Corrigendum


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DOI of original article: http://dx.doi.org/10.1016/j.mbs.2012.06.004
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In the proofs of Theorems 3.3 and 3.6 of our paper we cannot assume a special structure for the generator matrix \(Q\) with nearly identical rows (this would mean that the infinitesimal probability to jump into a new environment is independent of the current environment). Nevertheless, the idea of the proof, namely that the Lyapunov function can be written as a sum of functions, is correct.

To fix notation, we consider the switching differential equation

\[
\frac{dP_i(t)}{dt} = P_i(t) \left( \omega_i^k - \sum_{j=1}^{m} w_{ij}^k P_j(t) \right),
\]

where \(w_i^k > 0\) is the fitness value of genotype \(i\) in environment \((h, s)\) and

\[
\sum_{j=1}^{m} P_j(t) = 1.
\]

The switching process \(\sigma(t)\) is a Markov process with generator matrix \(Q(P)\) whose entries \(q_{kl}(P)\) are defined by

\[
P(\sigma(t + \Delta t) = k | \sigma(s)) \approx q_{kl}(P) \Delta t + o(\Delta t).\]

The elements \(q_{kl}\) of the generator matrix \(Q\) satisfy \(q_{kl} \geq 0\) for all \(k \neq l\) and \(q_{kk} = 0\) for every \(k \in M\).

**Theorem 1.** Assume that the generator matrix \(Q = (q_{kl})_{k,l=1}^n\) is irreducible and let \(\pi\) be its unique stationary distribution. Let \(P_t\) be the genotype with the highest mean fitness, that is

\[
\pi \cdot \omega_i > \pi \cdot \omega_j, \quad \text{for all } i = 2, \ldots, m
\]

Then the equilibrium \(\pi\) is asymptotically stable in probability and all other equilibria are unstable in probability.

**Proof.** For \(i = 2, \ldots, m\) we set \(a_i = \omega_i - \omega_1\) for the difference of fitness values with respect to genotype 1 and \(a_{i,1} = (a_{i,1}, \ldots, a_{i,m})\).

Using the constraint (2), we eliminate \(P_1\) and obtain the reduced systems

\[
\frac{dP_i(t)}{dt} = a_{i,1} P_i(1 - P_i) - P_i \sum_{j=2}^{m} a_{j,1} P_j,
\]

for \(i = 2, \ldots, m\) and \(k = 1, \ldots, n\). Note that for fixed environment \(k\) the linear part of this system has a diagonal structure. We define

\[
\beta_i := -\pi \cdot a_{i,1} > 0,
\]

with the last inequality holding true since genotype 1 has the higher mean fitness compared to every other genotype. For \(i = 2, \ldots, m\) we solve the systems of equations

\[
QC_i = a_{i,1} + \beta_i 1
\]

for the vector \(C_i = (c_i^1, \ldots, c_i^n)\) where 1 is the column vector with \(n\) entries 1. The right hand sides of these equations are orthogonal to the kernel of \(Q\) which is spanned by 1, hence there exist solutions.

For \(i = 2, \ldots, m\) and \(k = 1, \ldots, n\), we define

\[
V_i(P_i, k) = (1 - c_i^k)P_i^k, \quad P_i > 0.
\]

with \(0 < \gamma < 1\) yet to be selected, in such a way that all coefficients are positive. We have

\[
\mathcal{L}V_i(P_i, k) = \gamma (1 - c_i^k)P_i^{k-1}(a_{i,1} + o(1)) + \sum_{j=1}^{n} \gamma q_{ij} P_j (1 - c_i^k)P_i^k
\]

\[
= \gamma P_i (1 - c_i^k)a_{i,1} + \sum_{j=1}^{n} \gamma q_{ij} c_i^j + o(1)
\]

\[
= \gamma P_i (1 - c_i^k)a_{i,1} - a_{i,1} + \beta_i + o(1)
\]

\[
= \gamma P_i ( - c_i^k a_{i,1} + \pi \cdot a_{i,1} + o(1)).
\]

where we have made use of the fact that the row sums of \(Q\) are zero.

In order to make all the factors in parentheses negative, we have to choose \(0 < \gamma < 1\) such that the inequality

\[
\pi \cdot a_{i,1} < \gamma c_i^k a_{i,1}
\]

holds. By assumption (4), the left hand side of inequality (7) is negative. Therefore, for those indices \(i\) and \(k\) for which \(c_i^k a_{i,1} \geq 0\), no condition arises for \(\gamma\). If on the other hand \(c_i^k a_{i,1} < 0\), then we can select
\[ 0 < \gamma < \min_{i=2, \ldots, m} \left\{ \frac{\pi \cdot a_{i1}}{c_i^1 a_{i1}^k} : c_i^1 a_{i1}^k < 0 \right\}. \]

Although the \( c_i^k \) are not explicitly known, this is a minimum of finitely many positive numbers. The Lyapunov function is the sum of functions of a single variable

\[ V(P_1, \ldots, P_m, k) = \sum_{i=2}^m V_i(P_i, k) \]

and the condition of Proposition 8.6 in [1] follows from the linearity of the operator \( \mathcal{L} \) and the choice of \( \gamma \).

To prove the unstability in probability of equilibrium \( e_i \) for \( i > 1 \), we use the constraint (2) to eliminate \( P_i \). This results in the reduced systems

\[ \frac{dP_i(t)}{dt} = a_{i1}^1 P_i(1 - P_i) - P_i \sum_{j=1}^m a_{ij}^k P_j, \quad (8) \]

for \( l \neq i \) and \( a_{ij}^k = w_{ik} - w_{kl} \). For \( i = 2, \ldots, m \) let \( c_i = (c_i^1, \ldots, c_i^n) \) be the solution of

\[ Qc_i = a_{i1} - \beta_i 1. \]

We set

\[ V(P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_m, k) = V(P_1, k) = (1 - \gamma c_1^1)P_1^k, \quad P_1 > 0. \]

where \( 0 > \gamma > -1 \) has yet to be selected, small enough that all coefficients are positive. With a calculation similar to (6) we obtain

\[ \mathcal{L}V(P_1, k) = \gamma(1 - \gamma c_1^1)P_1^{-1} \left( a_{i1}^k P_1 + o(1) \right) + \sum_{j=1}^n q_{ij} \left( 1 - \gamma c_1^1 \right)P_1^k \]

\[ = \gamma P_1 \left( (1 - \gamma c_1^1) a_{i1}^k - \sum_{j=1}^n q_{ij} c_i^1 + o(1) \right) \]

\[ = \gamma P_1 \left( (1 - \gamma c_1^1) a_{i1}^k - \left( a_{i1}^k - \beta_i \right) + o(1) \right) \]

\[ = \gamma P_1 \left( -\gamma c_1^k a_{i1}^k + c_i a_{i1} + \pi \cdot a_{i1} + o(1) \right). \]

In order to make all the factors in parentheses positive (so that the entire expression becomes negative), we need to have

\[ 0 > \gamma > \max_{i=2, \ldots, m} \left\{ \frac{\pi \cdot a_{ij}}{c_i^1 a_{ij}^k} : c_i^1 a_{ij}^k < 0 \right\}. \]

The expressions whose maximum is taken are all negative since \( \pi \cdot a_{ij} > 0 \) by assumption (4). The condition of Proposition 8.7 in [1] is thereby verified. \( \Box \)

Acknowledgements

We are indebted to Professor Chao Zhu (University of Wisconsin - Milwaukee) for pointing out the mistake in the original article.

Reference