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High-Order Mixed Current Basis Functions for Electromagnetic Scattering of Curved Surfaces

Wei Cai[1]

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We construct high-order mixed current vector basis functions on an arbitrary curved surface which can be subdivided as a union of curved triangles and quadrilaterals. The objective is to construct vector basis (a) which consists of high-order polynomials of the surface parameterization variables on triangles and quadrilaterals, (b) part of the basis will have vanishing moments on the triangles and quadrilaterals. The first property will enable us to represent the current distribution over scatter surface with much less number of unknowns and larger patches of either triangular or quadrilateral shapes. The second property will achieve what wavelet basis does on an interval, but on a more general domain, namely, a sparse matrix representation for some integral operators.

KEY WORDS: Electromagnetic scattering; high-order method; integral equation method; Galerkin method.

1. INTRODUCTION

Integral equation formulation of electromagnetic scattering of conductive surfaces is a very popular approach for many applications including the parametric extraction for IC interconnects and computer packaging simulations [Nabors and White (1991)], and antenna calculations. The main advantage of the integral formulation is its flexibility in handling very complex geometry of the scatter surface and the automatic enforcement of

[1] Department of Mathematics, University of North Carolina at Charlotte, Charlotte, North Carolina 28223; e-mail: wcai@uncc.edu.

Sommerfeld exterior decaying conditions by the construction of proper Green's functions.

To represent the current vector field over conductor's surfaces, in many cases it is important to have a vector basis to insure the continuity of the normal component of the vector field across the interfaces among adjacent elements. The RWG basis function is the most used first order basis function for engineering applications [Rao *et al.* (1982)]. In this paper, we will extend such a basis to higher order with the required continuity across element interfaces. We consider an approach which could be applied to more general surfaces which can be subdivided as a union of curved triangles and quadrilaterals. Our objective is to construct vector basis (a) which will be of high-order polynomials of the surface parameterization variables on triangles and quadrilaterals, (b) part of the basis has vanishing moments on the triangles and quadrilaterals. The first property will enable us to represent the current distribution over scatter surface with less number of unknowns and larger patches of either triangular or quadrilateral shapes. The second property will achieve what wavelet basis does on an interval, but on a more general domain, namely, a sparse matrix representation of some integral operators. The ultimate goal is to obtain a smaller and more sparse matrix representations of the integral operator on general curved scatter surfaces.

Higher order current basis functions have been attempted by Wandzura (1992), but no systematic ways are presented to derive the basis functions so higher accuracy could be insured. We will construct a new type of current basis functions on arbitrary curved triangles and quadrilaterals, part of the basis functions will have vanishing moments. Such basis functions will be able to produce a more sparse matrix for the integral operator with slowly varying kernels, say, in the case of scattering at low frequency or scatters of few wavelengths. The vanishing moments are the key property of sparse wavelet representation of integral operators [see Beylkin *et al.* (1991); Alpert (1993)].

In Section 1, we will give the main framework for deriving high-order basis functions as Sherwin and Karniadakis (1995). In Section 2, we will present the matching condition in triangle/triangle and triangle/quadrilateral matches. In Section 3, we will introduce the Dubiner's orthogonal polynomials basis [Dubiner (1991)] and then construct a new set of basis functions in three types of modes: vertex modes, edge modes and interior modes, among them the interior modes will have vanishing moments. Sherwin and Karniadakis (1995) show different kinds of modes have been used for the solution of Navier-Stokes equation. In Section 4, we will formulate the matching conditions in terms of the coefficients of the basis functions.

2. TANGENTIAL CURRENT VECTOR FIELDS ON CURVED SURFACE PATCHES

2.1. Basic Notations

Let S be a curved triangle or quadrilateral surface in $\mathbb{R}^3$ and S is parameterized by $\mathbf{x}=\mathbf{x}(u_1, u_2)$, $(u_1, u_2)\in S^0$. In the case of curved triangle, S^0 will be a standard reference triangle in Fig. 1. Meanwhile, if S is a curved quadrilateral, S^0 will be a standard reference square in Fig. 1.

Tangential vectors: Tangential vectors $\partial_i\mathbf{x}$, $i=1,2$ are defined as

$$\partial_i\mathbf{x}=\frac{\partial\mathbf{x}}{\partial u_i} \qquad i=1,2 \tag{2.1}$$

For the sake of convenience, we also define a third tangential vector

$$\partial_3\mathbf{x}=\partial_1\mathbf{x}-\partial_2\mathbf{x} \tag{2.2}$$

Normal Vector: The outer normal vector $\mathbf{n}$ on S is defined by the right handed convention as

$$\mathbf{n}=\frac{\partial_1\mathbf{x}\times\partial_2\mathbf{x}}{\|\partial_1\mathbf{x}\times\partial_2\mathbf{x}\|} \tag{2.3}$$

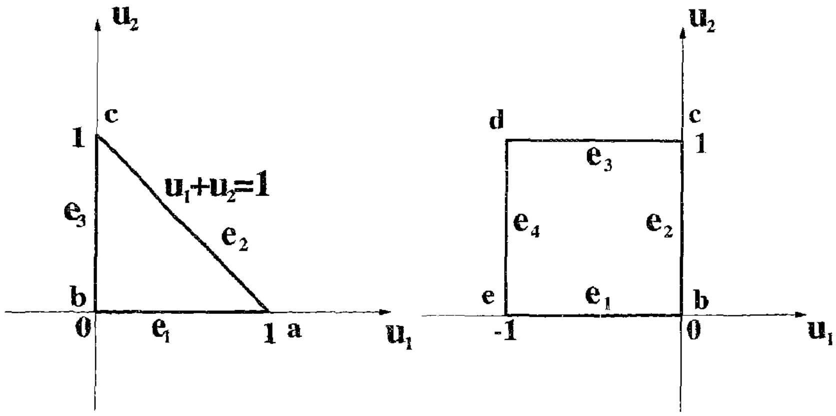

Fig. 1. Reference domain S^0.

Metric Tensor: The distance between two points on S parameterized by (u_1, u_2) and (u_1+du_1, u_2+du_2) is given by

$$(ds)^2 = g_{\mu\nu}(u)\, du_\mu\, du_\nu \tag{2.4}$$

where repeated indices imply summation

$$g_{\mu\nu} = \frac{\partial \mathbf{x}}{\partial u_\mu} \cdot \frac{\partial \mathbf{x}}{\partial u_\nu} \qquad 1 \leqslant \mu, \quad \nu \leqslant 2 \tag{2.5}$$

$\{g_{\mu\nu}\}$ is defined as the covariant tensor [Kreyszig (1991)]. The contravariant tensor $\{g^{\alpha\beta}\}$ is defined by

$$g_{\alpha\nu} g^{\nu\beta} = \delta_\alpha^\beta \tag{2.6}$$

where $\delta_\alpha^\beta = \{\begin{smallmatrix} 0 & (\alpha \neq \beta) \\ 1 & (\alpha = \beta) \end{smallmatrix}$ is the Knonecker symbol. The determinant of $\{g_{\mu\nu}\}$ is denoted by

$$g = \det\{g_{\mu\nu}\} = g_{11} g_{22} - g_{12}^2 = \|\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}\|^2 \tag{2.7}$$

Surface Element: The oriented differential surface element is given by

$$d\mathbf{S} = \partial_1 \mathbf{x} \times \partial_2 \mathbf{x}\, du_1\, du_2 \tag{2.8}$$

$$|d\mathbf{S}| = \|\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}\|\, du_1\, du_2 = \sqrt{g}\, du_1\, du_2 \tag{2.9}$$

where

$$\sqrt{g} = \|\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}\| = \sqrt{g_{11} g_{22} - g_{12}^2} \tag{2.10}$$

2.2. Tangential Vector Space $H_t(Div_s, S)$

In the MoM framework, the scatter surface S will be decomposed into a union of curved triangular or quadrilateral patches S_i parameterized by $\mathbf{x} = \mathbf{x}(u_1, u_2)$, $(u_1, u_2) \in S^0$ (Fig. 2). In each of the patch S_i, we will consider the current vector field $\mathbf{f}(u_1, u_2) = \mathbf{f}(\mathbf{x}(u_1, u_2))$ tangential to surface S in the following form

$$\mathbf{f} = M_1(u_1, u_2)\, \partial_1 \mathbf{x} + M_2(u_1, u_2)\, \partial_2 \mathbf{x}$$

where $M_1(u_1, u_2)$ and $M_2(u_1, u_2)$ are polynomials of parameterization variable u_1, u_2.

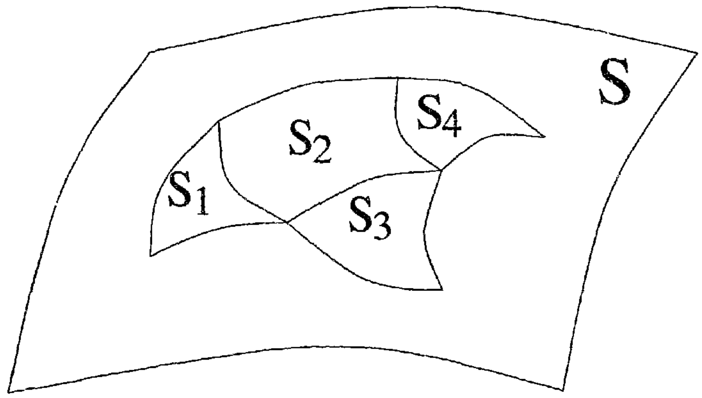

Fig. 2. Decomposition of general surface S.

The current vector field $\mathbf{f}(\mathbf{x})$ belongs to the Sobolev space $H_t(Div_s, S)$

$$H_t(Div_s, S) = \left\{ \mathbf{f}(\mathbf{x}), \mathbf{x} \in S \,\middle|\, \|\mathbf{f}\| \in L^2, \mathbf{f} \cdot \mathbf{n} = 0, \int_S |Div_s \mathbf{f}|^2 \, ds < \infty \right\} \tag{2.11}$$

where the surface divergence operator is defined as

$$Div_s \mathbf{f} = \frac{1}{\sqrt{g}} [\partial_1(\sqrt{g}\, M_1) + \partial_2(\sqrt{g}\, M_2)] \tag{2.12}$$

Lemma 1. Let $S = S_1 \cup S_2$ and $F = S_1 \cap S_2$ be the common interface and $\mathbf{t}_\perp$ is the vector perpendicular to F and tangential to S, then

$$\mathbf{f} \in H_t(Div_s, S) \Leftrightarrow \mathbf{f}|_{S_1} \cdot \mathbf{t}_\perp = \mathbf{f}|_{S_2} \cdot \mathbf{t}_\perp \qquad \text{on } F \tag{2.13}$$

Thus, the high-order current basis function to be constructed will have to satisfy condition (2.13).

2.3. Curved Triangular Patch

We can decompose $\mathbf{f}$ into a sum of 3 vector functions $1 \leqslant i \leqslant 3$ such that each function will only have non-zero normal component along only one edge, i.e.,

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 \tag{2.14}$$

where

$$\mathbf{f}_i \cdot \mathbf{t}_\perp^j = 0, \qquad \text{if} \quad i \neq j, \quad 1 \leqslant i, \quad j \leqslant 3$$

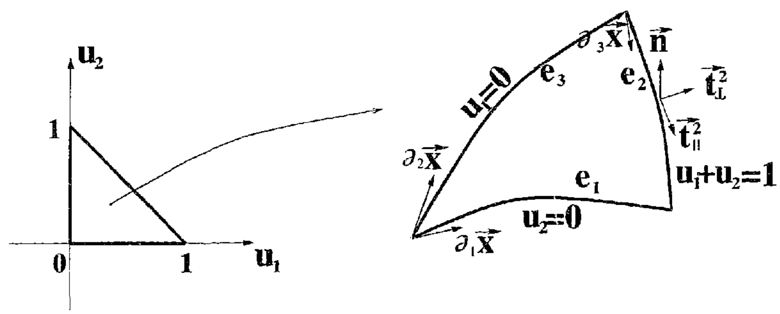

Fig. 3. Curved triangular surface.

$\mathbf{t}_{\perp}^{j}$ is the vector on edge e_j which is normal to e_j and tangential to S in Fig. 3.

We first consider the construction of $\mathbf{f}_2$ given in the following form

$$\mathbf{f}_2 = \frac{l_2}{\sqrt{g}} \left(P_{21}(u_1, u_2)\, \partial_1 \mathbf{x} + P_{22}(u_1, u_2)\, \partial_2 \mathbf{x} \right) \tag{2.15}$$

Let us assume that

e_1 is parameterized by $u_2 = 0$

e_2 is parameterized by $u_1 + u_2 = 1$

e_3 is parameterized by $u_1 = 0$

The requirement that $\mathbf{f}_2 \cdot \mathbf{t}_{\perp}^{1} = \mathbf{f}_2 \cdot \mathbf{t}_{\perp}^{3} = 0$ implies that

$$\begin{aligned} P_{21}(0, u_2) &= 0 \\ P_{22}(u_1, 0) &= 0 \end{aligned} \tag{2.16}$$

The tangential vector $\mathbf{t}_{\parallel}^{2}$ of curve e_2 is given as

$$\mathbf{t}_{\parallel}^{2} = \frac{\partial_3 \mathbf{x}}{\|\partial_3 \mathbf{x}\|} = \frac{\partial_1 \mathbf{x} - \partial_2 \mathbf{x}}{\|\partial_1 \mathbf{x} - \partial_2 \mathbf{x}\|} = \frac{\partial_1 \mathbf{x} - \partial_2 \mathbf{x}}{\sqrt{g_{11} + g_{22} - 2g_{12}}} \tag{2.17}$$

where $\sqrt{g_{11} + g_{22} - 2g_{12}}$ is the length element along e_2 and

$$l_2 = \int_0^1 \sqrt{g_{11} + g_{22} - 2g_{12}}\, du_1 \tag{2.18}$$

is the length of e_2.

$$\mathbf{t}_{\perp}^{2} = -\mathbf{t}_{\parallel}^{2} \times \mathbf{n} = -\frac{\partial_1 \mathbf{x} - \partial_2 \mathbf{x}}{\sqrt{g_{11} + g_{22} - 2g_{12}}} \times \frac{\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}}{\sqrt{g}}$$

$$= \frac{\partial_2 \mathbf{x} \times (\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}) - \partial_1 \mathbf{x} \times (\partial_1 \mathbf{x} \times \partial_2 \mathbf{x})}{\sqrt{g}\sqrt{g_{11} + g_{22} - 2g_{12}}}$$

$$= \frac{\partial_1 \mathbf{x}(g_{22} - g_{12}) + \partial_2 \mathbf{x}(g_{11} - g_{12})}{\sqrt{g}\sqrt{g_{11} + g_{22} - 2g_{12}}} \tag{2.19}$$

Therefore, the normal component of $\mathbf{f}_2$ along e_2 is

$$\mathbf{f}_2 \cdot \mathbf{t}_{\perp}^{2} = \frac{l_2}{\sqrt{g}}(P_{21}\,\partial_1 \mathbf{x} + P_{22}\,\partial_2 \mathbf{x}) \cdot \frac{\partial_1 \mathbf{x}(g_{22} - g_{12}) + \partial_2 \mathbf{x}(g_{11} - g_{12})}{\sqrt{g}\sqrt{g_{11} + g_{22} - 2g_{12}}}$$

$$= \frac{l_2}{\sqrt{g_{11} + g_{22} - 2g_{12}}}(P_{21}(u_1, u_2) + P_{22}(u_1, u_2)) \tag{2.20}$$

It is important to note that the normal component of $\mathbf{f}_2$ only depends on the geometry information of the edge e_2 itself, i.e., the length element $\sqrt{g_{11} + g_{22} - 2g_{12}}$ along e_2. This implies that a continuous matching in normal component of vector field $\mathbf{f}_2$ between two adjacent curved surface patches will be possible. This is the main idea of original work of Rao *et al.* (1982) in their first order current basis functions.

Similarly, we can consider $\mathbf{f}_1$, $\mathbf{f}_3$ as follows

$$\mathbf{f}_1 = \frac{l_1}{\sqrt{g}}(-P_{11}\,\partial_2 \mathbf{x} + P_{12}\,\partial_3 \mathbf{x}) \tag{2.21}$$

where

$$P_{11}(u_1, 1 - u_1) = 0, \qquad P_{12}(0, u_2) = 0 \tag{2.22}$$

$$\mathbf{t}_{\parallel}^{1} = \frac{\partial_1 \mathbf{x}}{\sqrt{g_{11}}} \tag{2.23}$$

$$\mathbf{t}_{\perp}^{1} = \frac{g_{12}\,\partial_1 \mathbf{x} - g_{11}\,\partial_2 \mathbf{x}}{\sqrt{g_{11}}\sqrt{g}} \tag{2.24}$$

$$\mathbf{f}_1 \cdot \mathbf{t}_{\perp}^{1} = \frac{l_1}{\sqrt{g_{11}}}(P_{11} + P_{12}) \tag{2.25}$$

and

$$\mathbf{f}_3 = \frac{l_3}{\sqrt{g}} (-P_{31}\, \partial_1 \mathbf{x} - P_{32}\, \partial_3 \mathbf{x}) \tag{2.26}$$

$$P_{31}(u_1, 1-u_1) = 0, \qquad P_{32}(u_1, 0) = 0 \tag{2.27}$$

$$\mathbf{t}_{\parallel}^3 = \frac{\partial_2 \mathbf{x}}{\sqrt{g_{22}}} \tag{2.28}$$

$$\mathbf{t}_{\perp}^3 = \frac{g_{12}\, \partial_2 \mathbf{x} - g_{22}\, \partial_1 \mathbf{x}}{\sqrt{g}\, \sqrt{g_{22}}} \tag{2.29}$$

$$\mathbf{f}_3 \cdot \mathbf{t}_{\perp}^3 = \frac{l_3}{\sqrt{g_{22}}} (P_{31} + P_{32}) \tag{2.30}$$

Plug $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ into (2.14), and using $\partial_3 \mathbf{x} = \partial_1 \mathbf{x} - \partial_2 \mathbf{x}$, we have

$$\begin{aligned} \mathbf{f} &= \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 \\ &= \frac{1}{\sqrt{g}} [(l_1 P_{12} + l_2 P_{21} - l_3 P_{31} - l_3 P_{32})\, \partial_1 \mathbf{x} \\ &\quad + (-l_1 P_{11} - l_1 P_{12} + l_2 P_{22} + l_3 P_{32})\, \partial_2 \mathbf{x}] \\ &= M_1(u_1, u_2)\, \partial_1 \mathbf{x} + M_2(u_1, u_2)\, \partial_2 \mathbf{x} \end{aligned} \tag{2.31}$$

where

$$\begin{aligned} M_1 &= \frac{1}{\sqrt{g}} (l_1 P_{12} + l_2 P_{21} - l_3 P_{31} - l_3 P_{32}) \\ M_2 &= \frac{1}{\sqrt{g}} (-l_1 P_{11} - l_1 P_{12} + l_2 P_{22} + l_3 P_{32}) \end{aligned} \tag{2.32}$$

2.4. Curved Quadrilateral Patch

Let S be a curved quadrilateral patch in Fig. 4 parameterized by $\mathbf{x} = \mathbf{x}(u_1, u_2)$ where

$$(u_1, u_2) \in S^0$$

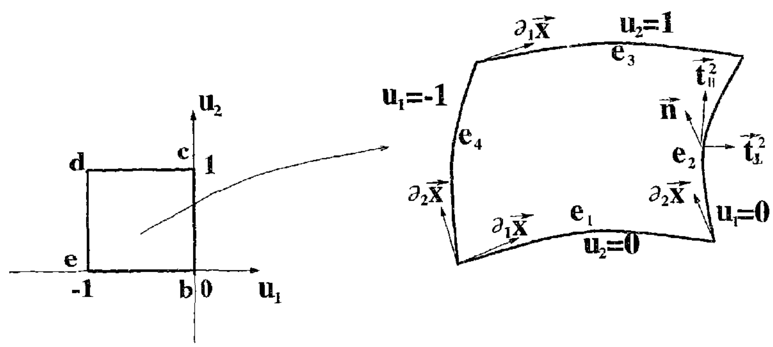

Fig. 4. Curved quadrilateral surface.

where S^0 is the reference rectangle. So on edges e_1, e_2, e_3, and e_4 we have

$$\begin{aligned} u_2 &= 0 && \text{on } e_1 \\ u_2 &= 1 && \text{on } e_3 \\ u_1 &= 0 && \text{on } e_2 \\ u_1 &= -1 && \text{on } e_4 \end{aligned} \tag{2.33}$$

and let ℓ_i denote the length of edge e_i, $1 \leqslant i \leqslant 4$.

Again, a tangential vector field on S can be decomposed into 4 components, each of the $\mathbf{f}_i$, $1 \leqslant i \leqslant 4$ only has nonzero normal components on e_i, namely,

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{f}_4 \tag{2.34}$$

$$\mathbf{f}_i \cdot \mathbf{t}_\perp^j = 0 \qquad \text{if} \quad i \neq j, \quad 1 \leqslant i, \quad j \leqslant 4 \tag{2.35}$$

Let us first consider $\mathbf{f}_2$

$$\mathbf{f}_2 = \frac{l_2}{\sqrt{g}} \left(Q_{21}(u_1, u_2)\, \partial_1 \mathbf{x} + Q_{22}(u_1, u_2)\, \partial_2 \mathbf{x} \right) \tag{2.36}$$

The requirement that $\mathbf{f}_2 \cdot \mathbf{t}_\perp^j = 0$, $1 \leqslant j \leqslant 4, j \neq 2$ implies that

$$\begin{aligned} Q_{21}(u_1, u_2) &= 0 && \text{if} \quad u_1 = -1 \\ Q_{22}(u_1, u_2) &= 0 && \text{if} \quad u_2 = 0 \quad \text{or} \quad u_2 = 1 \end{aligned} \tag{2.37}$$

Meanwhile

$$\mathbf{t}_{\parallel}^2 = \frac{\partial_2 \mathbf{x}}{\|\partial_2 \mathbf{x}\|} = \frac{\partial_2 \mathbf{x}}{\sqrt{g_{22}}} \tag{2.38}$$

$$\mathbf{n} = \frac{\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}}{\sqrt{g}}$$

$$\mathbf{t}_{\perp}^2 = \mathbf{t}_{\parallel}^2 \times \mathbf{n} = \frac{g_{22}\,\partial_1 \mathbf{x} - g_{12}\,\partial_2 \mathbf{x}}{\sqrt{g_{22}}\,\sqrt{g_{11}\,g_{22} - g_{12}^2}} \tag{2.39}$$

so

$$\mathbf{f}_2 \cdot \mathbf{t}_{\perp}^2 = \frac{l_2}{\sqrt{g_{22}}}\,Q_{21}(u_1, u_2) \tag{2.40}$$

Similarly

$$\mathbf{f}_1 = \frac{l_1}{\sqrt{g}}\,(Q_{11}(u_1, u_2)\,\partial_1 \mathbf{x} + Q_{12}(u_1, u_2)\,\partial_2 \mathbf{x}) \tag{2.41}$$

$$\begin{aligned} Q_{11} &= 0 \qquad \text{if} \quad u_1 = -1 \quad \text{or} \quad u_1 = 0 \\ Q_{12} &= 0 \qquad \text{if} \quad u_2 = 1 \end{aligned} \tag{2.42}$$

$$\mathbf{t}_{\parallel}^1 = \frac{\partial_1 \mathbf{x}}{\|\partial_1 \mathbf{x}\|} = \frac{\partial_1 \mathbf{x}}{\sqrt{g_{11}}}$$

$$\mathbf{t}_{\perp}^1 = \mathbf{t}_{\parallel}^1 \times \mathbf{n} = \frac{g_{12}\,\partial_1 \mathbf{x} - g_{11}\,\partial_2 \mathbf{x}}{\sqrt{g}\,\sqrt{g_{11}}}$$

$$\mathbf{f}_1 \cdot \mathbf{t}_{\perp}^1 = -\frac{l_1}{\sqrt{g_{11}}}\,Q_{12}(u_1, u_2)$$

and

$$\mathbf{f}_3 = \frac{l_3}{\sqrt{g}}\,(Q_{31}(u_1, u_2)\,\partial_1 \mathbf{x} + Q_{32}(u_1, u_2)\,\partial_2 \mathbf{x})$$

$$\begin{aligned} Q_{31} &= 0 \qquad \text{if} \quad u_1 = -1 \quad \text{or} \quad u_1 = 0 \\ Q_{32} &= 0 \qquad \text{if} \quad u_2 = 0 \end{aligned}$$

$$\mathbf{t}_{\|}^{3} = \frac{\partial_1 \mathbf{x}}{\sqrt{g_{11}}}$$

$$\mathbf{t}_{\perp}^{3} = \mathbf{n} \times \mathbf{t}_{\|}^{3} = -\frac{g_{12}\,\partial_1 \mathbf{x} - g_{11}\,\partial_2 \mathbf{x}}{\sqrt{g}\,\sqrt{g_{11}}}$$

$$\mathbf{f}_3 \cdot \mathbf{t}_{\perp}^{3} = \frac{l_3}{\sqrt{g_{11}}}\, Q_{32}(u_1, u_2)$$

and

$$\mathbf{f}_4 = \frac{l_4}{\sqrt{g}}\,(Q_{41}(u_1, u_2)\,\partial_1 \mathbf{x} + Q_{42}(u_1, u_2)\,\partial_2 \mathbf{x})$$

$$Q_{41} = 0 \qquad \text{if} \quad u_1 = 0$$

$$Q_{42} = 0 \qquad \text{if} \quad u_2 = 0 \quad \text{or} \quad u_2 = 1$$

$$\mathbf{t}_{\|}^{4} = \frac{\partial_2 \mathbf{x}}{\sqrt{g_{22}}}$$

$$\mathbf{t}_{\perp}^{4} = -\frac{g_{22}\,\partial_1 \mathbf{x} - g_{12}\,\partial_2 \mathbf{x}}{\sqrt{g_{22}}\,\sqrt{g}}$$

$$\mathbf{f}_4 \cdot \mathbf{t}_{\perp}^{4} = -\frac{l_4}{\sqrt{g_{22}}}\, Q_{41}(u_1, u_2)$$

3. HIGH-ORDER BASIS FUNCTIONS ON TRIANGLES AND QUADRILATERALS

3.1. Dubliner Orthogonal Polynomial Basis on Triangles

The Dubiner basis triangles is obtained by transforming Jacobian polynomials defined on intervals to form polynomials on triangles.

The nth order Jacobian polynomials $P_n^{\alpha,\beta}(x)$ on $[-1, 1]$ are orthogonal polynomials under Jacobian weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, i.e.,

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x^{\beta})\, P_l^{\alpha,\beta}(x)\, P_m^{\alpha,\beta}(x)\, dx = \delta_{lm} \tag{3.1}$$

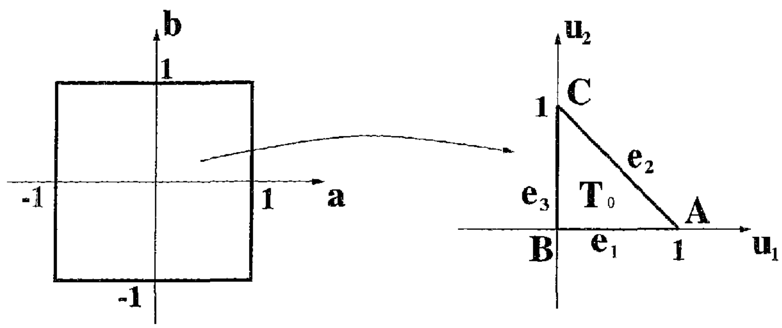

Fig. 5. Mapping between reference square and reference triangle.

To construct an orthogonal polynomial basis on a standard reference triangle T_0, we follow Dubiner (1991) and consider the mapping in Fig. 5 between the square

$$K = [-1, 1] \times [-1, 1]$$

and the triangle

$$T_0 = \left\{ (u_1, u_2) \mid 0 \leqslant \frac{u_1}{u_2} \leqslant 1, 0 \leqslant u_1 + u_2 \leqslant 1 \right\} \tag{3.2}$$

$$\begin{cases} u_1 = \dfrac{(1+a)(1-b)}{4} \\ u_2 = \dfrac{1+b}{2} \end{cases} \quad \text{or} \quad \begin{cases} a = \dfrac{2u_1}{1-u_2} - 1 \\ b = 2u_2 - 1 \end{cases} \tag{3.3}$$

The maps in (3.3) basically collapse the top edge of K into the top vertex (0, 1) of T_0. The Jacobian of the map

$$\begin{aligned} J &= \frac{\partial(a, b)}{\partial(u_1, u_2)} = \frac{4}{1-u_2} \\ J^{-1} &= \frac{\partial(u_1, u_2)}{\partial(a, b)} = \frac{1-b}{8} = \frac{1-u_2}{4} \end{aligned} \tag{3.4}$$

The Dubiner polynomial basis on T_0 is defined as

$$\begin{aligned} g_{lm}(u_1, u_2) &= P_l^{0,0}(a)(1-b)^l P_m^{2l+1,0}(b) \\ &= 2^l P_l^{0,0}\left(\frac{2u_1}{1-u_2} - 1\right)(1-u_2)^l P_m^{2l+1,0}(2u_2 - 1) \qquad (3.5) \\ (l, m) &\in P \end{aligned}$$

where

$$P = \left\{ (l, m) \quad \begin{matrix} 0 \leqslant l \leqslant L, 0 \leqslant m \leqslant M \\ 0 \leqslant l + m \leqslant M, L \leqslant M \end{matrix} \right\} \qquad (3.6)$$

We have the following properties of $\{g_{lm}\}$.

Lemma 2. $\mathscr{P}_g = \{g_{lm}(u_1, u_2)\}_{(l,m) \in P}$ forms an orthogonal polynomial set, i.e.,

$$(g_{lm}, g_{pq})_{T_0} = \tfrac{1}{2}\delta_{lp}\delta_{mq} \qquad (3.7)$$

and is complete in polynomial space $\mathscr{P}_L = \text{span}\{u_1^l u_2^m \mid (l, m) \in P\}$.

Proof. The orthogonality of (3.7) can be found [see Dubiner (1991)] and we will show the second part.

First note that

$$\text{span}\{P_l^{0,0}(x)\}_{l=0}^{L} = \text{span}\{x^l\}_{l=0}^{L}$$
$$\text{span}\{P_m^{2l+1,0}(x)\}_{m=0}^{M} = \text{span}\{x^m\}_{m=0}^{M}$$

therefore,

$$\begin{aligned} &\text{span}\{g_{lm}, (l, m) \in P\} \\ &= \text{span}\left\{P_l^{0,0}\left(\frac{2u_1 + u_2 - 1}{1 - u_2}\right)(1 - u_2)^l P_m^{2l+1,0}(2u_2 - 1)\right\}_{(l,m) \in P} \\ &= \text{span}\{(2u_1 + u_2 - 1)^l (2u_2 - 1)^m\}_{(l,m) \in P} \\ &= \text{span}\{\bar{u}_1^l \bar{u}_2^m\}_{(l,m) \in P} \end{aligned}$$

where

$$\begin{matrix} \bar{u}_1 = 2u_1 + u_2 - 1 & & u_1 = \dfrac{2\bar{u}_1 - \bar{u}_2 + 1}{4} \\ & \text{or} & \\ \bar{u}_2 = 2u_2 - 1 & & u_2 = \dfrac{1 + \bar{u}}{2} \end{matrix}$$

Let $u_1^l u_2^m \in \mathscr{P}_L$, then

$$u_1^l u_2^m = \left(\frac{2\bar{u}_1 - \bar{u}_2 + 1}{4}\right)^l \left(\frac{1 + \bar{u}_2}{2}\right)^m \tag{3.8}$$

Expanding (3.8) in terms of $\bar{u}_1$ and $\bar{u}_2$, the highest order terms are

$$\bar{u}_1^k \bar{u}_2^{m+l-k} \qquad 0 \leqslant k \leqslant l$$

and using the fact that $l + m \leqslant M$, we have for $0 \leqslant k \leqslant l$

$$\begin{aligned} \bar{u}_1^k \bar{u}_2^{m+l-k} &\in \text{span}\{\bar{u}_1^l \bar{u}_2^m\}_{(l,m) \in P} \\ &= \text{span}\{g_{lm}\}_{(l,m) \in P} \end{aligned}$$

Therefore, $\mathscr{P}_L \subset \mathscr{P}_g$.

3.2. Modified Dubiner Basis and Vanishing Moments

In this section, we will use the idea in the Dubiner's basis to construct basis functions to represent vector fields. The basis functions will be divided into three types of modes, namely, vertex modes, edge modes and interior modes. This approach has been used by Sherwin and Karniadakis (1995) in designing basis functions for solving Navier–Stokes equations. As our task is to solve integral equations of EM scattering with current vector fields as unknown, we would like to have some of the high-order basis function with vanishing moments. Referring to Fig. 5, we can define the following basis functions.

Vertex modes:

$$g_A = \frac{(1+a)}{2} \frac{(1-b)}{2} = u_1 \tag{3.9}$$

$$g_B = \frac{(1-a)}{2} \frac{(1-b)}{2} = 1 - u_1 - u_2 \tag{3.10}$$

$$g_C = \frac{1+b}{2} = u_2 \tag{3.11}$$

Note: Each of the vertex mode is a linear function and assume value 1 at one vertex and zero at other two vertices.

Edge Modes:

$$g_l^{e_1} = \frac{(1+a)}{2}\frac{(1-a)}{2} P_{l-2}^{1,1}(a)\left(\frac{1-b}{2}\right)^l$$

$$= u_1(1-u_1-u_2)(1-u_2)^{l-2} P_{l-2}^{1,1}\left(\frac{2u_1+u_2-1}{1-u_2}\right) \qquad 2 \leqslant l \leqslant L \quad (3.12)$$

$$g_m^{e_2} = \frac{(1+a)}{2}\frac{(1-b)}{2}\frac{(1+b)}{2} P_{m-2}^{1,1}(b)$$

$$= u_1 u_2 P_{m-2}^{1,1}(2u_2-1) \qquad 2 \leqslant m \leqslant M \quad (3.13)$$

$$g_m^{e_3} = \frac{(1-a)}{2}\frac{(1-b)}{2}\frac{(1+b)}{2} P_{m-2}^{1,1}(b)$$

$$= u_2(1-u_1-u_2)\, P_{m-2}^{1,1}(2u_2-1) \qquad 2 \leqslant m \leqslant M \quad (3.14)$$

Note: Each edge mode vanishes on two edges of the triangle.

Interior Modes:

$$g_{lm}^{\text{int}} = \frac{(1+a)}{2}\frac{(1-a)}{2}\frac{(1+b)}{2} P_{l-2}^{1,1}(a)\left(\frac{1-b}{2}\right)^l P_{m-1}^{l+1,1}(b)$$

$$= u_1 u_2(1-u_1-u_2)(1-u_2)^{l-2}$$

$$P_{l-2}^{1,1}\left(\frac{2u_1+u_2-1}{1-u_2}\right) P_{m-1}^{l+1,1}(2u_2-1) \quad (3.15)$$

$$(l,m) \in P' = \left\{(l,m) \;\middle|\; \begin{matrix} 2 \leqslant l \leqslant L \\ 1 \leqslant m \leqslant M \end{matrix}\; l+m \leqslant M\right\}$$

Note: Each interior mode vanishes on all edges. We have the following properties regarding these new basis functions.

Lemma 3. $\text{Span}(g_A(u_1,u_2),\ g_B(u_1,u_2),\ g_l^{e_1}(u_1,u_2),\ 2 \leqslant l \leqslant L\}|_{e_1} = \text{Span}\{u_1^l,\ u_1 \in [0,1],\ 0 \leqslant l \leqslant L\}$, $\text{Span}\{g_A(u_1,u_2),\ g_C(u_1,u_2),\ g_m^{e_2}(u_1,u_2),\ 2 \leqslant m \leqslant M\}|_{e_2} = \text{Span}\{u_1^m,\ u_1 \in [0,1],\ 0 \leqslant m \leqslant M\}$, $\text{Span}\{g_B(u_1,u_2),\ g_C(u_1,u_2),\ g_m^{e_3}(u_1,u_2),\ 2 \leqslant m \leqslant M\}|_{e_3} = \text{Span}\{u_2^m,\ u_2 \in [0,1],\ 0 \leqslant m \leqslant M\}$.

Lemma 4. $\text{Span}\{g_A,\ g_B,\ g_C,\ g_l^{e_1},\ 2 \leqslant l \leqslant L,\ g_m^{e_2},\ g_m^{e_3},\ 2 \leqslant m \leqslant M,\ g_{lm}^{\text{int}},\ (l,m) \in P'\}$ contains $\mathscr{P}_L$ of (3.6).

Proof. Let $P(u_1, u_2) \in \mathscr{P}_L$, then

$$P|_{e_1} \in \text{span}\{u_1^l\}_{l=0}^{L}$$
$$P|_{e_2} \in \text{span}\{u_1^m\}_{m=0}^{M} \qquad \text{using} \quad u_2 = 1 - u_1$$
$$P|_{e_3} \in \text{span}\{u_2^m\}_{m=0}^{M}$$

Meanwhile, from Lemma 3

$$\mathscr{P}_1 = \text{span}\{g_A, g_B, g_l^{e_1}, 2 \leqslant l \leqslant L\}|_{e_1} = \text{span}\{u_1^l\}_{l=0}^{L}$$
$$\mathscr{P}_2 = \text{span}\{g_A, g_C, g_m^{e_2}, 2 \leqslant m \leqslant M\}|_{e_2} = \text{span}\{u_1^m\}_{m=0}^{M}$$
$$\mathscr{P}_3 = \text{span}\{g_B, g_C, g_m^{e_2}, 2 \leqslant m \leqslant M\}|_{e_3} = \text{span}\{u_2^m\}_{m=0}^{M}$$

therefore, there exists $P_1 \in \mathscr{P}_1$, $P_2 \in \mathscr{P}_2$, $P_3 \in \mathscr{P}_3$ such that $\tilde{P} = P - P_1 - P_2 - P_3$ will be a linear polynomial along e_1, e_2, and e_3.

Let

$$P_4 = \tilde{P}(A)\, g_A(u_1, u_2) + \tilde{P}(B)\, g_B(u_1, u_2) + \tilde{P}(C)\, g_C(u_1, u_2)$$

then $\tilde{\tilde{P}} = \tilde{P} - P_4$ will vanish on all three edges, so we can write

$$\tilde{\tilde{P}}(u_1 u_2) = u_1 u_2 (1 - u_1 - u_2)\, Q(u_1, u_2)$$

where

$$Q(u_1 u_2) \in \text{span}\{u_1^l u_2^m, (l, m) \in P''\}$$
$$P'' = \left\{(l, m) \quad \begin{matrix} 0 \leqslant l \leqslant L-2, 0 \leqslant m \leqslant M-2 \\ 0 \leqslant l+m \leqslant M-3 \end{matrix}\right\}$$

On the other hand for $(l, m) \in P''$ following the same argument as in Lemma 2, it is easy to check,

$$\text{span}\{g_{lm}^{\text{int}}, (l, m) \in P'\} \subset \text{span}\{\bar{u}_1^l \bar{u}_2^m\}_{(l, m) \in P''} \cdot u_1 u_2 (1 - u_1 - u_2)$$
$$= \text{span}\{u_1^l u_2^m\}_{(l, m) \in P''} \cdot u_1 u_2 (1 - u_1 - u_2)$$

So, $\tilde{\tilde{P}}(u_1, u_2) \in \text{span}\{g_{lm}^{\text{int}}, (l, m) \in P'\}$.

Finally we have

$$P = \tilde{P} + P_1 + P_2 + P_3$$
$$= \tilde{\tilde{P}} + P_4 + P_1 + P_2 + P_3$$

thus the proof of the lemma. □

Theorem 1. (Vanishing moments of interior modes) For each interior mode $g_{lm}^{\text{int}}(u_1, u_2)$, $(l, m) \in P'$, we have

$$\int_{T_0} u_1^p u_2^q g_{lm}^{\text{int}}(u_1 u_2)\, du_1\, du_2 = 0$$

$$\text{if} \quad 0 \leqslant p \leqslant l-3 \quad \text{or} \quad 0 \leqslant p+q \leqslant m-2 \tag{3.16}$$

Proof.

As $\operatorname{span}\{u_1^p(1-u_2)^q\}_{0 \leqslant p \leqslant l-3,\, 0 \leqslant p+q \leqslant m-2} = \operatorname{span}\{u_1^p u_2^q\}_{0 \leqslant p \leqslant l-3,\, 0 \leqslant p+q \leqslant m-2}$

We consider

$$\int_{T_0} u_1^p(1-u_2)^q\, g_{lm}^{\text{int}}(u_1, u_2)\, du_1\, du_2$$

$$= \int_{-1}^{1} \int_{-1}^{1} \frac{(1+a)^p\,(1-b)^p}{4^p} \frac{(1-b)^q}{2^q} \frac{(1+a)}{2} \frac{(1-a)}{2} \frac{(1+b)}{2}$$

$$\times P_{l-2}^{1,1}(a) \left(\frac{1-b}{2}\right)^l P_{m-1}^{l+1,1}(b) \frac{1-b}{8}\, da\, db$$

$$= \frac{1}{2^{2p+q+6+l}} \int_{-1}^{1} (1+a)^{p+1}\,(1-a)\, P_{l-2}^{1,1}(a)\, da$$

$$\times \int_{-1}^{1} (1-b)^{p+q+l+1}\,(1+b)\, P_{m-1}^{l+1,1}(b)\, db$$

$$= \frac{1}{2^{2p+q+l+6}} I_1 I_2$$

where

$$I_1 = \int_{-1}^{1} (1-a)(1+a)(1+a)^p\, P_{l-2}^{1,1}(a)\, da$$

$$I_2 = \int_{-1}^{1} (1-b)^{l+1}\,(1+b)(1-b)^{p+q}\, P_{m-1}^{l+1,1}(b)\, db$$

From the orthogonality of Jacobian polynomial $P_m^{\alpha\beta}(x)$ we have

$$I_1 = 0 \qquad \text{if} \quad p \leqslant l-3$$

$$I_2 = 0 \qquad \text{if} \quad p+q \leqslant m-2$$

Thus the proof of the theorem.

3.3. High-Order Basis Function with Vanishing Moments on Quadrilaterals

Similarly, we will define high-order basis functions by three types of modes: vertex mode, edge mode, and interior mode. However, this is different from triangle case. The basis functions will be formed by direct tensor product of one-dimensional Jacobian polynomials.

For a reference square S^0 in Fig. 1, we define the following basis functions

Vertex Modes

$$N_B(u_1 u_2) = (1+u_1)(1-u_2) \tag{3.17}$$

$$N_C(u_1 u_2) = (1+u_1)\, u_2 \tag{3.18}$$

$$N_D(u_1 u_2) = -u_1 u_2 \tag{3.19}$$

$$N_E(u_1 u_2) = -u_1(1-u_2) \tag{3.20}$$

Edge Modes $2 \leqslant l \leqslant L$, $2 \leqslant m \leqslant M$

Side 1.

$$N_l^{e_1}(u_1, u_2) = -u_1(1+u_1)(1-u_2)\, P_{l-2}^{1,1}(2u_1+1) \tag{3.21}$$

Side 2.

$$N_m^{e_2}(u_1, u_2) = (1+u_1)\, u_2(1-u_2)\, P_{m-2}^{1,1}(2u_2-1) \tag{3.22}$$

Side 3.

$$N_l^{e_3}(u_1, u_2) = -u_1(1+u_1)\, u_2 P_{l-2}^{1,1}(2u_1+1) \tag{3.23}$$

Side 4.

$$N_m^{e_4}(u_1, u_2) = -u_1(1-u_2)\, u_2 P_{m-2}^{1,1}(2u_2-1) \tag{3.24}$$

Interior Modes $2 \leqslant l \leqslant L$, $2 \leqslant m \leqslant M$

$$N_{l,m}^{\text{int}}(u_1, u_2) = -u_1(1+u_1)\, u_2(1-u_2)\, P_{l-2}^{1,1}(2u_1+1)\, P_{m-2}^{1,1}(2u_2-1) \tag{3.25}$$

Again, we have the following lemmas regarding these basic functions.

Lemma 5. Span$\{N_B, N_C, N_D, N_E, N_l^{e_1}, N_m^{e_2}, N_l^{e_3}, N_m^{e_4}, N_{l,m}^{\text{int}}, 2 \leqslant l \leqslant L, 2 \leqslant m \leqslant L\} = \text{span}\{u_1^l u_2^m\}_{0 \leqslant l \leqslant L,\, 0 \leqslant m \leqslant M}$.

Theorem 2. (Vanishing moments of interior modes): For each interior mode $N_{lm}^{\text{int}}(u_1 u_2)$, we have

$$\iint_{S^0} u_1^p u_2^q N_{lm}^{\text{int}}(u_1 u_2)\, du_1\, u_2 = 0, \qquad \text{if} \quad p \leqslant l-3 \quad \text{or} \quad q \leqslant m-3 \tag{3.26}$$

Proof. As

$$\operatorname{span}\{(1+2u_1)^p (1-2u_2)^q\}_{0 \leqslant p \leqslant l-3,\, 0 \leqslant q \leqslant m-3}$$
$$= \operatorname{span}\{u_1^p u_2^q\}_{0 \leqslant p \leqslant l-3,\, 0 \leqslant q \leqslant m-3}$$

We consider

$$I = \iint_{S^0} (1+2u_1)^p (1-2u_2)^q N_{lm}^{\text{int}}(u_1 u_2)\, du_1\, du_2$$
$$= -\iint_{S^0} (1+2u_1)^p (1-2u_2)^q u_1(1+u_1)\, u_2(1-u_2)$$
$$\times P_{l-2}^{1,1}(2u_1+1)\, P_{m-2}^{1,1}(2u_2-1)\, du_1\, du_2$$

Let $\xi = 2u_1 + 1$, $\eta = 2u_2 - 1$

$$I = (-1)^q \int_{-1}^{1} \int_{-1}^{1} \xi^p \eta^q \frac{1}{16} (1-\xi^2)(1-\eta^2)\, P_{l-2}^{1,1}(\xi)\, P_{m-2}^{1,1}(\eta) \frac{1}{4}\, d\xi\, d\eta$$
$$= \frac{(-1)^q}{64} \int_{-1}^{1} (1-\xi)(1+\xi)\, \xi^p P_{l-2}^{1,1}(\xi)\, d\xi \int_{-1}^{1} (1-\eta)(1+\eta)\, \eta^q P_{m-2}^{1,1}(\eta)\, d\eta$$
$$= \frac{(-1)^q}{64} I_1 I_2$$

where

$$I_1 = \int_{-1}^{1} (1-\xi)(1+\xi)\, \xi^p P_{l-2}^{1,1}(\xi)\, d\xi = 0, \qquad \text{if} \quad p \leqslant l-3$$

and

$$I_2 = \int_{-1}^{1} (1-\eta)(1+\eta)\,\eta^q P_{m-2}^{1,1}(\eta)\,d\eta = 0, \qquad \text{if} \quad q \leqslant m-3$$

Therefore, $I = 0$ if $p \leqslant l-3$ or $q \leqslant m-3$.

4. HIGH-ORDER POLYNOMIAL REPRESENTATION OF CURRENT VECTORS ON CURVED SURFACES

In this section, we will assume that the surface of a scatter S has been subdivided into curved, triangular or quadrilateral patches as shown in Fig. 2. Our goal is to construct a tangential vector field using (2.31) or (2.34) with high-order polynomial coefficients P_{ij} or Q_{ij} such that the normal components across common interfaces between patches remain continuous. This requirement is needed for current fields on S so that there is no charge accumulation along interfaces.

4.1. Triangular and Triangular Patches Matching

Consider two curved triangular patches T^+ and T^- with a common interface ℓ in Fig. 6.

Let T^+ and T^- be parameterized, respectively, by

$$\mathbf{x} = \mathbf{x}^+(u_1, u_2): T_0 \to T^+ \tag{4.1}$$

$$\mathbf{x} = \mathbf{x}^-(u_1, u_2): T_0 \to T^- \tag{4.2}$$

We assume that the interface AC in both T^+ and T^- is parameterized by $u_1 + u_2 = 1$ and is labeled as side e_2^+ in T^+ and side e_2^- in T^-. The current

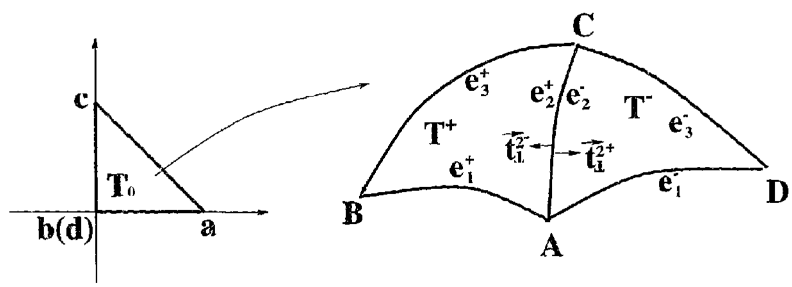

Fig. 6. Triangular/triangular patches matching.

basis function with no zero normal component along the edge AC only is defined as

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \dfrac{l}{\sqrt{g^+}}\left(P_1^+(u_1, u_2)\,\partial_1\mathbf{x} + P_2^+(u_1, u_2)\,\partial_2\mathbf{x}\right) \\ \quad \text{if} \quad \mathbf{x} = \mathbf{x}^+(u_1, u_2) \in T^+ \\ \dfrac{l}{\sqrt{g^-}}\left(P_1^-(u_1, u_2)\,\partial_1\mathbf{x} + P_2^-(u_1, u_2)\,\partial_2\mathbf{x}\right) \\ \quad \text{if} \quad \mathbf{x} = \mathbf{x}^-(u_1, u_2) \in T^- \end{cases}$$

where l is the length of edge AC.

Along AC, we have

$$\mathbf{x}^+(u_1, u_2) = \mathbf{x}^-(u_1, u_2)$$

$$\frac{d}{du_1}\mathbf{x}^+(u_1, u_2) = \frac{d}{du_1}\mathbf{x}^-(u_1, u_2)$$

$$\frac{\partial \mathbf{x}^+}{\partial u_1} - \frac{\partial \mathbf{x}^+}{\partial u_2} = \frac{\partial \mathbf{x}^-}{\partial u_1} - \frac{\partial \mathbf{x}^-}{\partial u_2}$$

$$\left\| \frac{\partial \mathbf{x}^+}{\partial u_1} - \frac{\partial \mathbf{x}^+}{\partial u_2} \right\| = \left\| \frac{\partial \mathbf{x}^-}{\partial u_1} - \frac{\partial \mathbf{x}^-}{\partial u_2} \right\|$$

In terms of the metric tensor of (2.5) we have

$$\sqrt{g_{11}^+ + g_{22}^+ - 2g_{12}^+} = \sqrt{g_{11}^- + g_{22}^- - 2g_{12}^-} \qquad \text{when} \quad u_1 + u_2 = 1 \tag{4.3}$$

According to (2.20), we have

$$\mathbf{f}\cdot\mathbf{t}_\perp^{2+} = \frac{l}{\sqrt{g_{11}^+ + g_{22}^+ - 2g_{12}^+}}\left(P_1^+ + P_2^+\right) \tag{4.4}$$

$$\mathbf{f}\cdot\mathbf{t}_\perp^{2-} = \frac{l}{\sqrt{g_{11}^- + g_{22}^- - 2g_{12}^-}}\left(P_1^- + P_2^-\right) \tag{4.5}$$

The normal components should negate each other, so we have

$$\mathbf{f}^+\cdot\mathbf{t}_\perp^{2+} = -\mathbf{f}^-\cdot\mathbf{t}_\perp^{2-} \tag{4.6}$$

And using (4.3), (4.4), and (4.5), we have the matching condition

$$P_1^+ + P_2^+ = -\left(P_1^- + P_2^-\right) \tag{4.7}$$

Now, we assume both P_k^+, P_k^-, $k=1,2$ are expressed in terms of the basis functions defined in Section 3 For $k=1,2$,

$$P_k^+(u_1,u_2) = \sum_{v_i \in \{A,B,C\}} a_{v_i}^k g_{v_i}(u_1,u_2) + \sum_{l=2}^{L} b_l^{1,k} g_l^{e_1^+}(u_1,u_2) + \sum_{m=2}^{M} b_m^{2,k} g_m^{e_2^+}(u_1,u_2) + \sum_{m=2}^{M} b_m^{3,k} g_m^{e_3^+}(u_1,u_2) + \sum_{(l,m)\in P'} c_{lm}^k g_{lm}^{\text{int}} \tag{4.8}$$

where the basis functions are defined in (3.9)–(3.15).

From (2.16) we have

$$b_m^{3,1}=0, \qquad 2\leqslant m\leqslant M \tag{4.9}$$

$$a_B^1=0, \qquad a_C^1=0 \tag{4.10}$$

$$b_l^{1,2}=0, \qquad 2\leqslant l\leqslant L \tag{4.11}$$

$$a_A^2=0, \qquad a_B^2=0 \tag{4.12}$$

Similarly we have

$$P_k^-(u_1,u_2) = \sum_{v_i \in \{A,D,C\}} \alpha_{v_i}^k g_{v_i} + \sum_{l=2}^{L} \beta_l^{1,k} g_l^{e_1^-} + \sum_{m=2}^{M} \beta_m^{2,k} g_m^{e_2^-} + \sum_{m=2}^{M} \beta_m^{3,k} g_m^{e_3^-} + \sum_{(l,m)\in P'} \gamma_{lm}^k g_{lm}^{\text{int}} \tag{4.13}$$

where the basis functions are defined in (3.9)–(3.15) (vertex mode $g_D = g_B$). Again, from (2.16) we have

$$\beta_m^{3,1}=0, \qquad 2\leqslant m\leqslant M \tag{4.14}$$

$$\alpha_D^1=0, \qquad \alpha_C^1=0 \tag{4.15}$$

$$\beta_l^{1,2}=0, \qquad 2\leqslant l\leqslant L \tag{4.16}$$

$$\alpha_A^2=0, \qquad \alpha_D^2=0 \tag{4.17}$$

The matching condition (4.7) on AC will be

Matching Condition A

Vertex Modes

$$a_A^1 = -\alpha_A^1 \tag{4.18}$$

$$a_C^2 = -\alpha_C^2 \tag{4.19}$$

Edge Modes on AC

$$\sum_{k=1}^{2}\sum_{m=2}^{M} b_m^{2,k} g_m^{e_2^+}(u_1,u_2) = -\sum_{k=1}^{2}\sum_{m=2}^{M} \beta_m^{2,k} g_m^{e_2^-}(u_1,u_2) \qquad \text{where} \quad u_1+u_2=1 \tag{4.20}$$

It can be shown that

$$g_m^{e_2^+}(u_1,u_2)=g_m^{e_2^-}(u_1,u_2) \qquad \text{where} \quad 2\leqslant m\leqslant M \quad \text{and} \quad u_1+u_2=1 \tag{4.21}$$

therefore, we have

$$b_m^{2,1}+b_m^{2,2}=-(\beta_m^{2,1}+\beta_m^{2,2}), \qquad 2\leqslant m\leqslant M \tag{4.22}$$

Finally, equations (4.18), (4.19), (4.22) form the additional constraints on the expansion coefficients of vector field in each triangle.

Next, we will give explicit formulas for the high-order current basis functions. It can be verified that the following functions P_1^+, P_2^+ and P_1^-, P_2^- will satisfy all the conditions (4.9)–(4.12), (4.14)–(4.17), (4.18), (4.19), and (4.22) (only edge modes over AC are needed as the edge basis functions are identified by their corresponding edges).

$$P_1^+(u_1,u_2)=I_n^a g_A(u_1,u_2)+\sum_{m=2}^{M}\frac{I_n^{(m)}-\tilde{I}_t^{(m)}}{2} g_m^{e_2^+}(u_1,u_2)+\sum_{(l,m)\in P'} c_{lm}^1 g_{lm}^{\text{int}}$$

$$P_2^+(u_1,u_2)=I_n^c g_C(u_1,u_2)+\sum_{m=2}^{M}\frac{I_n^{(m)}+\tilde{I}_t^{(m)}}{2} g_m^{e_2^+}(u_1,u_2)+\sum_{(l,m)\in P'} c_{lm}^2 g_{lm}^{\text{int}} \tag{4.23}$$

and coefficients P_1^-, P_2^- are given as

$$P_1^-(u_1,u_2)=-I_n^a g_A(u_1,u_2)+\sum_{m=2}^{M}\frac{-I_n^{(m)}-\hat{I}_t^{(m)}}{2} g_m^{e_2^-}(u_1,u_2) + \sum_{(l,m)\in P'} d_{lm}^1 g_{lm}^{\text{int}}$$

$$P_2^-(u_1,u_2)=-I_n^c g_C(u_1,u_2)+\sum_{m=2}^{M}\frac{-I_n^{(m)}+\hat{I}_t^{(m)}}{2} g_m^{e_2^-}(u_1,u_2) + \sum_{(l,m)\in P'} d_{lm}^2 g_{lm}^{\text{int}} \tag{4.24}$$

Unknowns for each edge AC are

$$I_n^a,\ I_n^c,\ I_n^{(m)},\ \tilde{I}_t^{(m)},\ \hat{I}_t^{(m)}, \qquad 2\leqslant m\leqslant M \tag{4.25}$$

and

$$c^1_{lm},\ c^2_{lm},\ d^1_{lm},{}^2_{lm}, \qquad (l, m) \in P' \tag{4.26}$$

- RWG Basis

If we assume that the normal component of the current basis function remains constant, i.e.,

$$I^a_n = I^c_n = I_n \tag{4.27}$$

and

$$\mathbf{f}(\mathbf{x}) = I_n \begin{cases} \dfrac{l}{\sqrt{g^+}} (g_A(u_1, u_2)\, \partial_1 \mathbf{x} + g_C(u_1, u_2)\, \partial_2 \mathbf{x}) \\ \qquad \text{if} \quad \mathbf{x} = \mathbf{x}^+(u_1, u_2) \in T^+ \\ \dfrac{l}{\sqrt{g^-}} (-g_A(u_1, u_2)\, \partial_1 \mathbf{x} - g_C(u_1, u_2)\, \partial_2 \mathbf{x}) \\ \qquad \text{if} \quad \mathbf{x} = \mathbf{x}^-(u_1, u_2) \in T^- \end{cases} \tag{4.28}$$

and for flat triangle patches, we have in T^+

$$\mathbf{x} = \mathbf{x}^+(u_1, u_2) = g_A(u_1, u_2)\, \mathbf{x}_A + g_B(u_1, u_2)\, \mathbf{x}_B + g_C(u_1, u_2)\, \mathbf{x}_C \tag{4.29}$$

$$\begin{aligned} \partial_1 \mathbf{x} &= \mathbf{x}_A - \mathbf{x}_B \\ \partial_2 \mathbf{x} &= \mathbf{x}_C - \mathbf{x}_B \end{aligned} \tag{4.30}$$

and in T^-

$$\mathbf{x} = \mathbf{x}^-(u_1, u_2) = g_A(u_1, u_2)\, \mathbf{x}_A + g_D(u_1, u_2)\, \mathbf{x}_D + g_C(u_1, u_2)\, \mathbf{x}_C \tag{4.31}$$

where $g_D(u_1, u_2) = g_B(u_1, u_2)$,

$$\begin{aligned} \partial_1 \mathbf{x} &= \mathbf{x}_A - \mathbf{x}_D \\ \partial_2 \mathbf{x} &= \mathbf{x}_C - \mathbf{x}_D \end{aligned} \tag{4.32}$$

thus, we have the RWG basis function

$$\mathbf{f}(\mathbf{x}) = I_n \begin{cases} \dfrac{l}{2A^+} (\mathbf{x} - \mathbf{x}_B) & \text{if} \quad \mathbf{x} = \mathbf{x}^+(u_1, u_2) \in T^+ \\ -\dfrac{l}{2A^-} (\mathbf{x} - \mathbf{x}_D) & \text{if} \quad \mathbf{x} = \mathbf{x}^-(u_1, u_2) \in T^- \end{cases} \tag{4.33}$$

where A^+ and A^- are the areas of triangles T^+ and T^-, respectively.

- First Order Basis

In this case, we allow the normal component of the current basis function to vary along the edge

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \dfrac{l}{\sqrt{g^+}} \left(I_n^a g_A(u_1, u_2)\, \partial_1 \mathbf{x} + I_n^c g_C(u_1, u_2)\, \partial_2 \mathbf{x} \right) \\ \quad \text{if} \quad \mathbf{x} = \mathbf{x}^+(u_1, u_2) \in T^+ \\ \dfrac{l}{\sqrt{g^-}} \left(-I_n^a g_A(u_1, u_2)\, \partial_1 \mathbf{x} - I_n^c g_C(u_1, u_2)\, \partial_2 \mathbf{x} \right) \\ \quad \text{if} \quad \mathbf{x} = \mathbf{x}^-(u_1, u_2) \in T^- \end{cases} \tag{4.34}$$

The unknowns for each edge AC are

$$I_n^a,\ I_n^c \tag{4.35}$$

- Second Order Basis

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \dfrac{l}{\sqrt{g^+}} \left\{ \left[I_n^a g_A(u_1, u_2) + \dfrac{I_n^{(2)} - \tilde{I}_t^{(2)}}{2}\, g_2^{e_2^+}(u_1, u_2) \right] \partial_1 \mathbf{x} \right. \\ \quad \left. + \left[I_n^c g_C(u_1, u_2) + \dfrac{I_n^{(2)} + \tilde{I}_t^{(2)}}{2}\, g_2^{e_2^+}(u_1, u_2) \right] \partial_2 \mathbf{x} \right\} \\ \quad \text{if} \quad \mathbf{x} = \mathbf{x}^+(u_1, u_2) \in T^+ \\ \dfrac{l}{\sqrt{g^-}} \left\{ \left[-I_n^a g_A(u_1, u_2) + \dfrac{-I_n^{(2)} - \hat{I}_t^{(2)}}{2}\, g_2^{e_2^-}(u_1, u_2) \right] \partial_1 \mathbf{x} \right. \\ \quad \left. + \left[-I_n^c g_C(u_1, u_2) + \dfrac{-I_n^{(2)} + \hat{I}_t^{(2)}}{2}\, g_2^{e_2^-}(u_1, u_2) \right] \partial_2 \mathbf{x} \right\} \\ \quad \text{if} \quad \mathbf{x} = \mathbf{x}^-(u_1, u_2) \in T^- \end{cases} \tag{4.36}$$

The unknowns for each edge AC are

$$I_n^a,\ I_n^c,\ I_n^{(2)},\ \hat{I}_t^{(2)},\ \tilde{I}_t^{(2)} \tag{4.37}$$

Higher order current basis function can be obtained by taking larger M, L and there are no interior mode unknowns for L, $M \leqslant 2$.

4.2. Triangular and Quadrilateral Patches Matching

Consider the patch of a curved quadrilateral patch Ω and a curved triangular patch T and they are parameterized separate by two mapping $\mathbf{x}_i(u_1, u_2)$, $i = 1, 2$, i.e.,

$$\mathbf{x}_1(u_1, u_2)\colon \Omega_0 \to \Omega \ (u_1, u_2) \in \Omega_0 = [-1, 0] \times [0, 1]$$
$$\mathbf{x}_2(u_1, u_2)\colon T_0 \to T (u_1, u_2) \in T_0 = \{(u_1, u_2) \mid 0 \leqslant u_1, u_2 \leqslant 1, 0 \leqslant u_1 + u_2 \leqslant 1\} \quad (4.38)$$

The edges of Ω and T are labeled as in Fig. 7. The common interface is BC, which is parameterized by $u_1 = 0$ and where

$$\mathbf{x}_1(0, u_2) = \mathbf{x}_2(0, u_2) \qquad 0 \leqslant u_2 \leqslant 1 \quad (4.39)$$

Therefore, along BC, we have

$$\frac{\partial \mathbf{x}_1}{\partial u_2} = \frac{\partial \mathbf{x}_2}{\partial u_2} \qquad 0 \leqslant u_2 \leqslant 1$$
$$\Rightarrow g_{22}^1 = g_{22}^2 \quad (4.40)$$

where $g_{ij}^k = \partial_i \mathbf{x}_k \cdot \partial_j \mathbf{x}_k$, $1 \leqslant k \leqslant 2$, $1 \leqslant i, j \leqslant 2$ denotes the metric tensor for Ω and T, respectively.

In order to have a vector field defined on $\Omega \cup T$ such that the normal components along BC matches, we consider the current basis function with no zero normal component along the edge BC which is defined by (2.36), (2.26) (the first index for the coefficients is ignored)

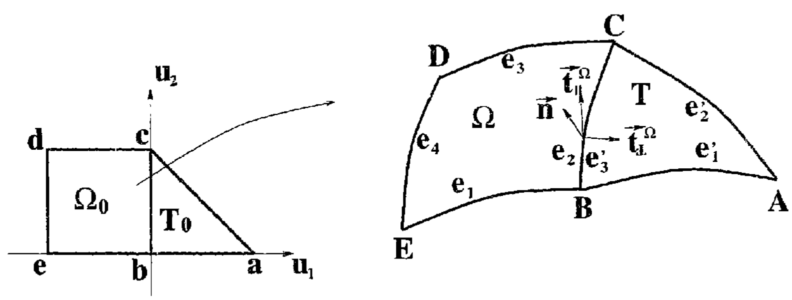

Fig. 7. Quadrilateral/triangular patch matching.

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \dfrac{l}{\sqrt{g^1}} (Q_1(u_1, u_2)\, \partial_1 \mathbf{x} + Q_2(u_1, u_2)\, \partial_2 \mathbf{x}) \\ \qquad \text{if} \quad \mathbf{x} = \mathbf{x}_1(u_1, u_2) \in \Omega \\ \quad -\dfrac{l}{\sqrt{g^2}} (P_1(u_1, u_2)\, \partial_1 \mathbf{x} + P_2(u_1, u_2)\, \partial_3 \mathbf{x}) \\ = \dfrac{l}{\sqrt{g^2}} [-(P_1 + P_2)\, \partial_1 \mathbf{x} + P_2 \partial_2 \mathbf{x}] \\ \qquad \text{if} \quad \mathbf{x} = \mathbf{x}_2(u_1, u_2) \in T \end{cases} \tag{4.41}$$

where l is the length of BC.

Thus, we have from (2.37)

$$Q_1(-1, u_2) = 0 \tag{4.42}$$

$$Q_2(u_1, 0) = 0 \tag{4.43}$$

$$Q_2(u_1, 1) = 0 \tag{4.44}$$

Similarly, we have from (2.27)

$$P_1(u_1, u_2) = 0 \qquad \text{on} \quad u_1 + u_2 = 1 \tag{4.45}$$

$$P_2(u_1, 0) = 0 \tag{4.46}$$

Let us find the normal component along common interface BC. First in Ω from (2.39), we have

$$\mathbf{t}_\perp^\Omega = \frac{g_{22}^1\, \partial_1 \mathbf{x}_1 - g_{12}^1\, \partial_2 \mathbf{x}_1}{\sqrt{g_{22}^1}\, \sqrt{g_{11}^1 g_{22}^1 - (g_{12}^1)^2}} \tag{4.47}$$

and from (2.40)

$$\mathbf{f} \cdot \mathbf{t}_\perp^\Omega = \frac{l}{\sqrt{g_{22}^1}} Q_1(u_1, u_2), \qquad u_1 = 0, \quad 0 \leqslant u_2 \leqslant 1 \tag{4.48}$$

On the other hand, in T from (2.29) we have

$$\mathbf{t}_\perp^T = -\frac{g_{22}^2\, \partial_1 \mathbf{x}_2 - g_{12}^2\, \partial_2 \mathbf{x}_2}{\sqrt{g^2}\, \sqrt{g_{22}^2}} \tag{4.49}$$

therefore, from (2.30), we have

$$\mathbf{f} \cdot \mathbf{t}_\perp^T = \frac{l}{\sqrt{g_{22}^2}} (P_1 + P_2), \qquad u_1 = 0, \quad 0 \leqslant u_2 \leqslant 1 \tag{4.50}$$

Therefore, to match the normal component we must have

$$\mathbf{f} \cdot \mathbf{t}_{\perp}^{\Omega} = -\mathbf{f} \cdot \mathbf{t}_{\perp}^{T} \tag{4.51}$$

by using (4.40) and (4.48), (4.50)

$$Q_1(0, u_2) = -[P_1(0, u_2) + P_2(0, u_2)] \tag{4.51}$$

Now by expanding $Q_i(u_1, u_2)$, $P_i(u_1, u_2)$, $1 \leqslant i \leqslant 2$ into high-order polynomial basis in Ω and T, respectively, we have in Ω (and assuming that $L = M$) for $k = 1, 2$

$$\begin{aligned} Q_k = & \sum_{v_i \in \{BCDE\}} \alpha_{v_i}^k N_{v_i}(u_1, u_2) + \sum_{s=1}^{4} \sum_{l=2}^{M} \beta_l^{s,k} N_l^{e_s}(u_1, u_2) \\ & + \sum_{l,m=2}^{M} v_{lm}^k N_{lm}^{\text{int}}(u_1, u_2) \end{aligned} \tag{4.53}$$

where the basis functions are defined in (3.17)–(3.25).

From (4.42) we have

$$\begin{aligned} \alpha_D^1 &= 0 \qquad \alpha_E^1 = 0 \\ \beta_l^{4,1} &= 0 \qquad 2 \leqslant l \leqslant M \end{aligned} \tag{4.54}$$

and from (4.43) and (4.44)

$$\alpha_B^2 = 0 \qquad \alpha_C^2 = 0 \tag{4.55}$$

$$\alpha_D^2 = 0 \qquad \alpha_E^2 = 0 \tag{4.56}$$

$$\beta_l^{1,2} = 0 \qquad \beta_l^{3,2} = 0 \qquad 2 \leqslant l \leqslant M \tag{4.57}$$

and in T we have for $k = 1, 2$

$$\begin{aligned} P_k(u_1, u_2) = & \sum_{v_i \in \{ABC\}} a_{v_i}^k g_{v_i}(u_1, u_2) \\ & + \sum_{l=2}^{L} b_l^{1,k} g_l^{e'_1}(u_1, u_2) + \sum_{m=2}^{M} b_m^{2,k} g_m^{e'_2}(u_1, u_2) \\ & + \sum_{m=2}^{M} b_m^{3,k} g_m^{e'_3}(u_1, u_2) + \sum_{(l,m) \in P'} c_{lm}^k g_{lm}^{\text{int}}(u_1, u_2) \end{aligned} \tag{4.58}$$

where the basis functions are defined in (3.9)–(3.15).

From (4.45) and (4.46), we have

$$a_A^1 = a_C^1 = 0 \tag{4.59}$$

$$b_m^{2,1} = 0, \qquad 2 \leqslant m \leqslant M \tag{4.60}$$

$$a_A^2 = a_B^2 = 0 \tag{4.61}$$

$$b_l^{1,2} = 0, \qquad 2 \leqslant l \leqslant L \tag{4.62}$$

Now along interface BC ($u_1 = 0$)

$$g_B = 1 - u_1 - u_2 = 1 - u_2 \tag{4.63}$$

$$g_C = u_2 \tag{4.64}$$

from (3.14),

$$\begin{aligned} g_m^{e_3^1} &= u_2(1 - u_1 - u_2)\, P_{m-2}^{1,1}(2u_2 - 1) \\ &= u_2(1 - u_2)\, P_{m-2}^{1,1}(2u_2 - 1) \end{aligned} \tag{4.65}$$

and

$$N_B = (1 + u_1)(1 - u_2) = 1 - u_2 \tag{4.66}$$

$$N_C = (1 + u_1)\, u_2 = u_2 \tag{4.67}$$

from (3.22)

$$N_l^{e_2} = u_2(1 - u_2)\, P_{l-2}^{1,1}(2u_2 - 1) \tag{4.68}$$

Therefore, the matching condition of (4.51) can be expressed as

Matching Condition B

Vertex Modes

$$\alpha_B^1 = -a_B^1 \tag{4.69}$$

$$\alpha_C^1 = -a_C^2 \tag{4.70}$$

Edge Modes on BC

$$\beta_l^{2,1} = -(b_l^{3,1} + b_l^{3,2}), \qquad 2 \leqslant l \leqslant M \tag{4.71}$$

It can be verified that the following functions P_1, P_2 and Q_1, Q_2 will satisfy conditions (4.54)–(4.57) for Q_1, Q_2 and (4.59)–(4.62) for P_1, P_2 and matching condition (4.69)–(4.71) (only edge modes over BC are needed)

$$Q_1(u_1, u_2) = I_n^b N_B(u_1, u_2) + I_n^c N_C(u_1, u_2) + \sum_{l=2}^{M} I_n^{(l)} N_l^{e_2}(u_1, u_2) + \sum_{2 \leqslant l, m \leqslant M} \gamma_{lm}^1 N_{lm}^{\text{int}} \tag{4.72}$$

$$Q_2(u_1, u_2) = \sum_{l=2}^{M} \hat{I}_t^{(l)} N_l^{e_2}(u_1, u_2) + \sum_{2 \leqslant l, m \leqslant M} \gamma_{lm}^2 N_{lm}^{\text{int}}$$

$$\begin{aligned} P_1(u_1, u_2) &= -I_n^b g_B(u_1, u_2) - \sum_{l=2}^{M} \frac{I_n^{(l)} - \bar{I}_t^{(l)}}{2} g_l^{e_3'}(u_1, u_2) + \sum_{(l,m) \in P'} c_{lm}^1 g_{lm}^{\text{int}} \\ P_2(u_1, u_2) &= -I_n^c g_C(u_1, u_2) - \sum_{l=2}^{M} \frac{I_n^{(l)} + \bar{I}_t^{(l)}}{2} g_l^{e_3'}(u_1, u_2) + \sum_{(l,m) \in P'} c_{lm}^2 g_{lm}^{\text{int}} \end{aligned} \tag{4.73}$$

Unknowns for each edge BC are

$$I_n^b, I_n^c, I_n^{(l)}, \bar{I}_t^{(l)}, \hat{I}_t^{(l)}, \qquad 2 \leqslant l \leqslant M \tag{4.74}$$

and

$$c_{lm}^1, c_{lm}^2, \qquad (l, m) \in P', \qquad \gamma_{lm}^1, \gamma_{lm}^2, \qquad 2 \leqslant l, \quad m \leqslant M \tag{4.75}$$

- Mixed RWG Basis

If we assume that the normal component of the current basis function remains constant, i.e.,

$$I_n^b = I_n^c = I_n \tag{4.76}$$

and

$$\mathbf{f}(\mathbf{x}) = I_n \begin{cases} \dfrac{l}{\sqrt{g^1}} [N_B(u_1, u_2) + N_C(u_1, u_2)] \, \partial_1 \mathbf{x} \\ \qquad \text{if} \quad \mathbf{x} = \mathbf{x}_1(u_1, u_2) \in \Omega \\ -\dfrac{l}{\sqrt{g^2}} [-(g_B(u_1, u_2) + g_C(u_1, u_2)) \, \partial_1 \mathbf{x} + g_C(u_1, u_2) \, \partial_2 \mathbf{x}] \\ \qquad \text{if} \quad \mathbf{x} = \mathbf{x}_2(u_1, u_2) \in T \end{cases} \tag{4.77}$$

And for flat triangle and quadrilateral patches, $\sqrt{g^2}=2A^T$, A^T denotes the area of T. In Ω

$$\begin{aligned}\partial_1 \mathbf{x}=\partial_1 \mathbf{x}_1 &= (1-u_2)(\mathbf{x}_B-\mathbf{x}_E)+u_2(\mathbf{x}_C-\mathbf{x}_D)\\ \partial_2 \mathbf{x}=\partial_2 \mathbf{x}_1 &= -u_1(\mathbf{x}_D-\mathbf{x}_E)+(1+u_1)(\mathbf{x}_C-\mathbf{x}_B)\end{aligned} \tag{4.78}$$

and in T

$$\begin{aligned}\partial_1 \mathbf{x}&=\partial_1 \mathbf{x}_2=\mathbf{x}_A-\mathbf{x}_B\\ \partial_2 \mathbf{x}&=\partial_2 \mathbf{x}_2=\mathbf{x}_C-\mathbf{x}_B\\ \partial_3 \mathbf{x}&=\partial_3 \mathbf{x}_2=\partial_1 \mathbf{x}_2-\partial_2 \mathbf{x}_2=\mathbf{x}_A-\mathbf{x}_C\end{aligned} \tag{4.79}$$

thus, we have the mixed RWG basis function

$$\mathbf{f}(\mathbf{x})=I_n\begin{cases}\dfrac{l}{\sqrt{g^1}}(1+u_1)[(1-u_2)(\mathbf{x}_B-\mathbf{x}_E)+u_2(\mathbf{x}_C-\mathbf{x}_D)]\\ \qquad \text{if} \quad \mathbf{x}=\mathbf{x}_1(u_1,u_2)\in\Omega\\ -\dfrac{l}{2A^T}(\mathbf{x}-\mathbf{x}_A)\\ \qquad \text{if} \quad \mathbf{x}=\mathbf{x}_2(u_1,u_2)\in T\end{cases} \tag{4.80}$$

- First Order Basis

$$\mathbf{f}(\mathbf{x})=\begin{cases}\dfrac{l}{\sqrt{g^1}}[I_n^b N_B(u_1,u_2)+I_n^c N_C(u_1,u_2)]\,\partial_1\mathbf{x}\\ \qquad \text{if} \quad \mathbf{x}=\mathbf{x}_1(u_1,u_2)\in\Omega\\ \dfrac{l}{\sqrt{g^2}}[I_n^b g_B(u_1,u_2)\,\partial_1\mathbf{x}+I_n^c g_C(u_1,u_2)\,\partial_3\mathbf{x}]\\ \qquad \text{if} \quad \mathbf{x}=\mathbf{x}_2(u_1,u_2)\in T\end{cases} \tag{4.81}$$

The unknowns for each edge BC are

$$I_n^b,\ I_n^c \tag{4.82}$$

- Second Order Basis

$$
\mathbf{f}(\mathbf{x}) = \begin{cases} \dfrac{l}{\sqrt{g^1}} \{ [I_n^b N_B(u_1, u_2) + I_n^c N_C(u_1, u_2) + I_n^{(2)} N_2^{e_2}(u_1, u_2) \\ \quad + \gamma_{22}^1 N_{22}^{\text{int}}(u_1, u_2)] \, \partial_1 \mathbf{x} + [\hat{I}_t^{(2)} N_2^{e_2}(u_1, u_2) + \gamma_{22}^2 N_{22}^{\text{int}}(u_1, u_2)] \, \partial_2 \mathbf{x} \} \\ \qquad \text{if} \quad \mathbf{x} = \mathbf{x}_1(u_1, u_2) \in \Omega \\ \dfrac{l}{\sqrt{g^2}} \left\{ \left[I_n^b g_B(u_1, u_2) + \dfrac{I_n^{(2)} - \tilde{I}_t^{(2)}}{2} g_2^{e'_3}(u_1, u_2) \right] \partial_1 \mathbf{x} \right. \\ \quad \left. + \left[I_n^c g_C(u_1, u_2) + \dfrac{I_n^{(2)} + \tilde{I}_t^{(2)}}{2} g_2^{e'_3}(u_1, u_2) \right] \partial_3 \mathbf{x} \right\} \\ \qquad \text{if} \quad \mathbf{x} = \mathbf{x}_2(u_1, u_2) \in T \end{cases} \tag{4.83}
$$

The unknowns for each edge BC are

$$
I_n^b, I_n^c, I_n^{(2)}, \hat{I}_t^{(2)}, \tilde{I}_t^{(2)}, \gamma_{22}^1, \gamma_{22}^2 \tag{4.84}
$$

while the interior unknowns γ_{22}^1, γ_{22}^2 are associated with each quadrilateral.

Higher order current basis function can be obtained by taking larger M and there are no interior mode unknowns for $M \leqslant 2$ for T.

5. CONCLUSIONS

In this paper, we have presented the construction of higher order polynomial basis for current vector field on arbitrary curved surfaces. The vector basis functions are constructed for curved surfaces made of triangular and/or quadrilateral patches and the current flow continuously along the normal directions of common interfaces between triangle/triangle or triangle/quadrilateral patches, so the resulting current vector field will belong to the Sobolev space $H_t(Div_s, S)$. The framework for the mixed high-order current basis functions can be used with other kind of mode basis functions such as the hierarchic basis functions defined by Szabo and Babuska (1991).

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