THE PRINCIPLE OF CLOSED WAVETRAINS, RESONANCE AND EFFICIENCY: PAST, PRESENT AND FUTURE

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ABSTRACT

The principle of closed wavetrains asserts the equivalence of the condition of a traveling wave closing onto itself in phase to the occurrence of a mode. This principle provides a direct conceptual link between spectral descriptions of dynamic responses and a path-based dynamic description. In this paper we present the history and development of the idea since d'Alembert first proposed traveling functional forms to solve the string equation. The subsequent argument between Daniel Bernoulli, Euler and him which led to the development of Fourier analysis and contemporary theories of partial differential equations. Other related developments include the development of chaos theory connected to Poincaré and others, asymptotic solutions associated with Rayleigh, Wenzel, Kramers, Brillouin, and Keller and Kac's famous isospectral problem. Then we discuss how the traveling functions have been utilized in the numerical simulation of musical instruments through work by Julius Smith, Karplus, Strong and other. This work has recently been extended to additional instruments types, in particular idiophones, which not only are more efficient than finite element based simulations but have the desirable property of stability and ease of interpretation under perturbations. We conclude with outlining possible research based on the advantages and drawbacks of the method.

This principle provides a direct conceptual link between modal synthesis and waveguide-style physical models. In addition it suggests the importance of understanding the link between geometry and modes for efficient dynamical simulation. This paper explores these connections with regards to banded waveguide models.

1. INTRODUCTION

How do musical instruments behave? What is responsible for the audible aspects of their behavior? How can we predict, recreate and modify this behavior to create music? These questions are at the very core of musical acoustics. In this paper we explore aspects of the evolution of mathematical formulation of the dynamical behavior of sounding objects in general and musical instruments in particular. Within this history a particular principle will be the main focal point of the discussion, namely the *principle of closed wavetrains* also called the *principle of equal phase closure*. Both these phrases can be found in [1] to describe the connection between a closed trajectory of a traveling wave and the occurrence of resonances. More precisely the principle asserts that if a traveling wave closing onto itself in phase a resonant response will result.

This principle is of interest precisely because it links spectral theory with mathematical modeling of the dynamics in a direct way. This is a rarely seen connection in approaches to musical acoustics questions. This paper in part hopes to draw attention to this link by exploring how it has developed over the years and suggest, how in part the separate developments of partial differential equations and functional analysis and dynamical theory can be understood. We will point at separate strands of past and ongoing research and suggest the benefit of bringing them together.

2. PAST: FROM D'ALEMBERT TO WAVEGUIDES

The history of the principle of closed wave trains probably starts in the 18th century with d'Alembert's famous first formulation of the perfectly elastic string in it's current form as partial differential equation. Along with it he also presented a solution of two functions traveling left and right[1]. This solution ensued an argument about the validity of the assumptions of the arbitrariness of these functions between d'Alembert, Euler and Daniel Bernoulli. Bernoulli proposed as an alternative solution the sum of sinusoids, yet left open how the coefficients were to be computed. This solution in turn came under criticism by both Euler and d'Alembert. Only with the work of Fourier that then solidified the idea of spectral analysis did Bernoulli's solution recover full acceptance [2].

This early development really produced two focus arguments. One was concerned with the nature of allowed functions in d'Alembert's solution. The other had to do with the details of spectral theory. The first

[1] Throughout this paper we will give preference to review literature if available. The reader is encouraged to follow the reference of the review articles of books for details and particular publications.

should then lead via the work of Cauchy, Dirichlet and others to modern functional analysis. These were however for the most part treated separate, a separation which is commonly still seen today.

Maybe this has to do with the localized description of Newton's form of classical mechanics which suggests forming the dynamical equations of a problem as a sum of local forces, which are then to be studied. Alternative forms of describing the classical mechanics were developed through the 18th and 19th century by Euler, Maupertuis, Lagrange, Hamilton and Jacobi. These forms, often called the Lagrangian and Hamiltonian mechanics, emphasizes the finding of trajectories of motion. In particular in the Hamilton's mechanics the evolution, the *flow*, of the dynamics is described by the variables of position and momentum which are driven by the gradient of the energy surface of the system called *the Hamiltonian* [3].

While acoustical problems were arguably an central driving force in the development of physics and mathematics during the 18th and 19th century the focus shifted around the turn of the 19th to the 20th century with the advent of quantum physics.

Physicist faced an inverse problem of sort to the earlier acoustical work. Instead of solving known constituent dynamical equations to match observed spectra, now spectra were observed (of for example black body radiation) and the task was to find dynamical equations that would yield these spectra under upon solution. Of course this happened after the classical dynamical equations failed to yield the correct results. At that time Bohr, Sommerfeld, Einstein and others devised quantization rules based on classical dynamics and Planck's energy quantization to describe the spectra. This work still is basis today for a field called semiclassical physics, which studies how classical methods can be utilized to find quantum mechanical solutions [4].

However there remained startlingly simple problems of classical dynamics for which no general solutions could be found with conventional methods. Among these problems were was the three-body problem as it applies to the motion of the moon in the constellation with the sun and earth. The first calculations date back to Newton, it was not until the late 19th century, that substantial progress was made by George William Hill. His calculation started with a known solution of periodic orbits, closed smooth trajectories, and he would investigate other solutions in its neighborhood. By finding that these periodic orbits are densely distributed among all possible trajectories, Poincaré suggested that periodic orbits were in fact the right starting points for these other possibilities. These possibilities were later found to be potentially chaotic [3].

If a dynamical system can be described by independent periodic orbits all allowed paths can be represented by a torus topology of the order of independent orbits. This was realized through work by Poincaré and Einstein around the turn of the century. Dynamical systems of this type are called *integrable*. Einstein used the torus structure to both argue that separation of variable is too stringent a condition for the classical solution of dynamical systems and that most dynamical systems don't have a corresponding torus structure. Regardless, a number of integrable systems have been solved by Keller and Rubinow and related work was performed by Brillouin. The procedure of calculating dynamical solutions through quantization on tori topologies are hence called *EBK quantization* after Einstein, Brillouin and Keller. Keller and Rubinow's work in particularly interesting because they provide explicit solutions for the cases of rectangular, circular and elliptical membranes [3]. Recently Chen, Coleman and Zhou presented solutions for stiff bars and plates using this approach [5].

There may be regions in the dynamics where a traveling waves assumption does not hold without reservations. In particular the solution can become singular in some regions. The development of methods treating this problem has a long and somewhat complex history. Early work goes back to Carlini, Liouville and Green and is hence sometimes called Liouville-Green approximation. The connection across singular regions was studied by Gans, Rayleigh, Jeffreys, Wenzel, Kramers and Brillouin and is hence often referred to as WKB or JWKB. The interested reader is referred to [6] for detailed historical context of mathematical developments until the 1970s. The following mathematical theories of relevance are often referred to as *asymptotic theory* and *catastrophe theory*.

These so-called *invariant tori* were then used to study the emergence of chaos in a more general setting by Kolmogoroff, Arnold and Moser in the 1950s and 60s. The *KAM theorem*, named after its contributors, establishes the robustness of tori under small perturbation as long as resonances are sufficiently separated. Similar to Poincaré's idea of starting with periodic orbits, they start with tori and study it's stability under perturbations [4].

There exists an intimate connection between the geometry of the dynamical system and the existence of invariant tori. Exact analytical solutions of arbitrary boundary contours is an unsolved problem in general and hence also this approach is yet limited in the type of geometries that have been solved for. This problem was highlighted by the question Kac posed in the 1960s: "Can we hear the shape of a drum?" [7]. While this question has been answered to be no for specific non-smooth constructions [8], the question remains open for most and very general classes of shapes (like all smooth boundaries).

It is not clear that acousticians were always aware of these developments and how they related to classical acoustic problems. However, similar ideas can be found in the acoustics literature of the time. Among the acousticians of the 20th century who developed the idea were Mead [9], Cremer and Heckl [1]. While they state that the approach of finding resonances by studying closed wavetrains works for higher-dimensional problems, only one dimensional problems of strings and bars are explicitly presented.

Keller is known in the violin acoustics community for his work on the bowed string and the friction characteristics he and Friedlander used in their calculations often bears their name. His other work seems to have gone mostly unnoticed. The development of chaos theory did find its way into acoustics [10].

The view of chaos however focuses its attention to the occurrence of chaotic states and the role of attractors and shows little connection to the particular dynamical system under consideration. The study of dynamical non-linearity has mostly focused around localized effects, predominantely excitation mechanisms, like the action of a violin bow on a string or the function of reeds in woodwind instruments [11].

With the advent of digital computers, numerical solutions to dynamical equations developed rapidly in the 20th century. This drive also was used in musical acoustics. Numerical methods that utilize propagation ideas in musical acoustics have focused on the one-dimensional wave-equation, which describes a large class of musical instruments in the western musical tradition [12]. McIntyre, Schumacher, and Woodhouse realized that many of these instruments can be studied as a concatenation of non-linear excitation and linear resonator. They realized that computational effort is focused on the reflection of traveling waves [13]. Karplus and Strong used circular buffers to simulate plucked string sounds which was soon realized to be an accurate discrete simulation of the one-dimensional wave-equation. This approach, usually called *waveguide synthesis* has since been extensively studied by Smith and others and has been used for a range of musical instruments [14, 15].

This sketch of the history of traveling wave physics is by the complexity of the pattern of individual contributions and by nature of the space available here somewhat limited. Many of the cited references contain a wealth of historical context, in particular Gutzwiller's text [3] presents the material in historical context and with focus on the individual contributions.

3. PRESENT: BANDED WAVEGUIDES AND THE CONVERGENCE OF PROBLEM-SOLUTIONS

Today we see in some sense a convergence of some of these past developments. In particular modern dynamical theory, spectral theory, numerical methods and filter theory are brought together.

The authors have proposed a method called *banded waveguides*, which attempts at combining the work of dynamic theory of traveling waves with numerical methods like digital waveguide synthesis. The core idea is to expand propagators into band-limited loop dynamics around resonances. The most basic filter structure capturing the idea is depicted in Figure 1. This way of depicting the method hides much of the connection of this structure to the past developed knowledge.

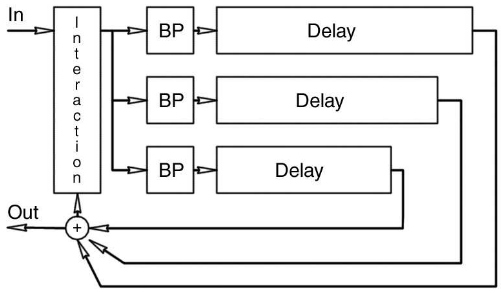

Figure 1: A simple banded waveguide system.

Conceptually this method brings various different ideas together. For one it is a spectral method for its spectral decomposition. It is also a numerical method for close trajectory dynamics for its discretization of loops.

A number of one and two-dimensional problems have been attempted using this method [16]. In particular, the approach is very useful for modeling idiophones of stiff material.

A similar development of the merging of spectral theory and dynamical theory can be observed in recent developments in mathematics where the connection of scattering phenomena on resonances is under scrutiny. In particular it is suggested that propagators are analyzed around resonances making the connection between spectral results and dynamical behavior the starting point of the investigations [17].

The problem of treating regions where the traveling wave approach apparently breaks down or is only approximate has seen much advance. There is ongoing research that studies how to make the asymptotic treatment exact [18, 19]. For entertaining popular treatments of these development, Michael Berry's series of articles are very recommended [20, 21]. Other related directions use complex-valued rays, which simplify similar problems [22].

Other significant developments include recent progress on the isospectral problem when Kac's question was finally answered negatively [8]. Loop dynamics are essential to the progress in this area.

In the realm of numerical methods in musical acoustics the connection between waveguide synthesis and finite differencing methods has seen detailed investigation by Bilbao [23]. Numerical simulations using waveguide synthesis has reached a level of maturity [14].

However, today, many of these ideas still are only developed in separation and can be seen as a partial convergence only. However, the they all seem to be reaching a certain level of individual maturity.

4. FUTURE

Much of the synthesis of these related strands of work has yet to happen and utilized in musical acoustics. In part the problem may be one of translation. This literature spans various areas from pure and applied mathematics, numerical methods, computer science and physics. The various disciplines are not necessarily aware of each others work and use different formalisms in the respective discussion. Yet a continued effort in bringing these ideas together has various advantages. For example waveguide style numerical methods have advantages over finite element methods. They are more efficient. Their stability conditions can be easily observed. The intuitive appeal of closed trajectory description over local description becomes clear when one wants to understand how perturbations to a system affect the response. This intuitive appeal has been praised by various writers in the field. Gutzwiller writes: "[..] all of these ideas deal with relatively elementary questions [..]; as soon as they are understood, some readers may be tempted to call them obvious because of their deceptive simplicity!" [3] Similarly Zworski writes: "The results are technically quite simple, at least by the standards of the subject, and the appeal of this study lies in its connection with applied problems." [17] Most of this appealing simplicity is however quite unknown to many and one might thus argue that looking into the future may first require connecting with past ideas that haven't been sufficiently recognized. Gutzwiller makes a related point when he writes: "Most scientists have not participated in the recent development of ideas related to chaos in Hamiltonian systems; they are usually not aware of the many different viewpoints and interpretations, the new problems and methods for their solution, and the novel applications to important experiments." [3] We hope that this paper helps in pointing towards knowledge that should be beneficial in future studies in musical acoustics.

5. REFERENCES

[1] L. Cremer, M. Heckl, and E. Ungar, *Structure-Borne Sound*, Springer Verlag, 2nd edition, 1988.

[2] I. Grattan-Guinness, "Daniel Bernoulli and the varieties of mechanics in the 18th century," *Nieuw Archief voor Wiskunde*, vol. 5, no. 1, pp. 242–249, September 2000.

[3] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer Verlag, New York, 1990.

[4] M. Brack and R. K. Bhaduri, *Semiclassical Physics*, vol. 96 of *Frontiers in Physics*, Addison-Wesley Publishing, 1997.

[5] G. Chen, M. P. Coleman, and J. Zhou, "Analysis of Vibration Eigenfrequencies of a Thin Plate by the Keller-Rubinow Wave Method I: Clamped Boundary Conditions With Rectangular or Circular Geometry," *SIAM Journal on Applied Mathematics*, vol. 51, no. 4, pp. 967–983, August 1991.

[6] F. W. J. Olver, *Asymptotics and Spectial Functions*, Academic Press, 1974.

[7] M. Kac, "Can One Hear the Shape of a Drum?," *American Mathematical Monthly*, vol. 73, no. 4, pp. 1–23, April 1966.

[8] T. A. Driscoll, "Eigenmodes of Isospectral Drums," *SIAM Review*, vol. 39, no. 1, pp. 1–17, March 1997.

[9] D. J. Mead, "Waves and Modes in Finite Beams: Application of the Phase-Closure Principle," *Journal of Sound and Vibration*, vol. 171, no. 5, pp. 695–702, 1994.

[10] W. Lauterborn and U. Parlitz, "Methods of chaos physics and their application to acoustics," *Journal of the Acoustical Society of America*, vol. 84, no. 6, pp. 1975–1993, December 1988.

[11] N. H. Fletcher, "The nonlinear physics of musical instruments," *Rep. Prog. Phys.*, vol. 62, pp. 723–764, 1999.

[12] N. H. Fletcher and T. D. Rossing, *The Physics of Musical Instruments*, Springer Verlag, New York, 2 edition, 1998.

[13] M. E. McIntyre, R. T. Schumacher, and J. Woodhouse, "On the oscillation of musical instruments," *J. Acoust. Soc. Am.*, vol. 74, no. 5, pp. 1325–1344, 1983.

[14] J. O. Smith, "Digital Waveguide Modeling of Musical Instruments," draft of unpublished online manuscript, available at `http://ccrma-www.stanford.edu/~jos/waveguide/`, 2002.

[15] P. R. Cook, *Real Sound Synthesis for Interactive Applications*, A K Peters, Ltd., July 2002.

[16] G. Essl, *Physical Wave Propagation Modeling for Real-Time Synthesis of Natural Sounds*, Ph.D. thesis, Princeton University, Princeton, NJ, November 2002, Available also as Princeton University Computer Science Technical Report TR-659-02 at `http://ncstrl.cs.princeton.edu/expand.php?id=TR-659-02`.

[17] M. Zworski, "Resonances in Physics and Geometry," *Notices of the American Mathematical Society*, vol. 46, no. 3, pp. 319–328, March 1999.

[18] J. P. Boyd, "The Devil's Invention: Asymptotic, Superasymptotic and Hyperasymptotic Series," *Acta Applicandae*, vol. 56, pp. 1–98, 1999.

[19] C. J. Howls, "Development of Exponential and Hyperasymptotics," in *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear*, C. J. Howls, T. Kawai, and Y. Takei, Eds. Kyoto University Press, 2000.

[20] M. Berry, "Why are special functions special?," *Physics Today*, vol. 54, no. 4, pp. 11–12, April 2001.

[21] M. Berry, "Singular Limits," *Physics Today*, vol. 55, no. 5, pp. 10–11, May 2002.

[22] S. J. Chapman, J. M. H. Lawry, J. R. Ockendon, and R. H. Tew, "On the Theory of Complex Rays," *SIAM Review*, vol. 41, no. 3, pp. 417–509, 1999.

[23] S. D. Bilbao, *Wave and Scattering Methods for the Numerical Integration of Partial Differential Equations*, Ph.D. thesis, Stanford University, May 2001.