AN EXACT ABSORBING BOUNDARY CONDITION FOR 2-D HELMHOLTZ EQUATIONS IN LAYERED MEDIA

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Abstract. In this paper we will present an exact absorbing boundary condition for solving 2-D exterior Helmholtz equation in a unbounded domain. The proposed boundary condition is based on the Dirichlet-to-Neumann mapping for the exterior strips surrounding the boundary of the rectangular computational domain. After the variables along the half-line interfaces are eliminated in terms of the solutions on the computational boundary, an exact boundary condition is obtained for the formulation of a finite element method of the exterior Helmholtz problem over the finite computational domain.

1. Introduction

For many scattering applications including the resonant waveguide, diffractive optics application such as the electromagnetic cavity problem and guided mode grating resonance filters and photonic crystal patterned semiconductor waveguide, we need to solve scattering of inhomogeneous scatter embedded in layered media. In order to apply finite difference or finite element methods, the computational domain needs to be truncated to a minimum finite region around the scatter. Therefore, an absorbing boundary condition is needed on the truncated computational domain boundary so no artificial wave reflection occurs there.

2000 *Mathematics Subject Classification.* 78A45.

The authors would like to thank the support of National Science Foundation (grant number: CCR-9972251, CCR-9988375, CCR-0098140 and CCR-0098275) for the work reported in this paper.

Many researches have been done for the case where the scatter is embedded in a homogeneous medium [1]-[4]. In this paper, we will present a new absorbing boundary condition for scattering embedded in layered media.

Waves propagate in the layered media in various modes such as guided modes, leaky and surface waves, and Rayleigh waves [6]. The Sommerfeld radiation condition [5] is replaced with different kinds of decay condition such as the upward going (or downward going) conditions [6] [8] . The formulation of boundary condition should accurately consider the effects of all these waves to arrive at the correct absorbing boundary condition.

2. The scattering problem

Consider a perfect conductor Ω_0 embedded in a layered medium. For simplicity, we assume that the medium has only three layers and that Ω_0 lies in the rectangle $D = \{x : \ -a < x_1 < a, \ \ 0 < x_2 < d\}$ for some positive constants a and d (see Figure 1). Then the dielectric constant $\epsilon(x)$ has the following form:

$$\epsilon(x) = \begin{cases} \epsilon_1, & x_2 > d, \\ \epsilon_2, & 0 < x_2 < d, \quad \text{and} \quad x = (x,x_2) \notin \Omega_0 \\ \epsilon_3, & x_2 < 0. \end{cases}$$

In this paper we only consider the transverse magnetic wave, i.e., the electric field $E = (0, 0, u(x_1, x_2))$. Let a plane wave $E_I = (0, 0, u_I)$ incident on $x_2 = d$, where $u_I = e^{i(\alpha x_1 - \beta x_2)}$, $\alpha = k_1 \sin(\theta)$, $\beta = k_1 \cos(\theta)$, and $\theta \in (-\pi/2, \pi/2)$, $k_1 = \omega\sqrt{\epsilon_1 \mu}$, where the permeability μ is assumed to be a constant. Then u solves the following Helmholtz equation derived from Maxwell's system of equations

$$\Delta u + k^2(x)u = 0 \text{ in } R^2 \backslash \overline{\Omega}_0 \qquad (2.1)$$

with the boundary condition

$$u = 0 \text{ on } \Gamma_0, \tag{2.2}$$

where $k^2(x) = \omega^2\mu\epsilon(x)$ with $\Re(k) > 0$ and $\Im(k) \geq 0$, Γ_0 is the boundary of Ω_0.

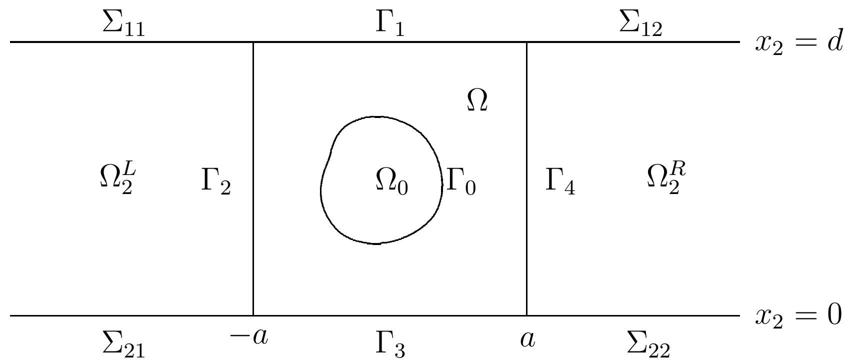

Figure 1. The geometry of the problem

In order to determine the physical solution, radiation conditions as x_2 goes to infinity have to be imposed on the scattered field. Here we shall adopt the radiation condition proposed in [8] [9]. Notice that the scattering field in the half planes $\Omega_1 = \{x : \ x_2 > d\}$ and $\Omega_3 = \{x : \ x_2 < 0\}$ are $u_s = u - u_I$ and u, respectively. We require that u satisfies the following radiation conditions:

$$u_S(x) = 2\int_{\Sigma_1} \frac{\partial G(k_1, x, y)}{\partial n(y)} u_S(y) ds(y), \quad x \in \Omega_1, \tag{2.3}$$

$$u(x) = 2\int_{\Sigma_2} \frac{\partial G(k_3, x, y)}{\partial n(y)} u(y) ds(y), \quad x \in \Omega_3, \tag{2.4}$$

where $\Sigma_1 = \{x : \ x_2 = d\}$ and $\Sigma_2 = \{x : \ x_2 = 0\}$, $n(x)$ is the upward or downward unit normal vector to Σ_1 or Σ_2 at x, and $G(k, x, y)$ is the fundamental solution of the Helmholtz equation with wave number k in R^2, i.e.,

$$G(k, x, y) = -\frac{i}{4} H_0^{(1)}(k|x - y|), \quad x \in R^2.$$

It follows from Remark 2.4 of [9] that the reflected filed $u_R = -e^{i(\alpha x_1+\beta(x_2-2d))}$ in Ω_1 has the following integral expression:

$$u_R(x) = -\int_{\Sigma_1} \frac{\partial G_1(x,y)}{\partial n(y)} u_R(y)ds(y) = -\int_{\Sigma_1} \frac{\partial G_1(x,y)}{\partial n(y)} u_I(y)ds(y), \quad x \in \Omega_1.$$

Hence, (2.3) becomes

$$u(x) = u_I(x) + u_R(x) + 2\int_{\Sigma_1} \frac{\partial G(k_1,x,y)}{\partial n(y)} u(y)ds(y), \quad x \in \Omega_1. \tag{2.5}$$

By taking normal derivatives in (2.5) and (2.4), we get the following radiation boundary conditions on Σ_1 and Σ_2:

$$\frac{\partial u(x)}{\partial n(x)} = 2\int_{\Sigma_1} \frac{\partial^2 G(k_1,x,y)}{\partial n(x)\partial n(y)} u(y)ds(y) + g(x), \quad x \in \Sigma_1, \tag{2.6}$$

$$\frac{\partial u(x)}{\partial n(x)} = 2\int_{\Sigma_2} \frac{\partial^2 G(k_3,x,y)}{\partial n(x)\partial n(y)} u(y)ds(y), \quad x \in \Sigma_2, \tag{2.7}$$

where $g(x) = -2i\beta e^{i(\alpha x_1-\beta d)}$. The above two equations are known as the Dirichlet-Neumann mapping. The hypersingular integral operators should be understood in the distribution sense. Now we can formulate our scattering problem as:

$$\begin{cases} \textit{Find } u \textit{ satisfies the Helmholtz equation (2.1) in } \{x:\ 0<x_2<d \textit{ and } x\notin\Omega_0\}, \\ \textit{the boundary condition (2.2) and the radiation boundary conditions (2.3) and} \\ \textit{(2.4).} \end{cases} \tag{2.8}$$

3. The problem on the bounded domain

In this section we shall reformulate the unbounded problem (2.8) into a problem on the bounded domain $\Omega = D\backslash\Omega_0$. To this end, we need to propose boundary conditions on Γ_j for $j = 1,\ 2,\ 3,\ 4$ (see Figure 1), which, together with Γ_0, forms the boundary of the domain Ω.

As shown in Figure 1, we shall use the following notations:

$$\Sigma_{11} = \{x:\ x_2 = d,\ x_1 \le -a\},\quad \Sigma_{12} = \{x:\ x_2 = d,\ x_1 \ge a\},$$

$$\Sigma_{21} = \{x:\ x_2 = 0,\ x_1 \le -a\},\quad \Sigma_{22} = \{x:\ x_2 = 0,\ x_1 \ge a\},$$

$$\Gamma_1 = \{x:\ x_2 = d,\ -a \le x_1 \le a\},\quad \Gamma_2 = \{x:\ x_1 = -a,\ 0 \le y \le d\},$$

$$\Gamma_3 = \{x:\ x_2 = 0,\ -a \le x_1 \le a\},\quad \Gamma_4 = \{x:\ x_1 = a,\ 0 \le y \le d\},$$

$$\Omega_2^R = \{x:\ x_1 > a,\ 0 < x_2 < d\},\quad \Omega_2^L = \{x:\ x_1 < -a,\ 0 < x_2 < d\}.$$

Define

$$N_{1j}\phi(x) = 2\int_{\Sigma_{1j}} \frac{\partial^2 G(k_1, x, y)}{\partial n(x)\partial n(y)}\phi(y)ds(y),\quad j = 1, 2,$$
$$N_1\phi(x) = 2\int_{\Gamma_1} \frac{\partial^2 G(k_1, x, y)}{\partial n(x)\partial n(y)}\phi(y)ds(y),$$
$$N_{2j}\phi(x) = 2\int_{\Sigma_{2j}} \frac{\partial^2 G(k_3, x, y)}{\partial n(x)\partial n(y)}\phi(y)ds(y),\quad j = 1, 2,$$
$$N_3\phi(x) = 2\int_{\Gamma_3} \frac{\partial^2 G(k_3, x, y)}{\partial n(x)\partial n(y)}\phi(y)ds(y).$$

Denote by $p_{ij}(x)$ the trace of $u(x)$ on Σ_{ij} for $i = 1, 2,\ j = 1, 2$. Then (2.6) and (2.7) can be rewritten as

$$\frac{\partial u(x)}{\partial n(x)} = N_{11}p_{11}(x) + N_1 u(x) + N_{12}p_{12}(x) + g,\quad x \in \Sigma_1, \tag{3.1}$$

$$\frac{\partial u(x)}{\partial n(x)} = N_{21}p_{21}(x) + N_3 u(x) + N_{22}p_{22}(x),\quad x \in \Sigma_2. \tag{3.2}$$

Now we study the solution in Ω_2^R. Let $G_0(x, y)$ be the Dirichlet Green's function for the Helmholtz equation in $\Omega_2 = \{x:\ 0 < x_2 < d\}$. Then

$$G^R(x, y) = G_0(x, y) - G_0(x', y)$$

is the Dirichlet Green's function in Ω_2^R, where $x' = (2R - x_1, x_2)$. Let $\nu(x)$ be the outward unit normal vector to the boundary Γ_2^R of Ω_2^R at x. It is not difficult to get following integral representation of $u(x)$ in Ω_2^R:

$$u(x) = -\int_{\Gamma_2^R} \frac{\partial G_2^R(x,y)}{\partial \nu(y)} u(y) ds(y), \quad x \in \Omega_2^R.$$

By taking normal derivative on Γ_2^R, we get

$$\frac{\partial u}{\partial n} = N_{12}^* p_{12} + N_{22}^* p_{22} + N_4 u, \quad x \in \Gamma_2^R, \tag{3.3}$$

where

$$N_4\phi(x) = -\int_{\Gamma_4} \frac{\partial^2 G^R(x,y)}{\partial n(x) \partial n(y)} \phi(y) ds(y), \quad N_{i2}^*\phi(x) = -\int_{\Sigma_{i2}} \frac{\partial^2 G^R(x,y)}{\partial n(x) \partial n(y)} \phi(y) ds(y), \; i = 1, 2.$$

Similarly, we have the integral representation of $u(x)$ in Ω_2^L and the integral equation on the boundary Γ_2^L of Ω_2^L:

$$u(x) = -\int_{\Gamma_2^L} \frac{\partial G^L(x,y)}{\partial n(y)} u(y) ds(y), \quad x \in \Omega_2^L, \tag{3.4}$$

$$\frac{\partial u}{\partial n} = N_{11}^* p_{11} + N_{21}^* p_{21} + N_2 u, \quad x \in \Gamma_2^L. \tag{3.5}$$

Combining (3.1), (3.2), (3.3) and (3.5), we get

$$N_{11} p_{11}(x) + N_1 u(x) + N_{12} p_{12}(x) + g = N_{11}^* p_{11} + N_{21}^* p_{21} + N_2 u, \quad x \in \Sigma_{11},$$

$$N_{11} p_{11}(x) + N_1 u(x) + N_{12} p_{12}(x) + g = N_{12}^* p_{12} + N_{22}^* p_{22} + N_4 u, \quad x \in \Sigma_{12},$$

$$N_{21} p_{21}(x) + N_3 u(x) + N_{22} p_{22}(x) = N_{11}^* p_{11} + N_{21}^* p_{21} + N_2 u, \quad x \in \Sigma_{21},$$

$$N_{21} p_{21}(x) + N_3 u(x) + N_{22} p_{22}(x) = N_{12}^* p_{12} + N_{22}^* p_{22} + N_4 u, \quad x \in \Sigma_{22}.$$

The above equations can be written into the following matrix form:

$$\mathcal{N} p = \sigma u + f \tag{3.6}$$

where $p = (p_{11}, p_{12}, p_{21}, p_{22})^T$, $f = (g, g, 0, 0)^T$ and

$$\mathcal{N} = \begin{pmatrix} N_{11} - N_{11}^* & N_{12} & -N_{21}^* & 0 \\ N_{11} & N_{12} - N_{12}^* & 0 & -N_{22}^* \\ -N_{11}^* & 0 & N_{21} - N_{21}^* & N_{22} \\ 0 & -N_{12}^* & N_{21} & N_{22} - N_{22}^* \end{pmatrix}, \quad \sigma = \begin{pmatrix} N_2 - N_1 \\ N_4 - N_1 \\ N_2 - N_3 \\ N_4 - N_3 \end{pmatrix}.$$

Then, assuming that $\mathcal{N}$ is invertible, we have

$$p = \mathcal{N}^{-1}(\sigma u + f) \equiv T_0(u, f).$$

Let

$$T_1(p, u) = N_{11} p_{11} + N_{12} p_{12} + N_1 u, \quad T_2(p, u) = N_{11}^* p_{11} + N_{21}^* p_{21} + N_2 u,$$

$$T_3(p, u) = N_{21} p_{21} + N_{22} p_{22} + N_3 u, \quad T_4(p, u) = N_{12}^* p_{12} + N_{22}^* p_{22} + N_4 u.$$

We have from (3.1), (3.2), (3.3) and (3.5) that

$$\begin{aligned} \frac{\partial u}{\partial n} &= T_1(T_0(u, f), u) + g \equiv \mathcal{T}_1 u + b_1, \quad x \in \Gamma_1, \\ \frac{\partial u}{\partial n} &= T_2(T_0(u, f), u) \equiv \mathcal{T}_2 u + b_2, \quad x \in \Gamma_2, \\ \frac{\partial u}{\partial n} &= T_3(T_0(u, f), u) \equiv \mathcal{T}_3 u + b_3, \quad x \in \Gamma_3, \\ \frac{\partial u}{\partial n} &= T_4(T_0(u, f), u) \equiv \mathcal{T}_4 u + b_4, \quad x \in \Gamma_4 \end{aligned}$$

where $b_1 = b_2 = g, b_3 = b_4 = 0$.

Finally, we have the following problem on the bounded domain Ω with the nonlocal boundary conditions on $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$:

$$\Delta u + k^2(x)u = 0, \quad x \in \Omega,$$

$$u = 0 \quad x \in \Gamma_0,$$

$$\frac{\partial u}{\partial n} = \mathcal{T}u + b, \quad x \in \Gamma,$$

where the subscript j was omitted.

4. Finite element method

For positive integer m, let $H_E^m(\Omega)$ be the closure of $C_E^\infty(\Omega) = \{u \in C^\infty(\Omega) : \ u(x) = 0 \text{ on } \Gamma_0\}$ with respect to the norm $\|\cdot\|_{m,\Omega}$ of Sobolev space $H^m(\Omega)$ ([7], [10]). As usual, We use $(\cdot,\cdot)$ to denote the inner product on $L^2(\Omega)(= H^0(\Omega))$. We denote by $\langle\cdot,\cdot\rangle$ the dual product between $H^{-s}(\Gamma)$ and $H^s(\Gamma)$ and the inner product on $L^2(\Gamma)$.

The variation form of problem (3.7)–(3.7) is: Find $u \in H_E^1(\Omega)$ such that

$$a(u,v) = F(v), \quad \forall v \in H_E^1(\Omega), \tag{4.1}$$

where

$$a(u,v) = (\nabla u \cdot \nabla v) - k_0^2(u,v) - \langle \mathcal{T}u, v\rangle, \quad F(v) = \langle b, v\rangle.$$

Let V_h be a finite element space in $H_E^1(\Omega)$ with meshsize h. Then the finite element approximation of (4.1) is: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in V_h.$$

To implement the above finite element, we need to find the the inverse of $\mathcal{N}$ numerically through solving system (3.6). The study in this aspect is undergoing.

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