can do after two finalists are determined. In a more general elimination contest with more players and more stages, the organizer can adjust seedings in all stages except the final stage. Our analysis here, though not directly applicable, can still offer useful insights for such more general elimination contests with collusion concerns.

There are two ways to arrange the two pairwise competitions in the group stage. The first is to assign the two collusive agents to compete against each other, which we denote as seeding $\{(1,2),(3,4)\}$, and the second seeding is denoted as $\{(1,3),(2,4)\}$ where the collusive agents are separated from competing against each other in the group stage.

Consider first the relatively simple seeding $\{(1,2),(3,4)\}$. Since only one agent will come out of the group stage, competition in the final stage, where an agent competes with a player, is similar to that in Proposition 1, with an expected equilibrium payoff of $K_1 = \frac{2-r_2}{4}v$ in the final stage. The agents' problem in the group stage is hence

$$\max_{b_1,b_2} K_1 - b_1 - b_2.$$

The agents then optimally bid $b_1 = b_2 = 0$, with an ex ante team expected payoff of $U_T = K_1$. Proposition 2 summarizes this equilibrium:

Proposition 2 *Suppose Assumption 1 holds. In the elimination contest with group-stage seeding* $\{(1,2),(3,4)\}$*, there is a unique pure-strategy SPE where*

$$\text{For } i \in \{1,2\}, \ j \in \{3,4\}, \ b_i = 0, \ b_j = \frac{r_1(2-r_2)}{16}v$$
$$B_i = B_j = \frac{r_2}{4}v \text{ for finalists agent } i \text{ and player } j$$

The equilibrium revenue for the organizer is

$$\Pi_S = \frac{r_1(2-r_2)}{8}v + \frac{r_2}{2}v.$$

Next, consider the seeding $\{(1,3),(2,4)\}$. This seeding creates more complexity because the collusive agents need to more carefully coordinate their group-stage bids. Intuitively, the agents want to maximize the probability of both agents entering the final stage, at which point, the agents can guarantee winning the prize and saving bidding cost in the final. Such an incentive changes the bidding behavior of the non-collusive players as well. Since either both agents enter the final and both bid 0 to win the prize, or exactly one agent enters the final and competes with a non-collusive player, or no agent reaches the final, the agents' maximization problem in the group stage is:

$$\max_{b_1,b_2} \frac{b_1^{r_1}}{b_1^{r_1}+b_3^{r_1}} \frac{b_2^{r_1}}{b_2^{r_1}+b_4^{r_1}} v + \frac{b_1^{r_1}b_4^{r_1}+b_2^{r_1}b_3^{r_1}}{\left(b_1^{r_1}+b_3^{r_1}\right)\left(b_2^{r_1}+b_4^{r_1}\right)} K_1 - b_1 - b_2,$$

where b_j, $j \in \{3,4\}$, is a non-collusive player's group-stage bid, b_1 and b_2 are the agents' group-stage bids, and $K_1 = \frac{2-r_2}{4}v$.[13]

One issue created by collusion with seeding $\{(1,3),(2,4)\}$ is that since the collusive agents will choose perfectly coordinated bids, the non-collusive players are actually playing against the same coalition in the group stage. An analysis of pure-strategy equilibrium now requires an additional restriction (Assumption 2) on the discriminatory parameters r_1 and r_2, in response to such a "change" in the number of players.

Assumption 2 *For any $r_1 > 1$, r_2 satisfies*

$$\frac{2r_2}{2-r_2} \leq \frac{r_1}{(r_1-1)^{\frac{1}{r_1}}} - r_1.$$

Assumption 2, imposed to guarantee the existence of a pure-strategy equilibrium in the elimination contest, requires r_2 to be not too large given a specific $r_1 > 1$. Technically, this assumption ensures the expected equilibrium payoff of a non-colluding player to be non-negative. To see this intuitively, as r_2 becomes larger, the expected prize K_1 decreases and gets closer to 0. The agents will then bid more aggressively in the group stage so as to avoid a rising bid cost in the final ($r_2 v/4$) when facing a non-collusive player. As a result, the non-colluding players will have no incentive to bid a positive amount if r_1 is also large, ruling out a pure-strategy equilibrium with positive bids.[14]

Proposition 3 characterizes the equilibrium for the seeding $\{(1,3),(2,4)\}$.

Proposition 3 *Suppose Assumptions 1 and 2 hold. In the elimination contest with group-stage seeding $\{(1,3),(2,4)\}$, there is a unique pure-strategy SPE where*

$$\begin{aligned} \textit{For } i &\in \{1,2\},\ j \in \{3,4\},\ b_i = \frac{r_1\lambda^{r_1-1}}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}v,\ b_j = \frac{r_1\lambda^{r_1}}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}v, \\ B_i &= \frac{r_2}{4}v \textit{ against a player, and } 0 \textit{ against the other agent, } B_j = \frac{r_2}{4}v \end{aligned}$$

and the organizer's equilibrium revenue is

$$\Pi_D = \frac{r_1(2-r_2)\lambda^{r_1-1}(1+\lambda)}{2(\lambda^{r_1}+1)^2}v + \left(1 - \frac{1}{(\lambda^{r_1}+1)^2}\right)\frac{r_2}{2}v,$$

[13] We will show in the proof of Proposition 3 that the agents actually bid the same amount in the group stage (and so do the players) so as to maximize their joint probability of entering the final.

[14] The necessity of Assumption 2 is formally established in Appendix, in Proof of Proposition 3.

where $\lambda := b_j/b_i$*, the ratio of group-stage bids of a player and an agent, solves*

$$\lambda^{r_1+1} - \lambda^{r_1} + \frac{2+r_2}{2-r_2}\lambda - 1 = 0.$$

In addition, we have $\lambda < 1$, $\frac{\partial\lambda}{\partial r_1} < 0$, *and* $\frac{\partial\lambda}{\partial r_2} < 0$.

Hence, in equilibrium, both agents coordinate on an identical group-stage bid $b_1 = b_2$ and in particular, each collusive agent bids strictly more than a player in the group stage ($\lambda < 1$), reflecting the collusive agents' incentives to bid aggressively in order to settle the final between themselves. In addition, the agents bid more aggressively (relative to the players) in the group stage if winning the group stage is "harder" (higher r_1) or winning the final stage is "harder" (higher r_2), both resulting from the collusive agents' purpose of securing winning the prize at minimal cost in the final stage.

3.3 Impact of Collusion

Our equilibrium characterization of various seedings in Propositions 2 and 3 makes it clear that collusion impacts both the organizer and the non-collusive players, relative to the equilibrium in the benchmark (Proposition 1). We now analyze such impacts in detail.

First, consider the organizer. Under seeding $\{(1,2),(3,4)\}$, exactly one agent will enter the final and the only negative impact of collusion to the organizer is the loss of group-stage bids from the agents. Under seeding $\{(1,3),(2,4)\}$, the organizer can potentially benefit from more aggressive bidding from the agents in the group stage, but will also lose bids from the final stage stochastically (when both agents are finalists).

The following Proposition 4 characterizes the organizer's optimal seeding:

Proposition 4 (Optimal Seeding for the Organizer) *In the elimination contest satisfying Assumptions 1 and 2, for each* $r_1 \in (0,2)$*, there is* $\bar{r}_2$ *such that assigning the agents to different pairwise competitions in the group stage, i.e., seeding* $\{(1,3),(2,4)\}$*, is optimal for all* $r_2 \in (0,\bar{r}_2)$*, while grouping the agents together, i.e., seeding* $\{(1,2),(3,4)\}$*, is optimal for all* $r_2 \in (\bar{r}_2, 2)$.

Proposition 4 implies that the organizer's optimal seeding depends crucially on the relative magnitudes of r_1 and r_2. Figure 1 plots the organizer's expected revenues from the two seedings and in the benchmark of Proposition 1.[15] Graphically, fixing r_1, the organizer's expected revenues from the two seedings cross exactly once in the interior of

[15]Notice that Proposition 4 holds under Assumption 2, which also imposes restrictions on the set of

the range for r_2. This result is intuitive: For a given r_1, a relatively small r_2 implies that the revenue generated in the final is less important and hence motivating incentives to bid aggressively in the group stage dominates, making the seeding of $\{(1,3),(2,4)\}$ optimal. On the other hand, a relatively big r_2 suggests that bids from the final are more important. The organizer then should prevent the collusive agents from meeting in the final, making $\{(1,2),(3,4)\}$ a preferred seeding.

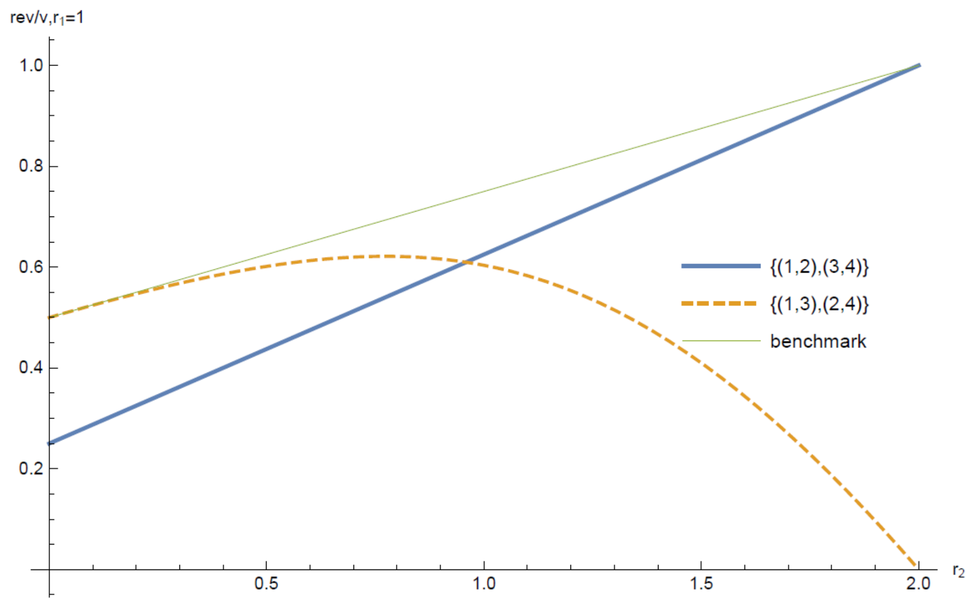

Figure 1: The Organizer's Optimal Seeding ($r_1 = 1$).

We next consider the impact of collusion on the organizer's expected revenue:

Proposition 5 (Collusion May Benefit the Organizer) *If $r_1 \leq 1$, the collusive elimination contest with either seeding generates a weakly lower revenue than the non-colluding benchmark. But for each $r_1 > 1$, there is a set of r_2 (near the neighborhood of 0) such that the collusive elimination contest with seeding $\{(1,3),(2,4)\}$ generates a strictly higher revenue than the non-colluding benchmark.*

Proposition 5 shows that collusion, while intrinsically bad for competitions, can paradoxically *benefit the organizer strictly* when r_1 is large and r_2 is small. Figure 2 illustrates a numerical example: when $r_1 = 1.5$, the revenue of the organizer with seeding $\{(1,3),(2,4)\}$ is strictly higher than in benchmark if $r_2 \in (0, 0.33)$. The rationale of such a counter-intuitive result is that when the collusive agents are separated in the group stage, they

admissible (r_1, r_2). We show in the proof of Proposition 4 that if r_1 is sufficiently larger than 1, the cutoff $\bar{r}_2$ will lie above all admissible r_2 under Assumption 2. Hence, whenever $r_1 > 1$ is sufficiently large, $\{(1,3),(2,4)\}$ is the organizer's optimal seeding for all r_2 satisfying Assumption 2.

bid more aggressively (than they do in the benchmark) in the hope to meet and save bid cost in the final. Such aggressive bidding is particularly beneficial to the organizer when bids from the group stage are important and loss from the final is minor to the organizer, i.e., when r_1 is large and r_2 is small. Notice that such a result cannot arise when the two stages have the same discriminatory power $r_1 = r_2$.

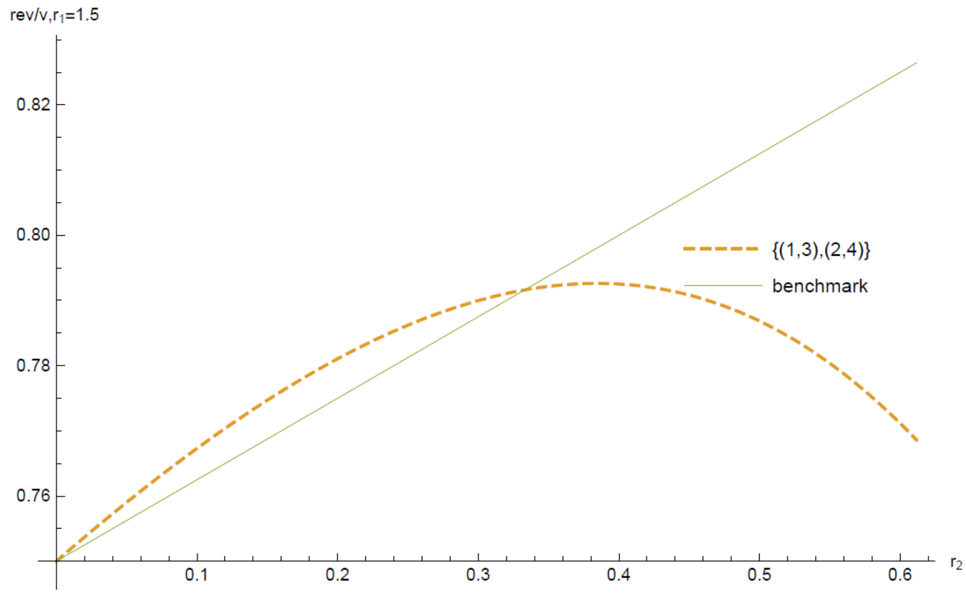

Figure 2: Collusion can Benefit the Organizer ($r_1 = 1.5$).

Proposition 6 summarizes the impact of collusion on the non-colluding players:

Proposition 6 (Impact of Collusion on Players) *Relative to the non-colluding benchmark, the elimination contest with seeding* $\{(1,2),(3,4)\}$ *does not affect the players' winning probability and expected payoffs, while the elimination contest with seeding* $\{(1,3),(2,4)\}$ *strictly decreases the players' winning probability and expected payoffs.*

Hence, unlike the organizer, the non-colluding players can never benefit from the agents' collusion. Under seeding $\{(1,2),(3,4)\}$, although each of the players still wins with probability $1/4$ and receives the same expected payoff as in the benchmark, this is achieved by bidding more than the agents. On the other hand, under seeding $\{(1,3),(2,4)\}$, each non-colluding player obtains a strictly lower payoff and wins with a probability strictly less than $1/4$. As a result, fairness of the competition will not be achieved regardless of the seeding.

4 Discussion

4.1 Sequential Group Competitions

An interesting variant to the previous elimination contest is to let the group-stage competitions be conducted sequentially. In practice, group-stage competitions in major sports events can either take place sequentially or simultaneously.[16] Presumably, different rules are adopted for different sports in reality due to logistic (such as budget, venue, and time) constraints, or revenue considerations. We investigate, from an incentive point of view, the effects of conducting group-stage competitions sequentially.

First, it is easy to see that when the seeding is $\{(1,2),(3,4)\}$, making the group-stage competitions sequential does not alter the equilibrium outcome in Proposition 2, due to that it is publicly known that the finalists always consist of one agent and one player.

Now consider seeding $\{(1,3),(2,4)\}$, and without loss of generality, that the pair $(1,3)$ competes first. By observing the winner from $(1,3)$, agent 2 and player 4 obtain additional information and can condition their behavior on such information. Intuitively, if agent 1 loses against player 3, agent 2 will then choose her bid against player 4 as she does in the benchmark, which is different from agent 2's equilibrium choice in Proposition 3. Such a consideration alters the incentives of all four players. Formally, denote the group-stage bids in this sequential game as $\{\mathtt{b}_1, \mathtt{b}_3, \mathtt{b}_{2T}, \mathtt{b}_{2P}, \mathtt{b}_{4T}, \mathtt{b}_{4P}\}$, where $\mathtt{b}_{2T}$ and $\mathtt{b}_{4T}$ are respectively agent 2's bid and player 4's bid after agent 1 (i.e., the team) wins against player 3, while $\mathtt{b}_{2P}$ and $\mathtt{b}_{4P}$ are their bids after winning by player 3. The team's group-stage (sequential) maximization problem can be written as:

$$\max_{\mathtt{b}_1} \left\{ p_T K_1 + p_T \max_{\mathtt{b}_{2T}} \{ \frac{\mathtt{b}_{2T}{}^{r_1}}{\mathtt{b}_{2T}{}^{r_1} + \mathtt{b}_{4T}{}^{r_1}} \frac{2+r_2}{4} v - \mathtt{b}_{2T} \} + p_P \max_{\mathtt{b}_{2P}} \{ \frac{\mathtt{b}_{2P}{}^{r_1} K_1}{\mathtt{b}_{2P}{}^{r_1} + \mathtt{b}_{4P}{}^{r_1}} - \mathtt{b}_{2P} \} - \mathtt{b}_1 \right\},$$

where $p_T = \frac{\mathtt{b}_1{}^{r_1}}{\mathtt{b}_1{}^{r_1} + \mathtt{b}_3{}^{r_1}}$ is the probability that agent 1 wins against player 3 and $p_p = 1 - p_T$.

Proposition 7 presents the equilibrium of the elimination contest with sequential group-stage competitions.[17] As before, the equilibrium bids are expressed implicitly as bid ratios and "b" and "$\mathtt{B}$" denote bids in the group stage and in the final respectively.

[16]For example, matches the group stage in the FIFA World Cup (which consists of round-robin tournaments) take place sequentially, except those in the last round. On the other hand, in the All England Open Badminton Championships, which is the world's oldest badminton tournament, multiple group-stage matches (such as R32, R16 and Quarter Finals) take place in neighboring courts simultaneously.

[17]For simplification, we only consider here the case with $r_1 \in (0,1]$. A pure-strategy SPE continues to exist when $r_1 > 1$, but one will have to impose a similar restriction on r_2 as in Assumption 2.

Proposition 7 *Suppose Assumption 1 holds and* $r_1 \in (0,1]$*. In the elimination contest with seeding* $\{(1,3),(2,4)\}$ *and the pair* $(1,3)$ *competing before the pair* $(2,4)$ *in the group stage, there is a unique pure-strategy SPE with* $(i \in \{1,2\}, j \in \{3,4\})$

$$\mathtt{b}_1 = \frac{r_1 {\lambda_1}^{r_1-1}}{(1+{\lambda_1}^{r_1})^2}\frac{2-r_2}{4}v, \mathtt{b}_3 = \lambda_1 \mathtt{b}_1,$$
$$\mathtt{b}_{2T} = \frac{r_1 k^{r_1}}{(1+k^{r_1})^2}\frac{2+r_2}{4}v, \mathtt{b}_{4T} = k\mathtt{b}_{2T}, \mathtt{b}_{2P} = \mathtt{b}_{4P} = \frac{r_1(2-r_2)}{16}v,$$
$$\mathtt{B}_i = \frac{r_2}{4}v \text{ against a player, } \mathtt{B}_i = 0 \text{ against an agent, } \mathtt{B}_j = \frac{r_2}{4}v$$

where $k := \frac{\mathtt{b}_{4T}}{\mathtt{b}_{2T}} = \frac{2-r_2}{2+r_2}$ *and* $\lambda_1 := \frac{\mathtt{b}_3}{\mathtt{b}_1}$ *is given by*

$$\lambda_1 = \frac{k(1+k^{r_1})^2}{\frac{2+r_1}{4}k(1+k^{r_1})^2 + 1 + (1-r_1)k^{r_1}}.$$

When $r_1 = r_2 = 1$*, the organizer's revenue is strictly lower than that in Proposition 3.*

According to Proposition 7, agent 2 bids a same amount as player 4 ($\mathtt{b}_{2P} = \mathtt{b}_{4P}$) if agent 1 loses against player 3. However, if agent 1 defeats player 3, agent 2 will bid more aggressively than player 4 ($k = \mathtt{b}_{4T}/\mathtt{b}_{2T} < 1$). Moreover, one can show that $\lambda > k$ and $\lambda_1 > k$–recall that $\lambda_1 = \mathtt{b}_3/\mathtt{b}_1$, and $\lambda = b_j/b_i$ is the equilibrium bid ratio in Proposition 3. This implies that after observing agent 1's winning, agent 2 bids more aggressively than she does in the elimination contest with simultaneous group-stage competitions ($\lambda > k$), while player 4 bids less compared to player 3's bid against player 1 ($\lambda_1 > k$). Hence, agent 1's victory brings good news to agent 2, but bad news to player 4, due to the fact that the collusive team's expected benefit in the final stage is now larger.[18]

The above discussion implies that the sequential group-stage competitions enable the collusive agents to better adjust agent 2's bid against player 4 according to the outcome of $(1,3)$ competition. Such flexibility, while benefiting the agents, can strictly hurt the organizer, as shown in Proposition 7 for the case of $r_1 = r_2 = 1$. A similar analytical comparison of the organizer's revenues for general r_1 and r_2 is difficult to obtain, given the inconvenient Tullock contest successful function. However, our various simulation exercises indicate that for general r_1 and r_2, the organizer indeed fares better in, and hence strictly prefers, the elimination contest with simultaneous group-stage competitions. The following Figure 3 modifies Figure 1 by adding the organizer's expected revenue from

[18]It is however analytically difficult to pin down the relationship of λ and λ_1 in terms of r_1 and r_2. Due to such complication, an analytical comparison between the organizer's revenues in Proposition 3 and Proposition 7 is also difficult, as we mention below.

sequential group-stage competitions, which lies below the expected revenue from the corresponding elimination contest with simultaneous group-stage competition $\{(1,3),(2,4)\}$.

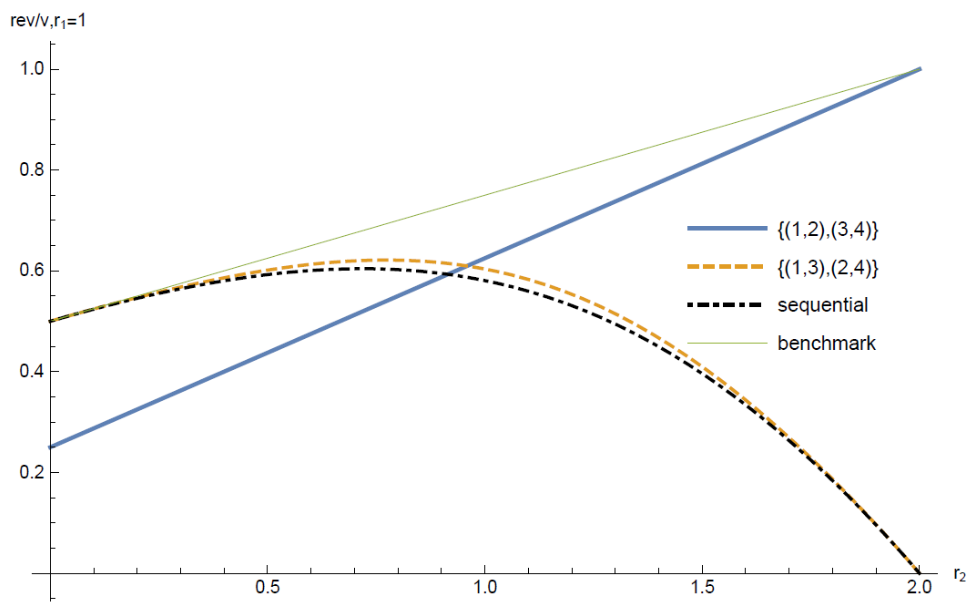

Figure 3: Sequential vs. Simultaneous Group-Stage Competitions ($r_1 = 1$).

4.2 One-Shot Contest vs. Elimination Contest

We now analyze the one-shot contest where the agents and the players compete in a one-round Tullock contest with $r = r_1 = r_2$, and we compare the corresponding collusive outcome with that in the elimination contest. This is a common and important exercise for optimal contest design and the standard result in the literature (e.g., Gradstein and Konrad 1999[13] and Fu and Lu 2012[10]) is that the one-shot or simultaneous contest generates more effort (revenue) for the organizer when the contest is discriminatory enough ($r > 1$), otherwise an optimally designed multi-stage contest is optimal.

Our contest, albeit having an exogenous structure, differs from the previous literature in that due to collusion, our players are heterogeneous. Such heterogeneity, as we have seen, introduces additional flexibility for the organizer to arrange players in the group stage, which leads to different implications in the comparison of the two contests. This is shown in our next Proposition 8, which characterizes the pure-strategy equilibrium of the one-shot contest and compares the two contests from the organizer's viewpoint:[19]

[19] The explicit equilibrium in Proposition 8 can be found in Appendix. Notice here that the requirement $r \in \left(0, \frac{3}{2}\right]$ is imposed for the one-shot contest where there are actually 3 independent "players" given the collusive agents 1 and 2, while Assumption 2 is imposed for the elimination contest in part (ii) of

Proposition 8 *Let* $r_1 = r_2 = r \in \left(0, \frac{3}{2}\right]$ *and Assumption 2 holds. There is a unique pure-strategy equilibrium outcome in the one-shot contest with bids* $(b_1^*, b_2^*; b_3^*, b_4^*)$ *where* $b_3^* = b_4^*$ *for all* r*,* $b_1^* = b_2^*$ *for* $r \in (0,1)$*,* $b_1^* + b_2^* = \frac{2v}{9}$ *for* $r = 1$*, and* $b_i = \frac{2rv}{9}$*,* $b_j^* = 0$ *for* $i, j \in \{1, 2\}$ *and* $r > 1$*. In addition, we have*

- (i) *Collusion, while beneficial in general, can hurt the collusive agents, relative to the non-collusive benchmark when* r *is in a neighborhood around* 1*.*

- (ii) *There is* $r^* < 1$ *such that for* $r \in [0, r^*)$*, the elimination contest with seeding* $\{(1,3),(2,4)\}$ *is optimal, while for* $r > r^*$*, the one-shot simultaneous contest is optimal for the organizer.*

Hence, in the equilibrium of the one-shot contest, depending on r, the agents either bid an identical amount or bid differently.[20] In addition, different from the elimination contest where collusion is always beneficial to the agents, the collusive agents' joint equilibrium payoff in the one-shot contest can be lower than their joint equilibrium payoff in the non-collusive benchmark in Lemma 1 when r is near 1. This is somewhat counter-intuitive and results from the fact that public collusion prompts the non-collusive players to bid more aggressively, which strictly hurts the agents when r is in an intermediate region.

It is interesting to contrast our results in Proposition 6 and Proposition 8 with the well-known *alliance formation puzzle* (see Chapter 7 of Konrad (2009)[17]), where players in an alliance face strategic disadvantages compared to individual players that do not belong to the alliance, due to free-riding incentives and future looming conflicts among members in the alliance. While our setting differs from Konrad (2009)[17] (Chapter 7) in that our collusive agents have a common goal and can perfectly coordinate their bids in the contest, our results show that even with perfect coordination and no future conflict, forming an alliance can either hurt or benefit collusive members in the alliance relative to the no collusion benchmark, depending on the underlying contest structure. As such, Proposition 6 and Proposition 8 provide interesting case studies for further analysis of resolving the alliance formation puzzle.

Finally, compared to the previous literature (Gradstein and Konrad 1999[13] and Fu and Lu 2012[10]), the elimination contest generates strictly higher revenue for the organizer when the contest rule is not discriminatory enough, but with a lower threshold, i.e.,

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Proposition 8. The two restrictions on r jointly imply that $r \in (0, 1.2184]$.

[20] For any fixed budget $b_1 + b_2 = B$, as the winning probability is increasing in ${b_1}^r + {b_2}^r$, the problem of the team is equivalent to maximizing ${b_1}^r + {b_2}^r$. If $r < 1$, x^r is concave, or ${b_1}^r + {b_2}^r \leq 2(\frac{b_1+b_2}{2})^r$ and the optimal choice is $b_1 = b_2 = \frac{B}{2}$, while if $r > 1$, x^r is convex, or ${b_1}^r + {b_2}^r \leq (b_1 + b_2)^r$ and it is optimal to let one agent bid 0.

for $r < r^*$ and $r^* < 1$. This is due to two counteracting factors: on one hand, heterogeneity enables the organizer to choose various seedings in the group stage, but on the other, such heterogeneity is due to collusion, which allows the agents to bid 0 regardless of seeding. As a result, the elimination contest is optimal for a smaller range of r. The following Figure 4 provides a numerical illustration that the elimination contest with seeding $\{(1,3),(2,4)\}$ is optimal for $r \in [0, r^*)$ with $r^* = 0.89$.

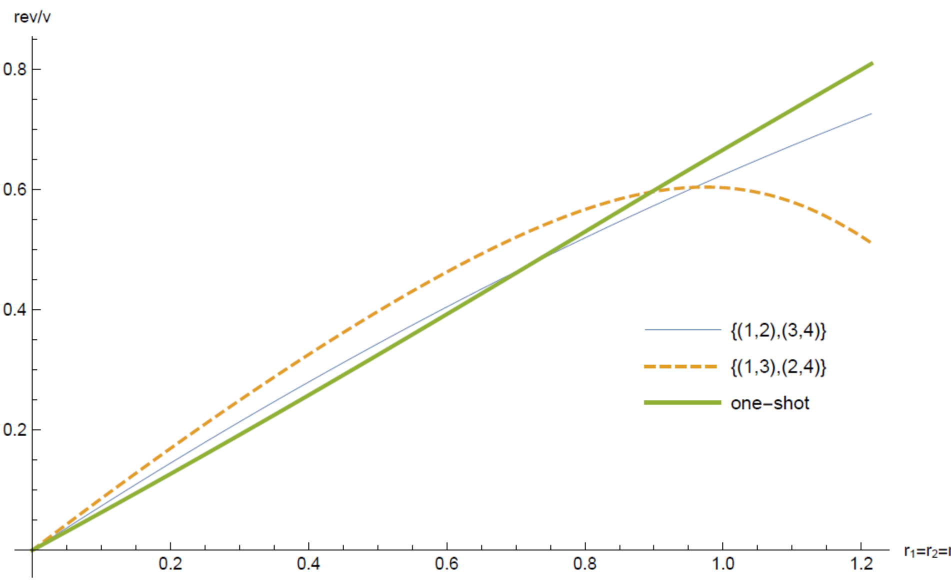

Figure 4: Revenue Comparison–One-Shot vs. Elimination ($r_1 = r_2 = r$).

4.3 Secret Collusion

We have assumed so far that collusion between the agents is publicly known, which is indeed an important and useful benchmark. We now briefly discuss the variant where collusion between agents 1 and 2 is secret, i.e., players 3 and 4 are ignorant about the collusion and treat the contest as a symmetric and non-collusive one.

First, if the seeding is $\{(1,2),(3,4)\}$, it is easy to see that the equilibrium outcome of the secret collusion is the same as public collusion in Proposition 2.

If the agents are assigned to different groups, now their opponents will not respond to the collusion and will play as in the benchmark in Proposition 1. With the absence of the discouraging effect from collusion, the competition in the group stage is now more intensive with secret collusion, in that the agents and the players all bid strictly more than their bids in Proposition 3. This further implies that the probability of both agents entering the final is reduced relative to that in Proposition 3. As a result, the revenue of the organizer is higher under secret collusion as his expected revenue in each stage is strictly higher. Another (technical) implication of secret collusion is that Assumption 2 is no longer required for equilibrium characterization, and the expected payoff of the players can be below 0 for $r_1 > 1$ and large r_2.

For the one-shot simultaneous contest, the players will bid as in the benchmark in Lemma 1 and the collusive agents now always benefit from (secret) collusion since the players will not respond to collusion by bidding more aggressively.

Overall, our discussion above indicates that secrete collusion, relative to public collusion, can either benefit or hurt collusive agents, depending on the contest structure. And our result here may have interesting implications to applications in practice.[21]

4.4 Large Discriminatory Powers

Among our results so far, the discriminatory powers (r_1, r_2) have played a critical role, in terms of both the existence of pure-strategy equilibria and the interpretations of our results. Given the importance of r_1 and r_2 in our setting, a natural question is whether our analysis and our results can be extended to the entire space of $(r_1, r_2) \in (0, \infty)^2$. This is particularly the case since all the competitions in our elimination contest are simple *pairwise* competitions and for complete-information Tullock contests, the literature has pointed out a clear path toward analyzing Nash equilibria for such contests with general discriminatory powers, especially for two-player Tullock contests (see, for example, Alcalde and Dahm (2010)[1], Wang (2010)[27], Ewerhart (2015, 2017)[7][8], and Feng and Lu (2017)[12]).[22]

We now discuss equilibrium characterization for our elimination contest with large r_1 and/or r_2.[23] As is clear from the literature cited above, we will now have to consider mixed-strategy equilibrium, and in particular, in some subgames, we will have a so-called "all-pay auction equilibrium," which is essentially a mixed-strategy equilibrium that exhibits the same properties and statistics (e.g., expected costs, expected payoffs and winning probabilities) as that of the corresponding all-pay auction.[24] First, consider

[21]Secrete collusion is perhaps particularly relevant for real-world geopolitical conflicts (for example, the Cuban Missile Crisis and the creation of the malicious computer virus Stuxnet, which is believed to have damaged the nuclear program of Iran) and online poker.

[22]We thank our anonymous referee for raising this issue to our attention.

[23]For brevity, we only summarize our results here. The explicit analysis is available upon request.

[24]See Alcalde and Dahm (2010)[1] and Wang (2010)[27] for detailed discussions on the "all-pay equilibrium." Ewerhard (2015)[7] identifies additional structural properties of mixed-strategy equilibria in Tullock contests with large discriminatory powers. In particular, he shows that any equilibrium in such (two-player) contest exhibits the same expected effort, winning probabilities, and payoffs.

seeding $\{(1,2),(3,4)\}$, where the collusive agents are in the same group competition. If $r_1 \in [2,\infty)$, or $r_2 \in [2,\infty)$, or both, then there is a mixed-strategy subgame perfect equilibrium where an "all-pay auction equilibrium" is played in the group/final stage, or both, depending on r_1, r_2. In addition, the organizer's expected payoff is either v (if $r_2 \geq 2$) or $\frac{2+r_2}{4}v$ (if $r_1 \geq 2$ and $r_2 < 2$). In this seeding, collusion only takes place in the group stage, which makes equilibrium characterization a simple extension of the results in the literature.

Next, consider the more complicated seeding $\{(1,3),(2,4)\}$. Recall that for this seeding, we have imposed Assumption 2 so as to analyze pure-strategy equilibrium. When $r_1 > 0$ and $r_2 \geq 2$, there is again a mixed-strategy equilibrium where an "all-pay auction equilibrium" will be played in the final if at least one player enters the final, with full rent dissipation. As a result, both non-collusive players will bid 0 in the group stage and the two agents wins the groups stage by bidding 0^+ (or with a proper tie breaking rule), leading to a zero revenue for the organizer. Things are much more complicated, however, when $r_1 \geq 2$ and $r_2 < 2$. According to the literature, if there is no collusion, the four players play an "all-pay auction equilibrium" in the group stage. When there is collusion and both agents' strategies are chosen by the team, we are not able to find a mixed-strategy equilibrium. The difficulty, as hinted already in Remark 1, comes from the fact that the two agents' (behavior strategies) can be arbitrarily correlated, making the team's strategy space too large to analyze. Finally, if both r_1, r_2 are in $(0,2)$ and Assumption 2 is violated, there is a "semi-pure-strategy" equilibrium, as in Wang (2010)[27], where the collusive agents play pure strategies and the non-collusive players randomize in the group stage. The complication here though is that expected bids and winning probabilities in this equilibrium are difficult to calculate.

In summary, for large discriminatory powers r_1, r_2, while some progress can be made in equilibrium characterization for our elimination contest, a full-blown analysis of the optimal seeding and impacts of collusion on various parties is difficult to obtain, due to both the complication in characterizing equilibrium in some cases, and the inconvenience in calculating equilibrium quantities in some other cases.[25] Nevertheless, the above analysis indeed elucidates the key technical difference between our setting and standard elimination contests. In addition, our discussion here also raises an open question of how to analyze mixed-strategy equilibrium in Tullock contests when some of the players can perfectly coordinate their strategies (as in, e.g., eSports competitions).

[25]It is worth pointing out that it is clear that whenever $r_2 \geq 2$, seeding $\{(1,3),(2,4)\}$, which leads to an equilibrium revenue of 0 to the organizer, is unambiguously worse than seeding $\{(1,2),(3,4)\}$.

5 Concluding Remarks

We have analyzed collusion among a pair of players and its impact on the organizer and non-collusive players in an elimination contest with its two stages modeled as Tullock contests with different discriminatory powers. We found that while collusion in general undermines fairness of the competition, the organizer's revenue can be either hurt or benefited by collusion and the organizer's optimal seeding can also vary, depending on the discriminatory powers of the Tullock contests. We have also discussed related issues such as sequential group-stage competitions, comparison between the elimination contest and its corresponding one-shot counterpart under collusion, and secret collusion.

Our results provide useful implications in coping with collusion in real-world competitions. For example, after the "disgrace of Gijón," FIFA revised the group-stage competitions so that the final two games in each group would take place simultaneously. Our result on comparing sequential and simultaneous group-stage competitions indeed demonstrates that the sequential format provides additional information, which facilitates collusive players to better coordinate their play. For other sports such as badminton, several Asian countries have been historically strong and typically have multiple top-ranked players than other countries. The Badminton World Federation changed the rule for the 2016 Olympic Games to limit the number of single players from each country to only two, likely with the aim to reduce collusion among players from a same country. Our results on seedings reveal that an additional measure to cope with collusion is to allow the organizer to choose seeding in the knock out stage (currently the seeding of the knock out stage is determined by a random draw)[26].

It is important to point out that our setting is a highly stylized one and although we have used several incidences of sports collusion as motivating examples, our setting (understandably) does not mirror collusion in sports perfectly. For example, for football matches in a FIFA World Cup, collusion in group stages is typically not geared toward saving effort, but rather toward securing a favorable match in later stages or in the final. In addition, even after two football teams collude in a group stage and meet in the final, it is unlikely that the two football teams would also collude in the final by exerting no effort. Nevertheless, we believe our analysis does illustrate some interesting and unexpected strategic implications that collusion can bring about to players and organizers in elimination contests.

Finally, given that our results are obtained in a highly stylized setting, it would be

[26] http://www.badmintonpanam.org/wp-content/uploads/2018/12/3.4.1.2-OG-Regulations-for-Badminton-Competition-Tokyo-2020-1.pdf

desirable to incorporate other more realistic features into our model, such as ex ante heterogeneous players (where seeding plays a more significant rule), incomplete information (where collusive players can not only coordinate their play but also share information among themselves), and a more elaborate dynamic contest structure. While these considerations without doubt complicate the analysis significantly, due to an increasing number of seedings and more complex fixed-point arguments for analytical solutions (as hinted in the conclusion of Groh et al. (2012)[14]), we believe such exercises will provide richer results to better understand and cope with collusion in contests in general.

Appendix

Proof of Proposition 1. First consider the final stage, where each finalist solves

$$B = \arg\max_b \frac{b^{r_2}}{b^{r_2} + B^{r_2}} v - b.$$

The corresponding first-order condition (FOC) and symmetry lead to optimal bid $B = \frac{r_2}{4} v$ and an expected payoff of $\frac{2-r_2}{4} v$, non-negative given Assumption 1. The corresponding second-order condition (SOC) is

$$-\frac{B^{r_2} b^{r_2 - 2} r_2 v \left((r_2 + 1) b^{r_2} + (1 - r_2) B^{r_2} \right)}{(B^{r_2} + b^{r_2})^3}$$

which is negative for $r_2 \leq 1$, i.e., the finalist's payoff function is strictly concave and B is the unique global maximizer. For $r_2 \in (1, 2)$, the payoff function, while not concave, first decreases in the neighborhood of $b = 0$, and then is single-peaked at $b = B$. Hence the only candidates for the global maximizer are $b = 0$ and $b = B$. Since bidding B is always at least as good as bidding 0 and both bidding 0 is not an equilibrium, the only symmetric equilibrium has both players bidding B. Finally, the analysis for the group stage can be done analogously. □

Proof of Proposition 3. The equilibrium in the final is immediate. Consider now the competition between the pair $(1, 3)$ in the group stage (the analysis for the pair $(2, 4)$ is symmetric). Given a bid profile (b_1, b_2, b_3, b_4), the FOCs for agent 1 and player 3 are

$$\begin{aligned}\frac{1}{v} &= \frac{r_1 {b_3}^{r_1} {b_1}^{r_1-1}}{({b_1}^{r_1}+{b_3}^{r_1})^2}\left(\frac{{b_2}^{r_1}}{{b_2}^{r_1}+{b_4}^{r_1}}\frac{2+r_2}{4}+\frac{{b_4}^{r_1}}{{b_2}^{r_1}+{b_4}^{r_1}}\frac{K_1}{v}\right),\\ \frac{1}{v} &= \frac{r_1 {b_1}^{r_1} {b_3}^{r_1-1}}{({b_1}^{r_1}+{b_3}^{r_1})^2}\frac{K_1}{v}.\end{aligned}$$

Define two group-stage bid ratios $\lambda_{31} := b_3/b_1, \lambda_{42} := b_4/b_2$–notice that there is no equilibrium where anyone bids 0 in the group stage. Dividing the above two FOCs and similarly for the FOCs of agent 2 and player 4, we obtain

$$\lambda_{31} = \frac{\lambda_{42}^{r_1}+1}{\lambda_{42}^{r_1}+\frac{2+r_2}{2-r_2}}, \quad \lambda_{42} = \frac{\lambda_{31}^{r_1}+1}{\lambda_{31}^{r_1}+\frac{2+r_2}{2-r_2}}. \tag{4}$$

It is easy to see that there is a solution $\lambda_{31} = \lambda_{42} = \lambda$, which is determined by

$$\lambda^{r_1+1} - \lambda^{r_1} + \frac{2+r_2}{2-r_2}\lambda - 1 = 0 \Leftrightarrow (1-\lambda)(1+\lambda^{r_1}) = \frac{2r_2}{2-r_2}\lambda. \tag{5}$$

Given λ, it is then impossible to have solutions with $\lambda_{31} \neq \lambda_{42}$ solving (4). To see that, suppose to the contrary that $(\lambda_{31}, \lambda_{42})$ with (WLOG) $\lambda_{31} < \lambda_{42}$ also solves (4). Then $\lambda_{31} < \lambda < \lambda_{42}$,[27] which leads to a contradiction as $\lambda_{31} = \frac{\lambda_{42}^{r_1}+1}{\lambda_{42}^{r_1}+\frac{2+r_2}{2-r_2}} > \lambda$ by $(1-\lambda)\lambda_{42}^{r_1}+1 > (1-\lambda)\lambda^{r_1}+1 = \frac{2+r_2}{2-r_2}\lambda$. We hence have a unique solution $\lambda_{31} = \lambda_{42} = \lambda$ solving (4).

We next argue that there is a unique $\lambda \in (0,1)$ solving (5) and we have $\frac{\partial\lambda}{\partial r_1}, \frac{\partial\lambda}{\partial r_2} < 0$.

By (5), λ is the intersection of the function $L(x) := (1-x)(1+x^{r_1})$ and a linear function $\frac{2r_2}{2-r_2}x$. By examining the sign of each function at any $x \in (-\infty, 0] \cup [1, +\infty)$, we have $\lambda \in (0,1)$. The uniqueness and derivative properties will then follow from the shape of $L(x)$, with $L'(x) = x^{r_1-1}(r_1-(r_1+1)x) - 1$ and $L''(x) = r_1 x^{r_1-2}(r_1-1-(r_1+1)x)$. We discuss three cases: (1) If $r_1 > 1$, $L'(x)$ is increasing on $(0, \frac{r_1-1}{r_1+1})$ and decreasing on $(\frac{r_1-1}{r_1+1}, 1)$. As $L'(\frac{r_1-1}{r_1+1}) = (\frac{r_1-1}{r_1+1})^{r_1-1} - 1 < 0$, we have $L'(x) < 0$ or $L(x)$ is decreasing on $(0,1)$. Since $\frac{2r_2}{2-r_2}x$ increases in x, we have that λ is unique. In addition, it is easy to verify that $\frac{\partial\lambda}{\partial r_1} < 0$ as x^{r_1} is decreasing in r_1 on $x \in (0,1)$, and $\frac{\partial\lambda}{\partial r_2} < 0$ as $\frac{2r_2}{2-r_2}$ is increasing in r_2. (2) If $r_1 = 1$, we have $L(x) = 1 - x^2$ and the analysis is the same as in (1). (3) If $r_1 < 1$, $L''(x) < 0$ on $(0,1)$, that $L(x) - \frac{2r_2}{2-r_2}x$ is first increasing (by $L'(0^+) = +\infty$) and concave. Thus λ is unique and we have the same properties of the derivatives. The proof here is then complete by observing that with any r_1, $x < \lambda \Leftrightarrow L(x) > \frac{2r_2}{2-r_2}x$.

Next, we verify that the above characterized bids are indeed optimal. For the players,

[27] Each of the two inequlities cannot be binding, otherwise we will have $\lambda = \lambda_{31} = \lambda_{42}$.

each bid b_j is optimal as long as the corresponding expected payoff is non-negative:

$$U_j = \frac{\lambda^{r_1}(\lambda^{r_1}+1-r_1)}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}v \geq 0 \Leftrightarrow \lambda^{r_1}+1-r_1 \geq 0.$$

Hence there is a cutoff r_2^* s.t. $\lambda^{r_1}+1-r_1=0$, i.e., $\Pi_j=0$. As $\frac{\partial\lambda}{\partial r_2}<0$, $\Pi_j \geq 0$ iff $r_2 \leq r_2^*$, corresponding to the inequality in Assumption 2 as $\frac{2r_2}{2-r_2}$ is increasing in r_2.

Meanwhile, verifying optimality of the agents' bids involves two complications: (i) the agents maximize joint payoffs, and (ii) the agents' strategy sets are larger. We 'divide and conquer' these complications via three steps:

1. Given b_3, b_4, any deviation $(b_1, b_2) = (0, b_2')$ is dominated by (b_2', b_2').
2. Any deviation to non-zero $b_1 \neq b_2$ while still satisfying the FOCs of the agents result in a local minimum, thus not profitable.
3. When $b_1 = b_2 = b$, the equilibrium b_i in Proposition 3 is a best response to (b_3, b_4).

Step 1. The team's expected payoff from bidding $b_1' = 0, b_2' > 0$ can be calculated as

$$u_T' = \frac{{b_2'}^{r_1}}{{b_2'}^{r_1}+{b_j}^{r_1}}\frac{2-r_2}{4}v - b_2' > 0,$$

which is dominated by $(b_1, b_2) = (b_2', b_2')$:

$$\left(\frac{{b_2'}^{r_1}}{{b_2'}^{r_1}+{b_j}^{r_1}}\right)^2 v + 2\frac{{b_2'}^{r_1}{b_j}^{r_1}}{({b_2'}^{r_1}+{b_j}^{r_1})^2}\frac{2-r_2}{4}v - 2b_2' = \left(\frac{{b_2'}^{r_1}}{{b_2'}^{r_1}+{b_j}^{r_1}}\right)^2 \frac{r_2}{2}v + 2u_T' > u_T'.$$

Hence, $(0, b_2')$ is not a profitable deviation for the agents (by **Step 3**, (b_2', b_2') is still suboptimal compared with $(\frac{b_j}{\lambda}, \frac{b_j}{\lambda})$).

Step 2. If there exists a profitable deviation with $b_1 \neq b_2$, then such (b_1, b_2) satisfies the FOCs, as deviating to $b_1 = 0$ is not profitable. Indeed, we can verify that any other strategy satisfying the FOCs of the agents will be a local minimum.[28]

Step 3. As $b_1 \neq b_2$ result in a local minimum, we only need to show that any deviation in the form of $(b_1, b_2) = (b, b)$ is not profitable. We can compare the utilities by checking the FOC and SOC with respect to b, which are respectively

$$\frac{r_1 {b_j}^{r_1} b^{2r_1-1}}{(b^{r_1}+{b_j}^{r_1})^3}\frac{2+r_2}{4}v + \frac{r_1 {b_j}^{2r_1} b^{r_1-1}}{(b^{r_1}+{b_j}^{r_1})^3}\frac{2-r_2}{4}v - 1,$$

$$\frac{r_1 {b_j}^{r_1} b^{r_1-2}}{(b^{r_1}+{b_j}^{r_1})^4}(2+r_2)v\left(-(r_1+1)b^{2r_1} + \frac{4(r_1r_2-1)}{2+r_2}{b_j}^{r_1}b^{r_1} - \frac{(2-r_2)(1-r_1)}{2+r_2}{b_j}^{2r_1}\right).$$

[28]Detailed arguments on this can be found in the working paper version (supplemental materials) on https://people.smu.edu/bchen/research/.

The sign of the SOC hinges on the quadratic function of b^{r_1} in the parentheses. For cases where $r_1 \in (0, 0.5] \cup [1, 2)$ or $r_2 \leq 1$, the fact that b_i is the only one satisfying the FOC is easy to verify by the properties of the quadratic function.[29] But for $r_1 \in (0.5, 1), r_2 \in (1, 2)$ with $r_1 r_2 > 1$, the SOC can have two zero points on $\mathbb{R}_{++}$. One can show that the first zero point, which is close to zero, is a local minimum of the FOC with a value above 0.[30] Hence, in this case the team's utility first increases in b and is single-peaked at $b_i = \frac{b_j}{\lambda}$.

The above three steps, together with the discussion on the players' incentives, verify that the bids in the proposition are optimal for all players. $\square$

Proof of Proposition 4. We first simplify $\Pi_D > \Pi_S$ as follows:

$$\Pi_D > \Pi_S \quad \Leftrightarrow \quad \frac{r_2}{2}v + \frac{r_1(2-r_2)\lambda^{r_1-1}(1+\lambda) - r_2}{2(\lambda^{r_1}+1)^2}v > \frac{r_1(2-r_2)}{8}v + \frac{r_2}{2}v$$
$$\Leftrightarrow \quad f(r_1, r_2) := -\lambda^{2r_1} + 2\lambda^{r_1} + 4\lambda^{r_1-1} - 1 - \frac{4r_2}{r_1(2-r_2)} > 0,$$

Given that $\Pi_D > \Pi_S \Leftrightarrow f(r_1, r_2) > 0$, our proof will be based on the function $f(r_1, r_2)$. We first show that the single crossing of Π_D and Π_S is guaranteed if $f(r_1, r_2)$ is strictly decreasing in r_2 on $(0, 2)$: For any fixed $r_1 \in (0, 2)$, as $r_2 \to 0$, $(1-\lambda)(1+\lambda^{r_1}) = \frac{2r_2}{2-r_2}\lambda \to 0$, that $\lambda \to 1$ and $\lim_{r_2 \to 0} \Pi_D = \frac{r_1}{2}v > \frac{r_1}{4}v = \lim_{r_2 \to 0} \Pi_S$. As $r_2 \to 2$, The line $\frac{2r_2}{2-r_2}\lambda$ is vertical that the intersection $\lambda \to 0$. Then $\lim_{r_2 \to 0} \Pi_D = \frac{3}{4}v < v = \lim_{r_2 \to 0} \Pi_S$. Hence, if $f(r_1, r_2)$ is strictly decreasing in r_2 on $(0, 2)$, then Π_D and Π_S cross exactly once at some $r_2 \in (0, 2)$ for any fixed r_1.

We now verify $df(r_1, r_2)/dr_2 < 0$ in the sequel. First, we calculate:

$$\frac{df(r_1, r_2)}{dr_2} = \left(-2r_1\lambda^{2r_1-1} + 2r_1\lambda^{r_1-1} + 4(r_1-1)\lambda^{r_1-2}\right)\frac{d\lambda}{dr_2} - \frac{8}{r_1(2-r_2)^2}.$$

For $r_1 \geq 1$, $-2r_1\lambda^{2r_1-1} + 2r_1\lambda^{r_1-1} = 2r_1\lambda^{r_1-1}(1-\lambda^{r_1}) > 0$ and $4(r_1-1)\lambda^{r_1-2} \geq 0$, thus $\frac{df(r_1,r_2)}{dr_2} < 0$ for all r_2 by $\frac{d\lambda}{dr_2} < 0$. Π_D and Π_S have a single crossing at $\bar{r_2} \in (0, 2)$ s.t. $\forall r_2 < \bar{r_2}$, $\Pi_D > \Pi_S$ and $\forall r_2 \geq \bar{r_2}$, $\Pi_D \leq \Pi_S$. Notice that, as Assumption 2 restricts r_2 to be not too large, for large enough r_1, all r_2 satisfying Assumption 2 is smaller than $\bar{r_2}$, which implies $\Pi_D > \Pi_S$.

For $r_1 < 1$, establishing $\frac{df(r_1,r_2)}{dr_2} < 0$ is more involved since $4(r_1-1)\lambda^{r_1-2} < 0$. First,

[29] The detailed discussion can again be found in our online supplemental materials.

[30] This can be shown (in the working paper version) via an amplification based on the value at $b_j/2$.

plug in the expression of $\frac{d\lambda}{dr_2} = -\frac{4\lambda}{(2-r_2)^2} \cdot \frac{1}{((r_1+1)\lambda - r_1)\lambda^{r_1-1} + \frac{2+r_2}{2-r_2}}$ to obtain

$$\begin{aligned}\frac{df(r_1,r_2)}{dr_2} &= -\frac{8}{(2-r_2)^2} \cdot \frac{r_1\lambda^{r_1}(1-\lambda^{r_1}) + 2(r_1-1)\lambda^{r_1-1}}{((r_1+1)\lambda - r_1)\lambda^{r_1-1} + \frac{2+r_2}{2-r_2}} - \frac{8}{r_1(2-r_2)^2} \\ &< -\frac{8}{(2-r_2)^2} \cdot \frac{2(r_1-1)\lambda^{r_1-1}}{((r_1+1)\lambda - r_1)\lambda^{r_1-1} + \frac{2+r_2}{2-r_2}} - \frac{8}{r_1(2-r_2)^2} \\ &= \frac{8}{r_1(2-r_2)^2} \cdot \frac{r_1(3-2r_1)\lambda^{r_1-1} - (r_1+1)\lambda^{r_1} - \frac{2+r_2}{2-r_2}}{((r_1+1)\lambda - r_1)\lambda^{r_1-1} + \frac{2+r_2}{2-r_2}},\end{aligned}$$

where the inequality is due to $r_1\lambda^{r_1}(1-\lambda^{r_1}) > 0$ as $\lambda \in (0,1)$.

We claim that actually $\lambda^{r_1-1} < \frac{2}{2-r_2}$.[31] With this inequality,

$$\frac{df(r_1,r_2)}{dr_2} < \frac{8}{r_1(2-r_2)^3} \cdot \frac{2r_1(3-2r_1) - (2-r_2)(r_1+1)\lambda^{r_1} - (2+r_2)}{((r_1+1)\lambda - r_1)\lambda^{r_1-1} + \frac{2+r_2}{2-r_2}}.$$

As $((r_1+1)\lambda - r_1)\lambda^{r_1-1} + \frac{2+r_2}{2-r_2} > 0$ by $\frac{d\lambda}{dr_2} < 0$, a sufficient condition for $\frac{df(r_1,r_2)}{dr_2} < 0$ is

$$2r_1(3-2r_1) < (2-r_2)(r_1+1)\lambda^{r_1} + (2+r_2). \tag{6}$$

Observe that $2r_1(3-2r_1) \leq (2+r_2)$ for $r_1 \in (0, 0.5]$ or $r_2 \in [0.25, 2)$, by the property of $2r_1(3-2r_1)$. A tedious but straightforward calculation shows that (6) also holds for $\{(r_1,r_2) | r_1 \in (0.5, 1), r_2 \in (0, 0.25)\}$.

In conclusion, $\frac{df(r_1,r_2)}{dr_2} < 0$ for all r_1, $r_2 \in (0,2)$. Hence, for each r_1, Π_D and Π_S cross exactly once at some $\bar{r}_2 \in (0,2)$, leading to the optimal seeding result. □

Proof of Proposition 5. Under seeding $\{(1,2),(3,4)\}$, the revenue is strictly lower than the benchmark since the agents bid 0 in the group stage. Hereafter, we consider revenue comparison under seeding $\{(1,3),(2,4)\}$. We will first establish the second part of Proposition 5, i.e., $\exists \epsilon > 0$ s.t. $\forall r_1 > 1$, $r_2 \in (0, \epsilon)$, $\Pi_D > \Pi_N$ or the organizer can benefit from collusion. First, notice that

$$\Pi_D > \Pi_N \Leftrightarrow g(r_1,r_2) := -\frac{\lambda^{2r_1}}{2} + \lambda^{r_1-1} - \frac{1}{2} > \frac{1}{r_1}\frac{r_2}{2-r_2},$$

[31] This can be proved by plugging $\lambda_0 = (\frac{2-r_2}{2})^{\frac{1}{1-r_1}}$ into the equation determining λ to show that $\lambda > \lambda_0$.

and $\lim_{r_2\to 0}\Pi_D = \lim_{r_2\to 0}\Pi_N$ as $\lim_{r_2\to 0} g(r_1,r_2) = 0 = \frac{1}{r_1}\frac{0}{2-0}$. We then calculate

$$\begin{aligned}
\left.\frac{d\frac{1}{r_1}\frac{r_2}{2-r_2}}{dr_2}\right|_{r_2=0} &= \left.\frac{1}{r_1}\frac{2}{(2-r_2)^2}\right|_{r_2=0} = \frac{1}{2r_1},\\
\left.\frac{dg(r_1,r_2)}{dr_2}\right|_{r_2=0} &= \left.-r_1\lambda^{2r_1-1}\frac{d\lambda}{dr_2} + (r_1-1)\lambda^{r_1-2}\frac{d\lambda}{dr_2}\right|_{r_2=0} = r_1\frac{1}{2} - (r_1-1)\frac{1}{2} = \frac{1}{2},
\end{aligned}$$

where the last equality is due to $\left.\frac{d\lambda}{dr_2}\right|_{r_2=0} = -\frac{1}{2}$.

As $r_1 > 1$, $\left.\frac{d\frac{1}{r_1}\frac{r_2}{2-r_2}}{dr_2}\right|_{r_2=0} = \frac{1}{2r_1} < \left.\frac{dg(r_1,r_2)}{dr_2}\right|_{r_2=0}$. By continuity, the inequality also holds when r_2 lies in a neighborhood around 0, implying $\Pi_D > \Pi_N$ in the same neighborhood.

For the first part on $r_1 \le 1$, by Proof of Proposition 4, we already know that $\Pi_D < \Pi_S < \Pi_N$ on $r_2 \in [1,2)$. Thus we only need to show $\Pi_D < \Pi_N$ on $r_2 \in (0,1)$. As $g(r_1,0) = \frac{1}{r_1}\frac{0}{2-0} = 0$, we prove the inequality by showing that $\frac{d\frac{1}{r_1}\frac{r_2}{2-r_2}}{dr_2} > \frac{dg(r_1,r_2)}{dr_2}$, i.e.,

$$\frac{1}{r_1}\frac{2}{(2-r_2)^2} > -r_1\lambda^{2r_1-1}\frac{d\lambda}{dr_2} + (r_1-1)\lambda^{r_1-2}\frac{d\lambda}{dr_2} = -\lambda^{r_1-2}\frac{d\lambda}{dr_2}(1-r_1+r_1\lambda^{r_1+1}).$$

Plug in the expression of $\frac{d\lambda}{dr_2}$ and after simplification, the equality is equivalent to

$$\begin{aligned}
&((r_1+1)\lambda - r_1)\lambda^{r_1-1} + \frac{2+r_2}{2-r_2} > 2r_1\lambda^{r_1-1}(1-r_1) + 2{r_1}^2\lambda^{2r_1}\\
\Leftrightarrow\ & 2{r_1}^2\lambda^{2r_1} + r_1(3-2r_1)\lambda^{r_1-1} - (r_1+1)\lambda^{r_1} < \frac{2+r_2}{2-r_2}.
\end{aligned}$$

As $\lambda^{r_1-1} < \frac{2}{2-r_2}$, it suffices to establish a sufficient condition

$$\begin{aligned}
& 2{r_1}^2\lambda^{2r_1} + r_1(3-2r_1)\lambda^{r_1-1} - (r_1+1)\lambda^{r_1}\\
<\ & 2{r_1}^2\lambda^{2r_1} + \frac{2r_1(3-2r_1)}{2-r_2} - (r_1+1)\lambda^{r_1} \le \frac{2+r_2}{2-r_2}.
\end{aligned} \tag{7}$$

Observe that $2{r_1}^2\lambda^{2r_1} - (r_1+1)\lambda^{r_1} \le 0$ by $\lambda \le 1$ and $r_1 \in (0,1]$. Then similar to the reasoning in Proof of Proposition 4, by the range of $2r_1(3-2r_1)$ and $2+r_2$, the inequality (7) automatically holds for $r_1 \in (0,0.5]$ or $r_2 \in [0.25,2]$.

For $r_1 \in (0.5, 1]$ and $r_2 \in (0, 0.25]$, we obtain a sufficient condition as

$$\begin{aligned} & 2{r_1}^2\lambda^{2r_1} + \frac{2r_1(3-2r_1)}{2-r_2} - (r_1+1)\lambda^{r_1} \le \frac{2+r_2}{2-r_2} \\ \Leftrightarrow \quad & \lambda^{r_1}(2{r_1}^2\lambda^{r_1} - (r_1+1)) + \frac{2r_1(3-2r_1) - 2 - r_2}{2-r_2} \le 0 \\ \Leftarrow \quad & \lambda^{r_1}(2{r_1}^2\lambda^{r_1} - (r_1+1)) + \frac{2r_1(3-2r_1) - 2}{2-0.25} \le 0, \end{aligned} \tag{8}$$

by $r_2 \in (0, 0.25]$ and $2r_1(3-2r_1) > 0$ for $r_1 \in (0.5, 1]$.

As λ^{r_1} is decreasing in r_2, for $r_2 \in (0, 0.25]$, the value of λ^{r_1} is in a connected set. Observe that for any r_1, the LHS of (8) is a quadratic function of λ^{r_1} with $2{r_1}^2 > 0$. Then for (8) to hold for $r_2 \in (0, 0.25]$, we only need to show it holds at $r_2 \to 0$ and 0.25, which can be verified by brute force.

In conclusion, for all $r_1 \in (0, 1]$ and $r_2 \in (0, 2)$, the sufficient condition (7) holds, that $\frac{d\frac{1}{r_1}\frac{r_2}{2-r_2}}{dr_2} > \frac{dg(r_1,r_2)}{dr_2}$, i.e., $g(r_1, r_2) \le \frac{1}{r_1}\frac{r_2}{2-r_2}$, which is equivalent to $\Pi_D < \Pi_N$. □

Proof of Proposition 6. Proposition 2 already implies that under seeding $\{(1, 2), (3, 4)\}$, each player's equilibrium payoff is still U_N, with a winning probability of 1/4, as in the non-colluding benchmark.

Under seeding $\{(1, 3), (2, 4)\}$, a player bids less than his opponent in the group stage ($\lambda < 1$). Consequently, a player has a lower chance to be a finalist. The explicit winning probability of the collusive agents can be calculated as

$$\frac{1}{(1+\lambda^{r_1})^2} + 2\frac{\lambda^{r_1}}{(\lambda^{r_1}+1)^2}\frac{1}{2} = \frac{1}{1+\lambda^{r_1}} > \frac{1}{2}.$$

Hence, each player wins with probability less than 1/4. For each player's expected payoff,

$$U_j = \frac{\lambda^{r_1}(\lambda^{r_1}+1-r_1)}{(\lambda^{r_1}+1)^2}\frac{2-r_2}{4}v < U_N \Leftrightarrow \frac{\lambda^{r_1}(\lambda^{r_1}+1-r_1)}{(\lambda^{r_1}+1)^2} < \frac{2-r_1}{4},$$

which always holds as the inequality is binding at $\lambda = 1$ and $\frac{\lambda^{r_1}(\lambda^{r_1}+1-r_1)}{(\lambda^{r_1}+1)^2}$ is increasing for all r_1, r_2 satisfying Assumptions 1 and 2:

$$\left(\frac{x(x+1-r_1)}{(x+1)^2}\right)' = \frac{(1+x-r_1)+r_1 x}{(x+1)^3} \ge \frac{r_1 x}{(x+1)^3} > 0 \text{ for } x > r_1 - 1. \qquad \square$$

Proof of Proposition 7. First, notice that the competitions between $(2T, 4T)$ and

$(2P, 4P)$ conditional on the result of competition between $(1, 3)$ are standard Tullock contests, with ratios of equilibrium bids being k and 1 respectively.

For agent 1 and player 3, solving the corresponding FOC yields the λ_1 in Proposition 7, which is uniquely determined by k. Hence, the pure-strategy equilibrium is unique. Moreover, verifying these bids being optimal is straightforward. Notice that the FOC for a general player x against y can be written as $\frac{r_1 {b_y}^{r_1} {b_x}^{r_1-1}}{({b_x}^{r_1}+{b_y}^{r_1})^2} K_x - 1$, where

$$K_1 = \left(\frac{1 + k^{r_1} - r_1 k^{r_1}}{(1+k^{r_1})^2} \frac{2+r_2}{4} + \frac{(2+r_1)(2-r_2)}{16} \right) v \text{ and } K_{2T} = \frac{2+r_2}{4} v.$$

and the other K_x is still K_1. Hence, all the SOCs have the same pattern as in the non-colluding benchmark, and the equilibrium bids are indeed optimal for any $r_1 \in (0, 2)$.

When $r_1 = r_2 = 1$, $\lambda = \sqrt{2} - 1, k = \frac{1}{3}, \lambda_1 = \frac{16}{39}$. Then $\Pi_D = \frac{\sqrt{2}}{4} v + \frac{1}{4} v \approx 0.60v$ and the expected revenue when the first stage is sequential is

$$\begin{aligned} \Pi_{SE} &= \mathtt{b}_1 + \mathtt{b}_3 + p_T(\mathtt{b}_{2T} + \mathtt{b}_{4T}) + p_P \cdot 2 \cdot \frac{1}{16} v + (1 - \frac{1}{(1+\lambda_1)(1+k)}) \frac{1}{2} v \\ &= \frac{156 + 149 + 206}{880} v \approx 0.58v < \Pi_D. \end{aligned}$$

More explicitly, the revenue differences from agent 1 and player 3 are negligible ($\approx 10^{-4} v$). However, the organizer suffers a major loss with sequential group competitions, mainly from a lower chance of collecting bids in the final. The organizer also suffers a minor loss in the group stage from a sharp decrease in $\mathtt{b}_{2P}$ after agent 1's defeat. □

Proof of Proposition 8. The agents' problem can be written as

$$\arg\max_{b_1, b_2} \frac{{b_1}^r + {b_2}^r}{{b_1}^r + {b_2}^r + 2{b_P}^r} v - b_1 - b_2,$$

As before, the agents' optimal strategy depends on r, or on the convexity/concavity of b^r.

For $r \in (0, 1)$, in any equilibrium (b_T, b_T, b_3, b_4), the FOCs of agent 1 and player 3 are

$$\begin{aligned} r b_T^{r-1} \frac{{b_3}^r + {b_4}^r}{(2{b_T}^r + {b_3}^r + {b_4}^r)^2} v &= 1, \\ r b_3^{r-1} \frac{{b_T}^r + {b_4}^r}{(2{b_T}^r + {b_3}^r + {b_4}^r)^2} v &= 1. \end{aligned}$$

With $\lambda_3 := \frac{b_3}{b_T}, \lambda_4 := \frac{b_4}{b_T}$, by dividing the corresponding FOCs, we obtain two equations, both of which can be rewritten as the intersection of two functions of λ_4

$$\lambda_4^{r-1} = \frac{\lambda_3^r + \lambda_4^r}{2 + \lambda_3^r} \quad \Rightarrow \quad \lambda_4^r = (2 + \lambda_3^r)\lambda_4^{r-1} - \lambda_3^r, \tag{9}$$

$$\lambda_3^{r-1} = \frac{\lambda_3^r + \lambda_4^r}{2 + \lambda_4^r} \quad \Rightarrow \quad \lambda_4^r = \frac{\lambda_3^{r-1}(2 - \lambda_3)}{(1 - \lambda_3^{r-1})}. \tag{10}$$

Observe that for $\lambda_4 > 0$, we have $\lambda_3 \in (1,2)$ and symmetrically $\lambda_4 \in (1,2)$. The reasoning is similar to that in the elimination contest. WLOG, assume $\lambda_3 \leq \lambda_4$. The case of $\lambda_3 = \lambda_4$ leads to the unique equilibrium in Proposition 8. Suppose there is an equilibrium with $\lambda_3 < \lambda_4$. Then $\lambda_4^r > \lambda_3^r$ and (10) imply $\lambda_3^r - 2\lambda_3 + 2 > 0$. Similarly, $\lambda_4^r - 2\lambda_4 + 2 < 0$. Let ξ denote the solution of $\lambda^r - 2\lambda + 2 = 0$. Then $\lambda_3 < \xi < \lambda_4$ as x^r is concave and $2x - 2$ is increasing. However, $\lambda_3^r - 2\lambda_3 + 2 > 0$ and (9) implies $\lambda_4 < \lambda_3$, a contradiction.[32] Hence, $\lambda_3 = \lambda_4 = \xi$ is the only equilibrium, where

$$b_i = \frac{r\xi^r}{2(1+\xi^r)^2}v, \quad b_j = \frac{r\xi^{r+1}}{2(1+\xi^r)^2}v,$$

and the expected payoffs of the team (U_C) and the organizer (Π_C) are

$$U_C = \frac{1 + (1-r)\xi^r}{(1+\xi^r)^2}v, \quad \Pi_C = \frac{r\xi^r(1+\xi)}{(1+\xi^r)^2}v.$$

The equilibrium bids are optimal since the utility functions are concave in bids by $r < 1$.

For $r \geq 1$, in any equilibrium $(0, b_2, b_3, b_4)$, the contest is equivalent to a 3-player Tullock contest. The FOC of any player i given bids (b_j, b_k) from opponents j, k is

$$rb_i^{r-1}\frac{{b_j}^r + {b_k}^r}{({b_i}^r + {b_j}^r + {b_k}^r)^2}v = 1.$$

With $\lambda_{32} := \frac{b_3}{b_2}, \lambda_{42} := \frac{b_4}{b_2}$, by dividing the corresponding FOCs, we obtain two equations

$$\lambda_{32}^{r-1} = \frac{\lambda_{32}^r + \lambda_{42}^r}{1 + \lambda_{42}^r}, \quad \lambda_{42}^{r-1} = \frac{\lambda_{32}^r + \lambda_{42}^r}{1 + \lambda_{32}^r},$$

which are familiar but more complicated since $r > 1$.[33] One can verify that there are

[32] Substituting $\lambda_4 = \lambda_3$ into (9) yield $LHS < RHS$ by $\lambda_3^r - 2\lambda_3 + 2 > 0$. Then $\lambda_4 < \lambda_3$ follows from increasing λ_4^r and decreasing $(2 + \lambda_3^r)\lambda_4^{r-1} - \lambda_3^r$.

[33] The discussion is more involved as $\lambda_{42}^r = (1 + \lambda_{32}^r)\lambda_{42}^{r-1} - \lambda_{32}^r$ can have two solutions, whose locations depend on the relative size of λ_{32} and $(r-1)^{-\frac{1}{r}}$. Using $\lambda_{32} \leq \lambda_{42} \Rightarrow \lambda_{32}^r \leq \lambda_{42}^r$, we conduct a simplification from discussing whether $\lambda_{32}, \lambda_{42} > 1$ and we then obtain the three candidates.

exactly three candidates for the equilibrium: (1) $\lambda_{32} = \lambda_{42} = 1$, which leads to an equilibrium where $b_1 = 0, b_2 = b_j = \frac{2r}{9}v$ and non-negativity of payoff $\frac{3-2r}{9}v$ implies $r \in [1, \frac{3}{2}]$, (2) $\lambda_{32} = \lambda_{42} = \eta$ where $\eta > 1$ solves $\lambda^r - 2\lambda + 1 = 0$. But player 2's bid leads to a local minimum of her payoff, and (3) $\lambda_{32} < 1 = \lambda_{42}$, where player 3's bid is a local minimizer, following the same logic as in the non-collusive benchmark. Therefore only candidate (1) survives, and its optimality results from the same logic as in the benchmark.

We now establish results (i) and (ii). First, at $r = 1$, the team's expected payoff is $\frac{v}{9} < \frac{v}{8} = 2U_O$. As U_C and $\frac{3-2r}{9}v$ are continuous in r on $(0,1)$ and $(1, \frac{3}{2}]$, we have result (i), i.e., collusion hurts the agents when r is in a neighborhood around 1.

Finally, for result (ii) on revenue comparison for the organizer,

$$\Pi_S = \frac{(6-r)rv}{8}, \; \Pi_D = \frac{r}{2}v + \frac{r(2-r)\lambda^{r-1}(1+\lambda) - r}{2(\lambda^r+1)^2}v, \; \Pi_C = \begin{cases} \dfrac{r\xi^r(1+\xi)}{(1+\xi^r)^2}v, & \text{if } r < 1 \\ \dfrac{2rv}{3}, & \text{if } r \geq 1 \end{cases}.$$

Denote the cutoff for optimal seeding as $\bar{r}$, then $\bar{r} < 1$.[34] For $r > 1$, $\Pi_C > \Pi_S \Leftrightarrow \frac{2}{3} > \frac{6-r}{8} \Leftrightarrow r > \frac{2}{3}$, which always holds, i.e., we also have $\Pi_C > \Pi_D$ as $\bar{r} < 1$. Therefore, for $r > 1$ the one-shot contest is optimal. By continuity, $\exists r^*$ s.t. on $r \in (r^*, 1)$, $\Pi_C > \Pi_D$. For general $r \in (0,1)$, one can show that $\exists r^{**} \in (0, \bar{r})$ such that $\Pi_C > \Pi_S \Leftrightarrow r \in (r^{**}, 1)$.[35] Hence, the elimination contest with seeding $\{(1,3),(2,4)\}$ is optimal for $r \in (0, r^{**})$ as $r^{**} < \bar{r}$. Finally, for $(r^{**}, \bar{r})$, one can verify that $\frac{\Pi_D}{rv}$ is decreasing and $\frac{\Pi_C}{rv}$ is increasing, that $\exists r^* \in (r^{**}, \bar{r})$ such that for $r \in (r^{**}, \bar{r})$, $\frac{\Pi_D}{rv} > \frac{\Pi_C}{rv} \Leftrightarrow r \in (r^{**}, r^*)$. Hence, Π_C and Π_D have a single crossing at r^*, which establishes result (ii) in Proposition 8. □

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[34] One can verify that when $r_1 = r_2$, there is a similar single crossing. And $\bar{r} < 1$ by checking the equilibria at $r_1 = r_2 = 1$.

[35] For $r \in (0,1)$, $\Pi_C > \Pi_S \Leftrightarrow h(r) > 0$, where $h(r) := -4(2-r)\xi^2 + 4(6-r)\xi + r - 22$. Then r^{**} exists as $h(r)$ is strictly increasing and $h(0) = -18 + 36 - 22 < 0, h(1) = -16 + 40 + 1 - 22 > 0$.

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