On the three-dimensional Singer Conjecture for Coxeter groups

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Abstract

We give a proof of the Singer conjecture (on the vanishing of reduced ℓ^2-homology except in the middle dimension) for the Davis Complex Σ associated to a Coxeter system (W, S) whose nerve L is a triangulation of $\mathbb{S}^2$. We show that it follows from a theorem of Andreev, which gives the necessary and sufficient conditions for a classical reflection group to act on $\mathbb{H}^3$.

1 Introduction

Let (W, S) denote a *Coxeter system*: S is a finite set of generators and for any pair $\{s, t\}$ of generators there is a particular relation $m_{st} \in \mathbb{N} \cup \{\infty\}$ such that $(st)^{m_{st}} = 1$ with the rule that $m_{st} = 1$ if and only if $s = t$; these are the only relations. (See [9] or [4]). Denote by L the nerve of (W, S). (L is a simplicial complex with vertex set S, the precise definition will be given in section 2.1.) In several papers (e.g., [3], [4], and [5]), M. Davis describes a construction which associates to any Coxeter system (W, S), a simplicial complex $\Sigma(W, S)$, or simply Σ when the Coxeter system is clear, on which W acts properly and cocompactly. The two salient features of Σ are that (1) it is contractible and (2) that it admits a cellulation under which the nerve of each vertex is L. It follows that if L is a triangulation of $\mathbb{S}^{n-1}$, Σ is an n-manifold.

The following conjecture is attributed to Singer.

Singer's Conjecture 1.1. *If M^n is a closed aspherical manifold, then the reduced ℓ^2-homology of $\widetilde{M}^n$, $\mathcal{H}_i(\widetilde{M}^n)$, vanishes for all $i \neq \frac{n}{2}$.*

For details on ℓ^2-homology theory, see [5], [6] and [8].

Now, if G is a torsion-free subgroup of finite index in W, then G acts freely on Σ and Σ/G is a finite complex. By (1), Σ/G is aspherical. Hence, if L is homeomorphic to an $(n-1)$-sphere, Davis' construction gives examples of closed aspherical n-manifolds and Conjecture 1.1 for such manifolds becomes the following.

Singer's Conjecture for Coxeter groups 1.2. *Let (W, S) be a Coxeter group such that its nerve, L, is a triangulation of $\mathbb{S}^{n-1}$. Then $\mathcal{H}_i(\Sigma) = 0$ for all $i \neq \frac{n}{2}$.*

Conjecture 1.1 holds for elementary reasons in dimensions ≤ 2. In [6], Davis and Okun show that 1.2 holds for $n = 3$ when (W, S) is *right-angled* (this means that generators either commute, or have no relation). They do this in (at least) two ways, one of which is a direct calculation of the reduced ℓ^2-homology using a Mayer-Vietoris argument (Chapter 10). We follow that method here, proving the result for *arbitrary* Coxeter systems with nerve $\mathbb{S}^2$. This paper is a precursor to a JSJ-decomposition for three-dimensional Davis manifolds, which the author details in [11], and from which Conjecture 1.2 follows as a Corollary. Also, in [10], he uses the three-dimensional case to establish 1.2 in the case (W, S) is *even* and L is a flag triangulation of $\mathbb{S}^3$.

2 The Davis complex and ℓ^2-homology

Let (W, S) be a Coxeter system. Given a subset U of S, define W_U to be the subgroup of W generated by the elements of U. A subset T of S is *spherical* if W_T is a finite subgroup of W. In this case, we will also say that the subgroup W_T is spherical. Denote by $\mathcal{S}$ the poset of spherical subsets of S, partially ordered by inclusion. Given a subset V of S, let $\mathcal{S}_{\geq V} := \{T \in \mathcal{S} | V \subseteq T\}$. Similar definitions exist for $<, >, \leq$. For any $w \in W$ and $T \in \mathcal{S}$, we call the coset wW_T a *spherical coset*. The poset of all spherical cosets we will denote by $W\mathcal{S}$.

2.1 The Davis complex

Let $K = |\mathcal{S}|$, the geometric realization of the poset $\mathcal{S}$. It is a finite simplicial complex. Denote by $\Sigma(W, S)$, or simply Σ when the system is clear, the geometric realization of the poset $W\mathcal{S}$. This is the Davis complex. The natural action of W on $W\mathcal{S}$ induces a simplicial action of W on Σ which is proper and cocompact. Σ is a model for $\underline{EW}$, a *universal space for proper W-actions.* (See Definition [4, 2.3.1].) K includes naturally into Σ via the map induced by $T \to W_T$. So we view K as a subcomplex of Σ, and note that K is a strict fundamental domain for the action of W on Σ.

The poset $\mathcal{S}_{>\emptyset}$ is an abstract simplicial complex. This simply means that if $T \in \mathcal{S}_{>\emptyset}$ and T' is a nonempty subset of T, then $T' \in \mathcal{S}_{>\emptyset}$. Denote this simplicial complex by L, and call it the *nerve* of (W, S). The vertex set of L is S and a non-empty subset of vertices T spans a simplex of L if and only if T is spherical. Define a labeling on the edges of L by the map $m : \text{Edge}(L) \to \{2, 3, \ldots\}$, where $\{s, t\} \mapsto m_{st}$. This labeling accomplishes two things: (1) the Coxeter system (W, S) can be recovered (up to isomorphism) from L and (2) the 1-skeleton of L inherits a natural piecewise spherical structure in which the edge $\{s, t\}$ has length $\pi - \pi/m_{st}$. L is then a *metric flag* simplicial complex (see Definition [4, I.7.1]). This means that any finite set of vertices, which are pairwise connected by edges, spans a simplex of L if an only if it is possible to find some spherical simplex with the given edge lengths. In other words, L is "metrically determined by its 1-skeleton."

For the purpose of this paper, we will say that labeled (with integers ≥ 2) simplicial complexes are *metric flag* if they correspond to the labeled nerve of some Coxeter system. We will often indicate these complexes simply with their 1-skeleton, understanding the underlying Coxeter system and Davis complex. We write Σ_L to denote the Davis complex associated to the nerve L of (W, S).

Special subcomplexes. Suppose A is a full subcomplex of L. Then A is the nerve for the subgroup generated by the vertex set of A. We will denote this subgroup by W_A. (This notation is natural since the vertex set of A corresponds to a subset of the generating set S.) Let $\mathcal{S}_A$ denote the poset of the spherical subsets of W_A and let Σ_A denote the Davis complex associated to (W_A, A^0). The inclusion $W_A \hookrightarrow W_L$ induces an inclusion of posets $W_A\mathcal{S}_A \hookrightarrow W_L\mathcal{S}_L$ and thus an inclusion of Σ_A as a subcomplex of Σ_L. Such a subcomplex will be called a *special subcomplex* of Σ_L. Note that W_A acts on Σ_A and that if $w \in W_L - W_A$, then Σ_A and $w\Sigma_A$ are disjoint copies of Σ_A. Denote by $W_L\Sigma_A$ the union of all translates of Σ_A in Σ_L.

A mirror structure on K. If L is the triangulation of an n-sphere, then we have a another cellulation of K and Σ. For each $T \in \mathcal{S}$, let K_T denote the geometric realization of the subposet $\mathcal{S}_{\geq T}$. K_T is a triangulation of a k-cell, where $k = n + 1 - |T|$. We then define a new cell structure on K by declaring the family $\{K_T\}_{T\in\mathcal{S}}$ to be the set of cells in K. We write K_L to indicate K equipped with this cellulation and note that it extends to a cellulation of Σ_L. Since our concern is the case L is a triangulation of $\mathbb{S}^2$, we assume this cellulation of Σ_L.

The boundary complex of K_L is combinatorially dual to L, so K_L has codimension 1 faces corresponding the elements of S. In fact, if L is *any* cell complex homeomorphic to $\mathbb{S}^2$, in the strict sense that any non-empty intersection of two cells is a cell, then L is combinatorially dual to the boundary complex of a 3-dimensional convex polytope, which we will denote by K_L. If the edges of L are labeled with integers ≥ 2, (e.g. L is the labeled nerve of a Coxeter system) then we assign dihedral angles to K_L so that the angle between faces dual to vertices s and t is π/m_{st}, where m_{st} is the label on the edge between s and t. This assignment defines a classical reflection group generated by the reflections in the faces of K_L with relations prescribed by the dihedral angles.

A cellulation of Σ by Coxeter cells. Σ has a coarser cell structure: its cellulation by "Coxeter cells." (References for this section include [4] and [6].) The features of the Coxeter cellulation are summarized by [4, Proposition 7.3.4]. We note here that, under this cellulation, the link of each vertex is L. It follows that if L is a triangulation of $\mathbb{S}^{n-1}$, then Σ is a topological n-manifold.

2.2 Previous results in ℓ^2-homology

Let L be a metric flag simplicial complex (see subsection 2.1), and let A be a full subcomplex of L. The following notation will be used throughout.

$$\mathfrak{h}_i(L) := \mathcal{H}_i(\Sigma_L) \tag{2.1}$$

$$\mathfrak{h}_i(A) := \mathcal{H}_i(W_L\Sigma_A) \tag{2.2}$$

$$\beta_i(A) := \dim_{W_L}(\mathfrak{h}_i(A)). \tag{2.3}$$

Here $\dim_{W_L}(\mathfrak{h}_i(A))$ is the von Neumann dimension of the Hilbert W_L-module $W_L\Sigma_A$ and $\beta_i(A)$ is the i^{th} ℓ^2-Betti number of $W_L\Sigma_A$. The notation in 2.2 and 2.3 will not lead to confusion since $\dim_{W_L}(W_L\Sigma_A) = \dim_{W_A}(\Sigma_A)$. (See [6] and [8]).

Given a simplicial complex L and a full subcomplex $A \subset L$, we say that A is *ℓ^2-acyclic*, if $\beta_i(A) = 0$ for all i.

Bounded geometry. The following result is proved by Cheeger and Gromov in [2]. Suppose that X is a complete contractible Riemannian manifold with uniformly bounded geometry (i.e. its sectional curvature is bounded and its injectivity radius is bounded away from 0.) Let Γ be a discrete group of isometries on X with $\mathrm{Vol}(X/\Gamma) < \infty$. Then $\dim_\Gamma(\mathcal{H}_k(\underline{E\Gamma})) = \dim_\Gamma(\mathcal{H}_k(X))$, where $\underline{E\Gamma}$ denotes a universal space for proper Γ actions, and $\mathcal{H}_k(X)$ denotes the space of L^2-harmonic forms on X. Of particular interest to us is the case where $X = \mathbb{H}^3$. For it is proved by Dodziuk in [7] that the L^2-homology of any odd-dimensional hyperbolic space, $\mathbb{H}^{2k+1}$, vanishes.

Euclidean Space. The Cheeger Gromov result also implies that if $\Sigma_L = \mathbb{R}^n$ for some n, then $\mathfrak{h}_*(L)$ vanishes.

Joins. If $L = L_1 * L_2$ where each edge connecting a vertex of L_1 with a vertex of L_2 is labeled 2, then $W_L = W_{L_1} \times W_{L_2}$ and $\Sigma_L = \Sigma_{L_1} \times \Sigma_{L_2}$. We may then use Künneth formula to calculate the (reduced) ℓ^2-homology of Σ_L, and the following equation from [6, Lemma 7.2.4] extends to our situation:

$$\beta_k(L_1 * L_2) = \sum_{i+j=k} \beta_i(L_1)\beta_j(L_2). \tag{2.4}$$

Suspensions. If $L = P * L_2$, where P is two points not connected by an edge and each join edge is labeled with 2, we call L a *right-angled suspension.* $\Sigma_P = \mathbb{R}$ and $\mathfrak{h}_i(P) = 0$ for all i ([6, Lemma 7.3.4]). Then by equation 2.4, L is ℓ^2-acyclic.

3 Andreev's theorem

In [1], Andreev gives the necessary and sufficient conditions for abstract 3-dimensional polytopes, with assigned dihedral angles in $\left(0, \frac{\pi}{2}\right]$, to be realized as (possibly ideal) convex polytopes in $\mathbb{H}^3$ (these conditions are listed below, Theorem 3.1). In order for this convex polytope to tile $\mathbb{H}^3$, the assigned dihedral angles must be integer submultiples of π.

Let L be a labeled nerve of a Coxeter system, homeomorphic to $\mathbb{S}^2$. K_L has assigned dihedral angles π/m_{st} as discussed in Section 2.1. So, if K_L satisfies

Theorem 3.1, then it follows that $\Sigma_L = \mathbb{H}^3$. However, it is possible that K_L does not satisfy Andreev's theorem. So, for the remainder of the paper, we will show how to apply Theorem 3.1 to a modification $[L-T]$ of L. (Here $[L-T]$ is a cell complex homeomorphic to $\mathbb{S}^2$ with labeled edges.) If $K_{[L-T]}$, with assigned dihedral angles corresponding to the edge labeling, satisfies Andreev's theorem, then it follows that $K_{[L-T]}$ is the strict fundamental domain for the action of a reflection group on $\mathbb{H}^3$.

Theorem 3.1. ([1, Theorem 2]) *Let P be an abstract three-dimensional polyhedron, not a simplex, such that three or four faces meet at every vertex. The following conditions are necessary and sufficient for the existence in $\mathbb{H}^3$ of a convex polytope of finite volume of the combinatorial type P with the dihedral angles $\alpha_{ij} \leq \frac{\pi}{2}$ (where α_{ij} is the dihedral angle between the faces F_i, F_j):*

(i) *If F_1, F_2 and F_3 are all the faces meeting at a vertex of P, then $\alpha_{12} + \alpha_{23} + \alpha_{31} \geq \pi$; and if F_1, F_2, F_3, F_4 are all the faces meeting at a vertex of P then $\alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} = 2\pi$.*

(ii) *If three faces intersect pairwise but do not have a common vertex, then the angles at the three edges of intersection satisfy $\alpha_{12} + \alpha_{23} + \alpha_{31} < \pi$.*

(iii) *Four faces cannot intersect cyclically with all four angles $= \pi/2$ unless two of the opposite faces also intersect.*

(iv) *If P is a triangular prism, then the angles along the base and top cannot all be $\frac{\pi}{2}$.*

(v) *If among the faces F_1, F_2, F_3 we have F_1 and F_2, F_2 and F_3 adjacent, but F_1 and F_3 not adjacent, but concurrent at one vertex and all three do not meet in one vertex, then $\alpha_{12} + \alpha_{23} < \pi$.*

The case where L is the boundary of a 3-simplex. If L is the boundary of a 3-simplex, then K_L is a 3-simplex and we are unable to apply Andreev's theorem. However, one can check that in this case W_L is one of the groups listed in Figure 2.2 or 6.2 of [9] (n=4). In fact, $\Sigma_L = \mathbb{E}^3$ or $\Sigma_L = \mathbb{H}^3$. Therefore, if L is the boundary of a 3-simplex, then it is ℓ^2-acyclic.

Applying Andreev's theorem. Suppose now that L is not the boundary of a 3-simplex. If s is a vertex of L, define the *link of s in L*, L_s, to be the subcomplex of L consisting of all closed simplices which are contained in simplices containing s, but do not themselves contain s. Define the *star of s in L*, $\mathrm{St}_L(s)$, to be the subcomplex of L consisting of all closed simplices which contain s.

The *valence* of a vertex s of L is the number of vertices in its link. We say that a vertex s is *3-Euclidean* if s has valence 3 and if s_0, s_1, s_2 are the vertices in this link, then

$$\frac{\pi}{m_{s_0 s_1}} + \frac{\pi}{m_{s_1 s_2}} + \frac{\pi}{m_{s_2 s_0}} = \pi.$$

We say that $s \in T$ is *4-Euclidean*, if s has valence 4 and if s_0, s_1, s_2, s_3 are the vertices in this link, then $m_{s_i s_{i+1}} = 2$ for $i = 0, 1, 2, 3 \pmod{4}$. We'll say that the vertex s is *Euclidean* if it is either 3- or 4-Euclidean.

Lemma 3.2. *Let s be a Euclidean vertex.*

(a) *If s is a 3-Euclidean vertex, then L_s and $\mathrm{St}_L(s)$ are full subcomplexes of L.*

(b) *If s is a 4-Euclidean vertex and L is not the suspension of a 3-gon, then L_s and $\mathrm{St}_L(s)$ are full subcomplexes of L.*

Proof. (a): This is immediate since L is not the boundary of a 3-simplex.

(b): For a 4-Euclidean vertex s, L_s and $\mathrm{St}_L(s)$ can only fail to be full if L is the suspension of a 3-gon. □

Lemma 3.3. *Suppose that L is not the suspension of a 3-gon and let s be a Euclidean vertex of L. Then L_s is ℓ^2-acyclic.*

Proof. $\Sigma_{L_s} = \mathbb{R}^2$. Thus $\beta_i(L_s) = 0$ for all i. □

Lemma 3.4. *Suppose that L is not the suspension of a 3-gon and let s be a Euclidean vertex of L. Then $\mathrm{St}_L(s)$ is ℓ^2-acyclic.*

Proof. Suppose that s is a 4-Euclidean vertex. Let [St] denote the complex obtained by capping off the boundary of $\mathrm{St}_L(s)$ with a square cell. Then $K_{[\mathrm{St}]}$ clearly satisfies condition (i) and satisfies conditions (ii)-(iv) vacuously. The only condition of Theorem 3.1 that $K_{[\mathrm{St}]}$ may fail to meet is (v).

If $K_{[\mathrm{St}]}$ does not satisfy this condition, then $\mathrm{St}_L(s)$ is a right-angled suspension and therefore ℓ^2-acyclic.

If $K_{[\mathrm{St}]}$ does satisfy condition (v), then $K_{[\mathrm{St}]}$ can be realized as an ideal, convex polytope in $\mathbb{H}^3$, the ideal vertex dual to the square face of [St]. The resulting reflection group is $W_{\mathrm{St}_L(s)}$, and by the results in Section 2.2, $\beta_i(\mathrm{St}_L(s)) = 0$ for all i.

Now suppose that s is a 3-Euclidean vertex. If each edge in $(\mathrm{St}_L(s) - L_s)$ is labeled 2, then $\Sigma_{\mathrm{St}_L(s)} = [-1,1] \times \mathbb{R}^2$ ($\Sigma_s = [-1,1]$), and by equation 2.4, $\mathfrak{h}_i(\mathrm{St}_L(s))$ vanishes. Otherwise, let [St] denote the complex obtained by capping off the boundary of $\mathrm{St}_L(s)$ with a triangular cell. The resulting reflection group, $W_{\mathrm{St}_L(s)}$, is one of the Coxeter groups shown in Figure 6.3 of [9], the non-compact hyperbolic Coxeter groups ($n = 4$). It acts properly as a classical reflection group on $\mathbb{H}^3$ with fundamental chamber $K_{[\mathrm{St}]}$, a simplex of finite volume with one ideal vertex corresponding to the added triangular face of [St]. Therefore $\beta_i(\mathrm{St}_L(s)) = 0$ for all i. □

Let C be a 3-circuit in L and let s_0, s_1, s_2 be the vertices in this circuit. We say that C is an *empty Euclidean* 3 *circuit* if C is not the link of a vertex and if

$$\frac{\pi}{m_{s_0 s_1}} + \frac{\pi}{m_{s_1 s_2}} + \frac{\pi}{m_{s_2 s_0}} = \pi.$$

It follows from L being metric flag that C is a full subcomplex.

Let C be a 4-circuit in L. Order the vertices in this circuit s_0, s_1, s_2, s_3 so that s_i and s_{i+1} are connected by an edge of the circuit and s_i and s_{i+2} are not connected by an edge of the circuit ($i = 0, 1, 2, 3 \mod(4)$). We say C is an *empty Euclidean* 4*-circuit* if (a) C is not the link of a vertex, (b) C is not the boundary of the union of two adjacent 2-simplices, and (c) $m_{s_i s_{i+1}} = 2$ ($i = 0, \ldots, 3 \bmod(4)$). It follows from (b) and the fact that L is metric flag that C is a full subcomplex.

Lemma 3.5. *Suppose that L has no empty Euclidean 4-circuits and that L is not the suspension of a 3, 4, or 5-gon. Then no two Euclidean vertices of L are connected by an edge.*

Proof. First, since L is a metric flag, no two 3-Euclidean vertices are connected by an edge.

Second, suppose that s and s' are 4-Euclidean vertices which are connected by an edge. Then the star of that edge is the configuration pictured in Figure 1. The indicated vertices v and v' cannot coincide, since if they did L would be the suspension of a 3-gon. The top and bottom vertices cannot be connected by an edge, since then $\{t, b, s'\}$ would be a spherical subset, and since L is metric flag, it would not be a triangulation of $\mathbb{S}^2$. Let C be the boundary of the star in the figure. If C is the boundary of two adjacent 2-simplices, then L is the suspension of a 4-gon. If C is the link of a missing vertex, then L is the suspension of a 5-gon. Otherwise, C is an empty Euclidean 4-circuit, a contradiction.

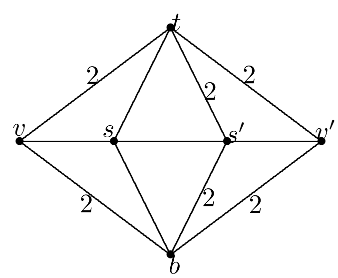

Figure 1: Two 4-Euclidean vertices connected by an edge.

Lastly, suppose that s, a 3-Euclidean vertex, and s', a 4-Euclidean vertex, are connected by an edge. Then the star of that edge is the configuration pictured in Figure 2. Since L is metric flag, $\{r, t, b\}$ is the vertex set of a simplex of L and thus L is the suspension of a 3-gon, with s and r the suspension points, a contradiction.

□

Lemma 3.6. *Suppose L is not the suspension of a 3-gon. Let T be a set of Euclidean vertices of L, no two of which are connected by an edge. Then $\beta_i(L) = \beta_i(L - T)$ for all i.*

Proof. Let s be a Euclidean vertex of L. By Lemma 3.2, L_s and $St_L(s)$ are full subcomplexes. Consider the Mayer-Vietoris sequence:

$$\ldots \to \mathfrak{h}_i(L_s) \to \mathfrak{h}_i(St_L(s)) \oplus \mathfrak{h}_i(L - s) \to \mathfrak{h}_i(L) \to \mathfrak{h}_{i-1}(L_s) \to \ldots$$

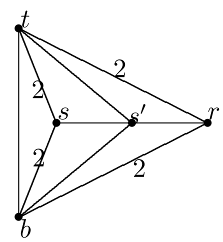

Figure 2: 3-Euclidean vertex connected to 4-Euclidean vertex.

By Lemmas 3.3 and 3.4, $\mathfrak{h}_i(L_s)$ and $\mathfrak{h}_i(St_L(s))$ vanish for all i. The result follows. □

Suppose C is an empty Euclidean 3- or 4-circuit in L. Then C separates L into two 2-disks, D_1 and D_2. Let L_1 and L_2 denote the result of capping off D_1 and D_2, respectively (where "capping off" means adjoining a cone on the boundary, with edges each labeled 2). Let $s_1 \in L_1$ and $s_2 \in L_2$ denote the newly introduced cone points. These are Euclidean vertices. Since C is an empty circuit, the two resulting triangulations, L_1 and L_2, each have fewer vertices than does L. With this set up, we have the following lemma.

Lemma 3.7. *$\beta_i(L) = \beta_i(L_1) + \beta_i(L_2)$ for all i. As a result, $\mathfrak{h}_*$ vanishes for L if and only if it vanishes for both L_1 and L_2.*

Proof. Consider the Mayer-Vietoris sequence for Σ_L:

$$\dots \mathfrak{h}_i(C) \to \mathfrak{h}_i(L_1 - s_1) \oplus \mathfrak{h}_i(L_2 - s_2) \to \mathfrak{h}_i(L) \to \mathfrak{h}_{i-1}(C) \to \dots$$

$\Sigma_C = \mathbb{R}^2$ so $\mathfrak{h}_*(C)$ vanishes. Thus $\beta_i(L) = \beta_i(L_1 - s_1) + \beta_i(L_2 - s_2)$ for all i. By Lemma 3.6, we have that $\beta_i(L_j - s_j) = \beta_i(L_j)$ for all i and for $j = 1, 2$. The desired equality is obtained. □

Eliminating Euclidean vertices. Suppose L is not the suspension of a 3-, 4-, or 5-gon and that L has no empty Euclidean 3- or 4-circuits. Let T denote the set of Euclidean vertices of L. Consider a cellulation $[L - T]$ of $\mathbb{S}^2$ obtained by replacing stars of 4-Euclidean vertices by square cells and by replacing stars of 3-Euclidean vertices by triangular cells. Then either L is a suspension of a 6-gon formed from coning on the boundary of Figure 3, we refer to these as L_6-triangulations, or $[L - T]$ is a well-defined 2-dimensional cell complex homeomorphic to $\mathbb{S}^2$ with triangular and square faces in the strict sense that any nonempty intersection of two cells is a cell.

Lemma 3.8. *L_6-triangulations are ℓ^2-acyclic.*

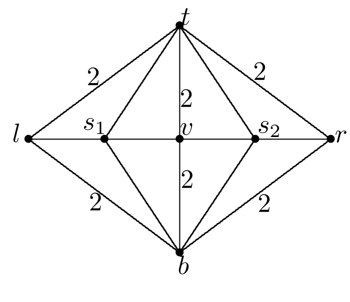

Figure 3: Stars of two 4-Euclidean vertices intersecting in adjacent edges.

Proof. Any L_6-triangulation is the union of the star of a 4-Euclidean vertex and the configuration in Figure 3, with intersection the boundary of the figure. Figure 3 can be decomposed as $\mathrm{St}_L(s_1) \cup \mathrm{St}_L(s_2)$ with intersection being a right-angled suspension. The desired result follows from Mayer-Vietoris. □

Theorem 3.9. *Suppose that L is not the boundary of a 3-simplex and not an L_6-triangulation. Suppose also that*

(a) *L has no empty Euclidean 3 or 4-circuits, and*

(b) *L is not the suspension of a 3-, 4- or 5-gon.*

Let T denote the set of Euclidean vertices of L and let $[L - T]$ be the cellulation of $\mathbb{S}^2$ obtained by replacing stars of vertices in T by triangular or square cells.

Then $K_{[L-T]}$ can be realized as a (possibly ideal), convex polytope in $\mathbb{H}^3$. (The ideal vertices correspond to the square or added triangular faces of $[L - T]$, i.e. to the Euclidean vertices of L.) The resulting classical reflection group is the Coxeter group W_{L-T}.

Proof. If $K_{[L-T]}$ is a 3-simplex, then the Coxeter group W_{L-T} is one of the non-compact hyperbolic Coxeter groups shown in Figure 6.3 of [9]. Then W_{L-T} acts on $\mathbb{H}^3$ with fundamental chamber $K_{[L-T]}$, a finite volume simplex with ideal vertices dual to the added triangular faces of $[L - T]$. Otherwise, we prove that $K_{[L-T]}$ satisfies the conditions of Andreev's theorem.

$[L - T]$ is a cell-complex with triangular and square faces, so $K_{[L-T]}$ has no more than three or four faces meeting at any vertex. Condition (i) is immediate under our hypothesis. The remaining conditions refer to certain configurations of faces of the polytope.

L contains no Euclidean vertices nor any empty Euclidean 3- or 4-circuits, so it follows that every 3- or 4-prismatic element in $K_{[L-T]}$ satisfies condition (ii) or (iii). L is not the suspension of a 3-gon, so the only way $K_{[L-T]}$ can be a triangular prism is if T is nonempty and $[L - T]$ is the suspension of a 3-gon. Then since we replaced the stars of some 3-Euclidean vertices of L with triangular cells whose three edge labels m_1, m_2 and m_3 have the property that $\frac{\pi}{m_1} + \frac{\pi}{m_2} + \frac{\pi}{m_3} = \pi$, we know that not every suspension line is labeled 2. Thus, $K_{[L-T]}$ satisfies condition (iv).

To verify condition (v) we note that if two faces F_1 and F_3 of $K_{[L-T]}$ intersect at a vertex, but are not adjacent, then this vertex must have valence 4. So this vertex corresponds to a square cell of $[L-T]$, where each edge is labeled 2, and the two faces are dual to opposite corners f_1 and f_3 of the square. The configuration in condition (v) has a third face, F_2, adjacent to both the previous two. So its dual vertex, f_2, is connected to both f_1 and f_3 in $[L-T]$, and if either $m_{f_1 f_3} \geq 3$ or $m_{f_2 f_3} \geq 3$, condition (v) is satisfied. So suppose that both $m_{f_1 f_2}$ and $m_{f_2 f_3}$ equal 2. The square in $[L-T]$ corresponds to the star of Euclidean vertex in L. If v denotes one of the remaining corners of the square, then the vertices f_1, f_2, f_3, v mark out a 4 circuit in L, each of whose edges is labeled 2. Since $[L-T]$ is a well-defined cell-complex, this circuit cannot be the link of a missing vertex (two edges of added squares would coincide). But L does not contain empty Euclidean 4-circuits, so f_2 is connected to v (f_1 and f_3 are not connected because in the set-up of condition (v), F_1 and F_3 are non-adjacent faces). This means that L contains a configuration pictured in Figure 1, which according to Lemma 3.5 is impossible. □

Lemma 3.10. *Suppose that L is the suspension of 3-,4- or 5-gon. Then $\mathfrak{h}_*(L)$ vanishes.*

Proof. If K_L satisfies the conditions of Andreev's theorem, then we are done. So we consider cases in which K_L does not satisfy the conditions of Andreev's theorem.

Case 1: Suppose that L is the suspension of a 3-gon. Then the only conditions K_L may fail to meet are (ii) and (iv). Suppose K_L does not satisfy (ii). Then the suspension points, s and s', are 3-Euclidean vertices. $L = St_L(s) \cup St_L(s')$, with $St_L(s) \cap St_L(s) = L_s$, the link of s. Each piece is full in L and ℓ^2-acyclic, (Lemmas 3.2, 3.3 and 3.4). So by Mayer-Vietoris, L is ℓ^2-acyclic.

Now suppose that K_L satisfies (ii) but does not satisfy (iv). Then in L, every suspension line is labeled 2. Thus L is a right-angled suspension and $\mathfrak{h}_*(L)$ vanishes.

Case 2: Suppose that L is the suspension of a 4-gon. Then K_L immediately satisfies conditions (i), (ii), (iv) and (v) of Andreev's theorem. Suppose that K_L does not satisfy condition (iii). Then L has at least two 4-Euclidean vertices, denote them s and s', and these can be arranged so that they are the suspension points. Then $L = St_L(s) \cup St_L(s')$ with $St_L(s) \cap St_L(s') = L_s$. Each piece is full in L and ℓ^2-acyclic. The result follows from Mayer-Vietoris.

Case 3: Lastly, suppose that L is the suspension of a 5-gon. Again, K_L immediately satisfies conditions (i),(ii),(iv) and (v) of Andreev's theorem. If K_L does not satisfy condition (iii), then L has at least one 4-Euclidean vertex, v. First, suppose that this vertex is not connected by an edge to any other 4-Euclidean vertex. Replace the star of this vertex with a square cell, and denote

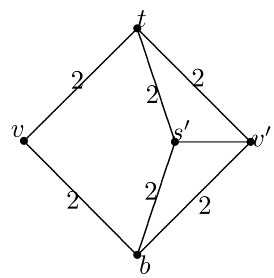

Figure 4: Lemma 3.10

this cell complex by $[L - v]$. The only condition $K_{[L-v]}$ may fail to meet is (v). If $K_{[L-v]}$ satisfies this condition, then v is the only Euclidean vertex, and $K_{[L-v]}$ can be realized as an ideal convex polytope in $\mathbb{H}^3$. (The ideal vertex corresponding to the square face of $[L - v]$.) The resulting reflection group is W_{L-v}, and $\mathfrak{h}_i(L - v)$ vanishes for all i. Hence, by Lemma 3.6, $\mathfrak{h}_i(L)$ also vanishes.

If $K_{[L-v]}$ does not satisfy condition (v), then L decomposes as the $St_L(v)$ and the configuration in Figure 1. The intersection is L_v. Figure 1 decomposes as the star of s, which is a Euclidean vertex, and the configuration in Figure 4, a right-angled suspension. The intersection is L_s. All of these parts are full in L and ℓ^2-acyclic. Use Mayer-Vietoris to determine that L is ℓ^2-acyclic.

Next, suppose that v is connected to another Euclidean vertex. Then in L, there is at most one vertex v' of the 5-gon not connected to the suspension points by edges labeled 2. But, it is itself a 4-Euclidean vertex. So L decomposes as the $\mathrm{St}_L(v')$ and a right-angled suspension, with intersection L'_v. Each piece is full in L and ℓ^2-acyclic. Use Mayer-Vietoris to determine that L is ℓ^2-acyclic. □

Main Theorem 3.11. *Let L be a metric flag triangulation of $\mathbb{S}^2$. Then $\mathfrak{h}_i(L) = 0$ for all i.*

Proof. We may assume L is not the boundary of a 3-simplex, not an L_6-triangulation, and not the suspension of a 3-,4- or 5-gon. If L has no empty Euclidean 3- or 4-circuits, then by Theorem 3.9, and the results in Section 2.2, $\mathfrak{h}_i(L-T)$ vanishes for all i, where T denotes the set of Euclidean vertices. Hence, by Lemmas 3.5 and 3.6, $\mathfrak{h}_i(L)$ also vanishes.

In every other case, L has an empty Euclidean 3- or 4-circuit which we can use to decompose L as, $L = L_1 \diamond L_2$. Since L_1 and L_2 each have fewer vertices than does L, this process must eventually terminate. The theorem follows from Lemma 3.7. □

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