ITERATED FUNDAMENTAL SOLUTION SIMULATIONS OF SECTIONS OF CIRCULAR MEMBRANES

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Abstract

We propose iterating the fundamental solution of the 2-D wave equation to achieve a simulation of sections of a circular drum-head. The main advantage of this approach is the structural preservation of all features of the fundamental solution. The method uses the time-domain, hence displaying both stationary as well as evolutionary behavior immediately and is intended to be the conceptual 2-D analogue case of 1-D McIntyre-Schumacher-Woodhouse or waveguide-type time-domain simulations. These also preserve the structure of the fundamental solution, which in the 1-D case corresponds to traveling impulses and steps.

Simulations show the form and scale of the well-known wake following disturbances of an initial excitation, but also further, rarely discussed features like a decaying residue in the neighborhood of a displacement excitation singularity which do not occur near velocity excitations.

The method solely relies on evaluation of functions and features of the solution can be resolved up to numerical precision. The method is generalizable to any domain-shape with known reflection behavior but is not computationally efficient. The performance degrades linearly with the order of reflections and hence this method in its current form is not useful for long-time simulation of excitations or real-time sound synthesis.

INTRODUCTION

The wave equation in two dimensions has a more intricate solution than the counterparts in one and three dimensions. The reason for this is the general and rather peculiar insight in geometry, functional analysis and differential equations, that geometry in odd dimensions behaves qualitatively different than in even dimensions [Courant & Hilbert, 1968, Stein & Shakarchi, 2003].

In the case of the wave equation in the plane, the result of this difference manifests itself in various ways. One is that Huygens principle holds in three dimensions but not in two [Stein & Shakarchi, 2003]. A related result in analysis is that solutions of the wave equation in one and three dimensions can form so-called lacunae, which are sharp fronts and no front behind them, but that this is not the case in

two dimensions [Atiyah et al., 1970, Atiyah et al., 1973]. In general there are two sets of solutions, those that hold for all odd dimensional wave equations and those that hold for even dimensional ones [Courant & Hilbert, 1968].

The structure of the solution of the one-dimensional wave equation is ultimately the reason for the success of time-domain methods like the McIntyre-Schumacher-Woodhouse method [McIntyre et al., 1983] or the digital waveguide synthesis approach [Smith, 1997]. The solution has a particularly simple form that makes extremely efficient and stable simulations of arbitrary excitation functions feasible. In detail, there are two parts to the simplicity of the structure of the solution of the one-dimensional wave equation. One is geometric and one is functional.

The geometric part refers to the way reflections factor into the complexity of the geometry of the evolution equations. In one dimensions the reflection does not carry any additional complications with it. Reflected disturbances stay on the same confined domain and a single closed loop can represent all of the data. In two dimensions the geometry of wave fronts generally will form self-intersections as well as singularities, which are semi-cubical cusps. However it has been show [Arnold, 1990] that this is the only type of singularity that can occur for wave fronts in the plane. Another important aspect is that only a subset of the evolution trajectories of disturbances (often called characteristics) form finite periodic orbits [Essl et al., 2004]. The geometric aspect of wave evolution of wave fronts on plane domains has been explored in detail in a series of previous studies [Essl, 2006]. It is noteworthy that in three dimensions the purely geometric part of the wave equation under reflection is more complicated than in two because of the possibility of more complex singularities forming.

The functional part pertains to the actual form of the solution in response to various functional forms of a disturbance. As said in three dimensions Huygens principle holds, which means that impulsive disturbances (or disturbances of any particular shape) will travel at constant speed without change of the functional shape [Stein & Shakarchi, 2003]. In one dimensional this is almost true. Velocity excitations will lead to a step function, which however, by differentiation leads to impulses again and hence can be simulated as propagating impulses in the wave variable in the digital waveguide synthesis [Smith, 1997]. The functional solution of the two dimensional wave equation is qualitatively different. An impulsive excitation will not stay impulsive, even under differentiation or integration. This functional part is called wake and is of course well known [Graff, 1991]. The goal here is to to find ways to simulate this well-known structure directly, or try to preserve this structure through numerical simulation specifically under reflections.

STRUCTURE OF THE WAVE EQUATION

The wave equation in two dimensions without external forces and dissipation reads:

$$\frac{\partial^2 y}{\partial t^2} - c^2 \left(\frac{\partial^2 y}{\partial {x_1}^2} + \frac{\partial^2 y}{\partial {x_2}^2} \right) = 0 \tag{1}$$

The fundamental solution of the wave equation in the plane without boundaries is [Graff, 1991, Egorov et al., 1999]:

$$y_f^d(x,t) = \frac{H(\pm ct - |x|)ct}{2\pi\sqrt{|x|}((ct)^2 - |x|^2)^{\frac{3}{2}}} \tag{2}$$

$$y_f^v(x,t) = \frac{H(\pm ct - |x|)}{2\pi c\sqrt{(ct)^2 - |x|^2}} \tag{3}$$

where equation (2) is the response to an impulsive displacement and equation (3) is the response to an impulsive velocity. $H(\cdot)$ is a step function also called the Heaviside distribution.

These equations have two parts to them. One is the support of the solution. This part is encoded in the Heaviside function of equations (2–3).The other is the functional shape created by an impulse through propagation inside this support.

The first part encodes what we will call the *wavefront*. The geometric evolution of the wave front was discussed at length previously [Essl, 2006, Essl, 2005b, Essl, 2005a]. The second part will be called the *wake*. Which is the central part of this work.

Wave fronts are the point of first arrival of a disturbance in response to an impulse (or a convolution of impulses). The argument of the Heaviside functions in (2) and (3) forms a circle for constant time. The radius of the circle expands linearly with time. Hence the wavefront inhabit a cone with the tip at the point of the impulsive excitation. Any section of this cone will give a wavefront at a certain time, which is a line embedded in the plane. Point of the wavefront propagate along straight lines called *rays*. Hence one can think of rays as the direction of propagation of waves and wave fronts as the points of arrival on them.

The geometric methods as well as the current discussion of waves maintains the picture that wavefronts and waves are supported on rays. This allows one to discuss local behavior as a sum of one-dimensional domains rather than as two-dimensional objects simplifying observations. Additionally rays can be rendered up to numerical precision without need of discretization of the domain itself. Hence the position will be accurate, up to numerical accuracy on such rays and there is no numerical dispersion by construction. Furthermore geometric features that come from crossing of rays (such as cusp singularities) are preserved even if finite numbers of rays are used. This is different to meshed method where cusps and other fine features are subject to mesh dispersion and there is no guarantee for their preservation under discretization.

For space reasons I cannot present the details of the algorithm here. These details can be found in [Essl, 2006, Essl, 2005b]. Using this method also complicated shapes can be simulated. If the domain is sufficiently simple the reflection code can be replaced by an iterative function of reflection angle and reflection point. For details see [Essl, 2006].

METHOD OF ITERATED WAKE SUMMATION

The method uses repeated summation of the fundamental solution with increasing numbers of reflections to calculate the solution of an excitation. This in the core is the image-source method on the line. But because a point source not staying a

point source, one has to sum the respective contributions of the function at each point. With each order of reflection two more image-sources (one per boundary) can bring its domain of influence into the actual range of the domain, hence the length of the summation increases by two per reflection order, one by specific reflection.

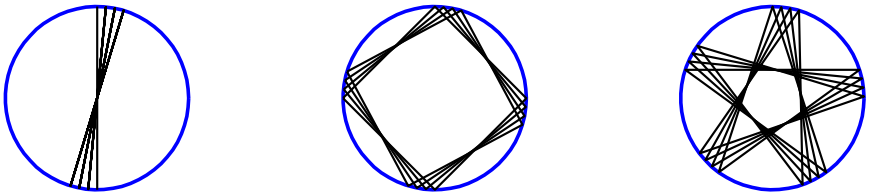

Figure 1: Reflected rays on a circular domain are repeated line segments under rotation.

The functional aspect of the method is the same for all cases. The functional contribution to a point on a ray are all the wake points from image sources and the original source on a point of the domain. On a circular domain one ray follows a regular trajectory and the line segment the reflected ray forms with the domain stays the same as depicted in Figure 1. For this reason also the relative position of the image source stays the same and we can unfold all reflected rays into an extended straight line where we keep track of the reflections.

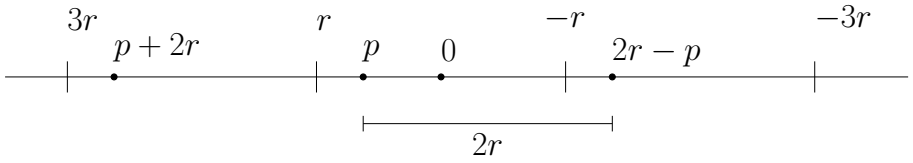

Figure 2: Derivation of the image source contribution to the interior of the domain. Vertical lines mark repeated real and virtual boundaries of the domain. Markers represent the center of the ray as well as the excitation point and its repeated image outside the domain.

From this one sees there there are two types of contribution for a ray through an excitation point — one for each direction of propagation. The simplest case of summation are rays passing through the center of the domain. In this case the solution has a circular symmetry and repeated summation on one ray describes the full solution of the problem.

It is easy to derive from the unfolded ray the following formula for rays passing through the center of the circle (see Figure 2):

$$y^+(x,t) = \sum_{n=0}^{l^+} (-1)^n y_f(|x-p| + n \cdot 2r, t) \tag{4}$$

$$y^-(x,t) = \sum_{m=0}^{l^-} (-1)^m y_f(m \cdot 2r - |x-(2r-p)|, t) \tag{5}$$

$$y(x,t) = y^+(x,t) + y^-(x,t), \tag{6}$$

where p is the position of the excitation point from the center of the ray, x is coordinate position of the ray relative to the center of the ray, t is time since the excitation started. r is the length from the center of a ray to the boundary of the circle. $y^+(\cdot)$ is the solution of the wave propagating in the positive direction along a ray, having reflected l^+ times and $y^-(\cdot)$ is the solution of the wave propagating in the negative direction along a ray, having reflected l^-. It is easy to see that $|l^+ - l^-| \leq 1$.

In the case of an excitation at the center of a circular domain (i.e. $p = 0$) this further simplifies to:

$$y^+(x,t) = \sum_{n=0}^{l^+} (-1)^n y_f(x + n \cdot 2r, t) \tag{7}$$

$$y^-(x,t) = \sum_{m=0}^{l^-} (-1)^m y_f((m+1) \cdot 2r - x, t) \tag{8}$$

$$y(x,t) = y^+(x,t) + y^-(x,t) \tag{9}$$

In the case of general points on the domain for off-center excitations, the summation becomes more complex. Rather than summing reflected contributions from on the same ray, one sums the wake contribution of all rays whose wavefront has over crossed a point inside the domain for all points of interest.

PROPERTIES OF THE SOLUTION FOR CENTER EXCITATIONS

Figures 3 and 4 show the radial section of a circular domain with increasing order of reflections. The figures show snap-shots when the initial center excitation repeatedly crosses the half radius point $r/2$ with increasing time and order of reflections.

As one can already see from the fundamental solution of velocity excitations (2) alone, the wavefront itself is singular. However in the case of velocity excitations the solution is smooth everywhere else. After the first reflection (Figure 3 top right) one sees that the interior part between the reflected wave fronts retains a negative deflection solely from the contribution of the wake of the unreflected part of the solution. This part, while decreasing in time, remains present in the summation for all orders of reflection. A similar effect can be seen in the bottom right of the same figure after a second reflection occurred. The positive deflection in between the wave fronts now consists of contributions of both the original and the first reflected wake.

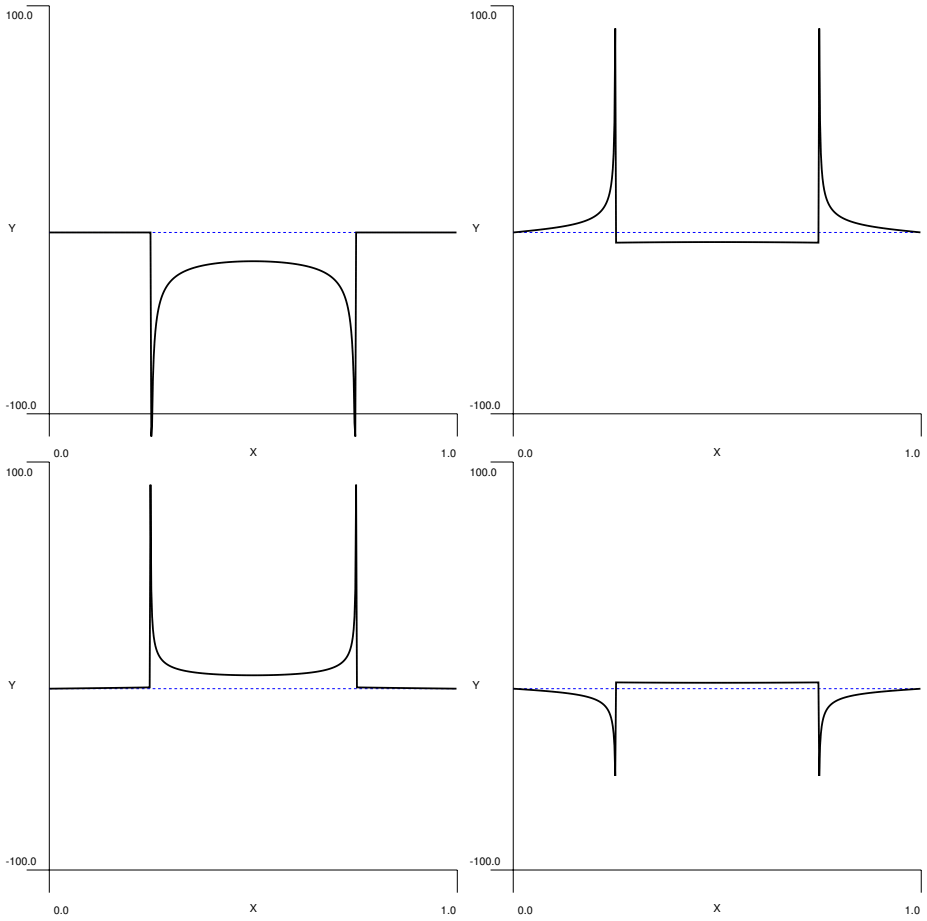

Figure 3: Velocity Excitation: Excitation when crossing half-radius ($x/4$) of the domain in increasing order.

The displacement excitation had an additional singularity at the excitation point as can already be seen in the fundamental solution (3). This singularity persists over time due to the contribution of the original wake. Source images can be thought of having such a singularity, but because those lie strictly outside the geometric domain of the solution of interest, they do never contribute to the actual solution. Otherwise the solution behaves qualitatively in a similar way as the velocity excitation. Quantitatively, velocity excitations have larger amplitude contributions over time than displacement excitations, away from singularities. This difference is at least three orders of magnitude as can be seen comparing the scales of Figures 3 and 4.

REMARKS ON PERFORMANCE

This method is not computationally efficient. The main reason is immediate from equations (4-5). The order of reflections l^+, l^- increase linearly with time with each spatial point calculated, whereas classical numerical methods are constant with respect to time and only depend on the discretization itself. A further contributing factor is geometric. In the case of center excitation symmetry reduces the problem to just one ray. In the general case, a dense set of rays have to be calculated. While the

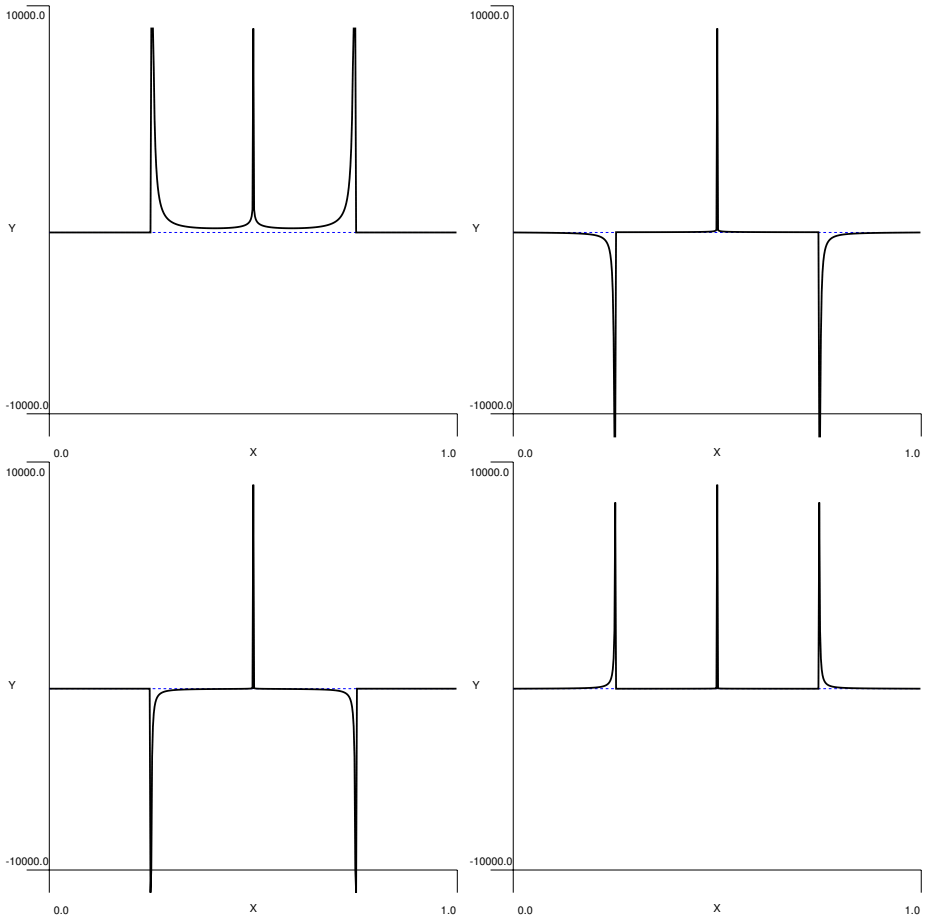

Figure 4: Displacement Excitation: Excitation when crossing half-radius (x/4) of the domain in increasing order.Figure caption

number of rays can be held constant, hence there is no chance in asymptotic behavior of the algorithm, the increased computational cost per time step is significantly.

CONCLUSIONS

By iterating the fundamental solution of the two-dimensional wave equation, one can calculate the complete wake contribution of the solution on a bounded domain by iterated summation with increasing order of reflections. This yields a direct rendering of the time domain behavior of the solution without the need of functional transformations or decompositions and even singularities in the solution are directly represented without dissipation introduced by discrete numerical schemes.

Future work are attempts to find reductions in complexity of direct use of the fundamental solution for simulation with respect to both increased reflection and geometric properties. Furthermore the question how to relate the time-domain picture of the fundamental solution under reflection to the well-known functional decomposition solution involving Hankel transforms and yielding the well-known Fourier-Bessel form of the solution is a difficult open problem.

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