Ergodicity and loss of capacity for a random family of concave maps

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Collaborator

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Overview of the talk

- random dynamical systems and invariant probability measures
- existence and uniqueness theorems
- the quadratic family \( f_\lambda(x) = x(1 + \lambda - x) \)
- method of universal conjugacy
- loss of capacity
- numerical examples, examples for Cantor measures
- outlook

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Random concave maps
For single maps on metric spaces $f : S \to S$ we look for fixed points or periodic points and their stability.

Often the maps contain a “parameter”, and then the special orbits are also parameter-dependent.
Randomness is “uncertainty” about the parameter. A common assumption is that the noise is an independent, identically distributed (iid) random variable.

What replaces the fixed points of the individual maps?
Bhattacharya and Rao (1993), Athreya and Dai (2000, 2002) studied the “classical” logistic map

\[ f_\lambda(x) = \lambda x(1 - x) \]

with \( \lambda \) is a random variable taking values in some subset of [0, 4]. Uniqueness and non-uniqueness results for invariant measures.

Randomized versions of the Beverton-Holt equation

\[ x_{n+1} = \frac{\nu K_n x_n}{K_n + (\nu - 1)x_n}, \]

Haskell and Sacker (2005), Bezandry et al. (2008). “Loss of capacity” phenomenon - varying environments are detrimental.
Let \((S, d)\) be a complete metric space and \((K, \mathcal{B}, m)\) a probability space. For every \(\lambda \in K\), let \(f_\lambda : S \rightarrow S\) a \((\mathcal{B} \otimes \mathcal{B}(S))\)-measurable map, where \(\mathcal{B}(S)\) denotes the Borel \(\sigma\)-algebra on \(S\).

We want to apply the family of maps \((f_\lambda)_{\lambda \in K}\) in an independent identically distributed sequence. For convenience, we use both notations \(f_\lambda(x) = f(x, \lambda)\) interchangeably.

\(m\) will be called the \textit{parameter measure} and is supposed to be fixed throughout.

\textit{L. Arnold, Random Dynamical Systems, 1998} (RDSs in much greater generality)
The skew product

The sequence space $\Omega = K^\mathbb{N}$ has a natural $\sigma$-algebra $\mathcal{F} = \mathcal{B}^\mathbb{N}$ generated by cylinder sets $A_1 \times A_2 \times \ldots$ with $A_i \in \mathcal{B}$, of which only finitely many are proper subsets of $K$ and all others are $K$. We construct a random dynamical system on the measurable product space $(S \times \Omega, \mathcal{B}(S) \otimes \mathcal{F})$ as follows. For a sequence $\mathbf{\lambda} = (\lambda_1, \lambda_2, \ldots)$ let $\theta$ denote the right shift $\theta(\lambda_1, \lambda_2, \ldots) = (\lambda_2, \lambda_3, \ldots)$ and $\pi_1 : \Omega \to K$ the projection onto the first coordinate $\pi_1(\lambda_1, \lambda_2, \ldots) = \lambda_1$. We consider the skew product map $T : S \times \Omega \to S \times \Omega$ given by

$$T(x, \mathbf{\lambda}) = (f(x, \pi_1(\mathbf{\lambda})), \theta(\mathbf{\lambda})).$$
Let $X_0$ be an $S$-valued random variable independent of $\lambda$ and

$$X_n = f^n(X_0, \lambda)$$

the repeated application of $f$ with the first $n$ entries in the random sequence $\lambda$. Then

$$P(x, G) = m\{\lambda \in K : f_\lambda(x) \in G\}$$

is a transition probability for $x \in S$ and any Borel set $G \in \mathcal{B}(S)$.

The transition operators

For a bounded measurable functions \( \phi \in L^\infty(S) \) let

\[
P\phi(x) = \int_S \phi(y) P(x, dy) = \int_K \phi(f_\lambda(x)) m(d\lambda).
\]

and for a Borel measure \( \mu \) on \( S \) let

\[
P^* \mu(G) = \int_S P(x, G) \mu(dx) = \int_K \mu(f_\lambda^{-1}(G)) m(d\lambda).
\]

A probability measure \( \mu \) on \( S \) is called \( P^* \)-invariant if \( P^* \mu = \mu \).
Proposition

If $S$ is compact and all $f_\lambda : S \to S$ are continuous then there exists at least one invariant probability measure.

For an arbitrary probability measure $\eta$ on $S$ define the Cesàro means

$$\eta_n = \frac{1}{n} \sum_{k=0}^{n-1} (P^*)^k \eta, \quad n = 1, 2, \ldots$$

and use Prokhorov’s Theorem to extract a weakly convergent subsequence. This limit is then shown to be $P^*$-invariant.

Ergodicity

A \( P^* \)-invariant probability measure \( \mu \) is **ergodic**, if any Borel set \( G \) whose indicator function satisfies \( P\chi_G = \chi_G \) is \( \mu \)-trivial, that is \( \mu(G) \in \{0,1\} \). This is equivalent to the statement that for every \( \phi \in L^\infty(S) \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(\phi) = \int_S \phi(y) \mu(\text{d}y).
\]

If there are two $P^*$-invariant probability measures $\mu_1 \neq \mu_2$, then they must be singular. On the other hand, if the invariant probability measure is unique, it is ergodic.

Uniqueness of the invariant measure has been proved by Dubins and Freedman (1966) for two situations, namely

- monotone maps of the unit interval that satisfy a certain “splitting condition”, and
- maps on a complete metric space that are contractions in the logarithmic average.

R. Bhattacharya, M. Majumdar, Random Dynamical Systems, 2007
Asymptotic stability

On the space $\mathcal{P}(S)$ of all Borel probability measures, convergence in distribution is metrizable by the total variation norm, given by

$$\|\mu\| = \sup_{A \in \mathcal{B}(S)} \mu(A) - \inf_{A \in \mathcal{B}(S)} \mu(A).$$

The operator $P^*$ is called asymptotically stable if

$$\lim_{n \to \infty} \| (P^*)^n \mu - \mu^* \| = 0,$$

for all $\mu \in \mathcal{P}(S)$, where $\mu^*$ is the unique fixed point of $P^*$. 
First strategy: “Logarithmic” contractions

Let $L^n$ denote the (random) Lipschitz coefficient of $f^n(\cdot, \lambda)$,

$$L^n = \sup \left\{ \frac{d(f^n(x, \lambda), f^n(y, \lambda))}{d(x, y)} : x \neq y \right\}.$$

Proposition

Assume that there exists an $n \geq 1$ such that

$$E[\log L^n] < 0,$$

where $E$ denotes the expectation with respect to the product measure $m^\otimes n$. Then the Markov process $(X_n)$ has a unique invariant probability and is stable in distribution.

R. Bhattacharya, M. Majumdar, Random Dynamical Systems, 2007
Let $\nu \neq 0$ be a measure on $S$. A Markov process $\{X_n\}_{n=0}^\infty$ on $S$ with transition probability $P$ is *Harris irreducible* or $\nu$-*irreducible* if for every $x \in S$ and every $B \in \mathcal{B}(S)$ with $\nu(B) > 0$, there is an $n \geq 1$ such that $P^n(x, B) > 0$.

For any Harris irreducible process there is a sequence $\{E_0, E_1, \ldots, E_{d-1}\}$ of non-empty disjoint sets such that, for any $x \in E_i$, $P\left( x, E_{i+1}^C \right) = 0$ (modulo $d$). If $d = 1$ then the Markov process is called *aperiodic*. 
Second strategy: Irreducibility and aperiodicity

Proposition

If a Markov process is Harris irreducible and aperiodic with invariant measure $\mu^*$ then for any $\mu \in \mathcal{P}(S)$

$$\lim_{n \to \infty} \| (P^*)^n \mu - \mu^* \| = 0.$$

E. Nummelin, General Irreducible Markov Chains and Nonnegative Operators, 1984
Let $\lambda \in K \subset [0, 1]$ and
\[
f_\lambda(x) = x(1 + \lambda - x).
\]
Let
\[
\lambda_- = \min \text{ supp } m, \quad \lambda_+ = \max \text{ supp } m
\]
be the support boundaries of the parameter measure and $\Gamma = [\lambda_-, \lambda_+]$. Then $f_\lambda(\Gamma) \subset \Gamma$ for all $\lambda \in \Gamma$. 
The “shark fin” family with the example $\lambda_- = \frac{1}{4}, \lambda_+ = \frac{3}{4}$. 

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Random concave maps
Consider a growth law with a noisy state-dependent growth parameter given by
\[ x_{n+1} = \theta_n x_n \]
where \( \theta_n \) is a random variable supported on \([1 - x_n, 2 - x_n]\). If
- \( x_n \) is small (\( x_n \approx 0 \)), then likely \( \theta_n > 1 \),
- \( x_n \) is large (\( x_n \approx 1 \)), then likely \( \theta_n < 1 \).
Using \( \lambda_n = \theta_n + x_n - 1 \), then we obtain \( f_\lambda(x) = x(1 + \lambda - x) \).
If \( f'_{\lambda_+}(\lambda_-) < 1 \), then \( |f'(x)| < 1 \) for all \( x, \lambda \in \Gamma \). Then the family \( f_{\lambda} \) has a unique invariant measure \( \mu^*_m \) for every probability measure \( m \) with support boundaries \( \lambda_- \) and \( \lambda_+ \), and is stable in distribution. This is the case if

\[
\lambda_+ < 2\lambda_-.
\]
Compositions of depth 4 of two maps, $\lambda_- = \frac{1}{10}, \lambda_+ = \frac{9}{10}$.

There is no control of large values of $|(f^n(\cdot, \lambda))'|$. 
The interval $\Gamma = [\lambda_-, \lambda_+]$ is *compressible*, if there exists a homeomorphism $h : \Gamma \to \Gamma$ such that the conjugates

$$g_\lambda = h \circ f_\lambda \circ h^{-1}$$

satisfy

$$|g_\lambda'(x)| < 1$$

for all $x, \lambda \in \Gamma$. 

**Compressible intervals**
Let $\alpha > 0$ and define a homeomorphism $h_\alpha : \Gamma \to \Gamma$ by

$$h_\alpha(x) = \lambda_- + (\lambda_+ - \lambda_-) \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^\alpha.$$

The inverse and the derivative are, respectively,

$$h^{-1}_\alpha(x) = h^{-1}_\alpha(x), \quad h'_\alpha(x) = \alpha \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\alpha-1}.$$
The good old chain rule

\[ g'_\lambda(x) = h'_\alpha(f_\lambda(h^{-1}_\alpha(x))) \cdot f'_\lambda(h^{-1}_\alpha(x)) \cdot h'_{\alpha^{-1}}(x) \]

\[ = \left( \frac{f_\lambda(h^{-1}_\alpha(x)) - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\alpha - 1} \left( 1 + \lambda - 2h^{-1}_\alpha(x) \right) \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1 - \alpha}{\alpha}}. \]

A useful observation is that the homeomorphisms are convex or concave and that the conjugates are also concave.
To show the concavity of the conjugates:

For $\lambda = \lambda_-$ we have

$$g''_{\lambda_+}(x) = -(\lambda_+ - \lambda_-)^2 \left( \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} \left( \lambda_- \left( \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - \lambda_+ \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + 1 \right) \right)^{\alpha+1} \left( \alpha - 2\alpha\lambda_+ \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + \lambda_- \left( \alpha \left( 2 \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - 1 \right) + 1 \right) \right) \left( \alpha (x - \lambda_-)^2 \left( \lambda_- \left( \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - \lambda_+ \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + 1 \right) \right)^{-3}.$$  

We observe for the individual terms that, since $0 < \lambda_- \leq x \leq \lambda_+ < 1$ and $\lambda_- < \lambda_+$,

$$\frac{x - \lambda_-}{\lambda_+ - \lambda_-} > 0,$$

$$\lambda_- \left( \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - \lambda_+ \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + 1 = 1 - \lambda_- + \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} (\lambda_- - \lambda_+) \geq 1 - \lambda_+ > 0,$$

$$\alpha - 2\alpha\lambda_+ \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + \lambda_- \left( \alpha \left( 2 \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - 1 \right) + 1 = 1 + \alpha - (1 + \alpha)\lambda_- + 2\alpha \left( \frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} (\lambda_- - \lambda_+) \geq 1 + \alpha - (1 + \alpha)\lambda_- + 2\alpha(\lambda_- - \lambda_+) = 1 + \alpha + (\alpha - 1)\lambda_- - 2\alpha\lambda_+ = 1 - \lambda_- + \alpha(1 + \lambda_- - 2\lambda_+) > 0.$$
Theorem

Let the pair $0 < \lambda_- < \lambda_+ \leq 1$ satisfy $\lambda_+ < \max \left\{ \frac{2 + \lambda_-}{3}, 2\lambda_- \right\}$.

Then the interval $[\lambda_-, \lambda_+]$ is compressible and the family $(f_\lambda)$ admits a unique invariant probability measure $\mu^*$ for every parameter measure $m$ whose support has the interval hull $[\lambda_-, \lambda_+]$.
The conjugacy of the transition operators

**Proof.** Let $P^*$ and $Q^*$ be the transition operators of the families $(f_\lambda)$ and $(g_\lambda)$, respectively. Since

\[ Q^* h_\alpha \mu = \int_\Gamma \mu h_\alpha^{-1} g_\lambda^{-1} m(d\lambda) = \int \mu f_\lambda^{-1} h_\alpha^{-1} m(d\lambda) = h_\alpha P^* \mu, \]

the operators are related by the same conjugacy.

**Remark.** This fails if the homeomorphism would depend on the parameter of the maps, i.e. $g_\lambda = h_\lambda \circ f_\lambda \circ h_\lambda^{-1}$. 😞
Measures with full interval support

Theorem

Let $0 < \lambda_- < \lambda_+ \leq 1$ and assume in addition that $\text{supp } m = \Gamma = [\lambda_-, \lambda_+]$. Then there is a unique probability measure that is a fixed point of the operator $P^*$ which also has support $\Gamma$, and the operator $P^*$ is asymptotically stable.

Proof. For every pair of open subsets $\tilde{E} \subset E \subset \Gamma$, then $m(\tilde{E}) = p > 0$. We show that there exists an $l \geq 0$ such that for any Borel measure $\mu$,

$$(\text{supp } (P^*)^l \mu) \cap E \neq \emptyset$$

and also that the Markov process is aperiodic. □
A two-parameter family of homeomorphisms of $\Gamma$ is given by

$$h_{\alpha,\beta}(x) = \lambda_- + (\lambda_+ - \lambda_-) \left( 1 - \left( \frac{\lambda_+ - x}{\lambda_+ - \lambda_-} \right)^\beta \right)^\alpha.$$
Compression of larger intervals

Here $\lambda_- = 0.2$, $\lambda_+ = 0.996$ and $\alpha = \beta = 0.52$ (but this is now only by numerical evidence).
If $[\lambda_-, \lambda_+]$ is compressible by the homeomorphism $h_{\alpha, \beta}$, then so is the subinterval $[\lambda_-, \tilde{\lambda}_+]$ with $\tilde{\lambda}_+ < \lambda_+$, using the same $h_{\alpha, \beta}$ (again, no proof is available).
Loss of capacity

Note that $f_\lambda(\lambda) = \lambda$ is the globally attracting fixed point of each individual map.

**Theorem**

Assume that $f(\cdot, \lambda)$ is strictly concave and $f(x, \cdot)$ is concave. Let $\mu$ denote the unique invariant measure and

$$\overline{\lambda} = \int_K \lambda \, m(d\lambda), \quad \overline{x} = \int_{\Gamma} x \, \mu(dx).$$

Then $\overline{x} \leq \overline{\lambda}$ with equality if and only if $m$ is a Dirac measure.

**Proof.** Applying Jensen’s inequality twice gives

$$\overline{x} = \int_{\Gamma} x \, \mu(dx) = \int_{\Gamma} x \, P^* \mu(dx) = \int_{\Gamma} \int_K f(x, \lambda) \, m(d\lambda) \, \mu(dx)$$

$$\leq \int_{\Gamma} f(x, \overline{\lambda}) \, \mu(dx) \leq f(\overline{x}, \overline{\lambda}).$$
Special case: Bernoulli distribution with $m(\{\lambda_\leftarrow\}) = p \in (0, 1)$ and $m(\{\lambda_\rightarrow\}) = 1 - p$.

**Example.** Let $\lambda_\leftarrow = \frac{1}{2}$ and $\lambda_\rightarrow = \frac{7}{8}$. Then

$$\max_{x \in \Gamma} f_{\lambda_\leftarrow}(x) < \min_{x \in \Gamma} f_{\lambda_\rightarrow}(x),$$

Both maps are strict contractions, so there is a unique invariant probability measure $\mu^*$ on $[\lambda_\leftarrow, \lambda_\rightarrow]$. 
The gap between the ranges
The support of the invariant measure

For every word $\lambda^{(n)} \in \{-, +\}^n$ let $I_{\lambda^{(n)}} = f^n(\Gamma, \lambda^{(n)})$ and let

$$J_n = \bigcup_{\lambda^{(n)} \in \{-, +\}^n} I_{\lambda^{(n)}}$$

where the union is taken over all possible sequences of symbols $-$ and $+$ of length $n$.

The nested sequence $\{J_n\}$ consists of closed, non-empty sets with

$$J = \bigcap_{n=1}^{\infty} J_n \supset supp \mu^*.$$

We also have $J \subset supp \mu^*$ and therefore $J = supp \mu^*$. $J$ is completely disconnected and contains no isolated points.
Let $\Gamma = [0, 1]$ and select $\frac{1}{2} < b < a < 1$. We define functions

$$f_1(x) = \begin{cases} \frac{bx}{a} & \text{if } 0 \leq x \leq a \\ b - \frac{b(x - a)}{1 - a} & \text{if } a \leq x \leq 1 \end{cases},$$

$$f_2(x) = \begin{cases} 1 - \frac{bx}{1 - a} & \text{if } 0 \leq x \leq 1 - a \\ 1 - b + \frac{b(x - 1 + a)}{a} & \text{if } 1 - a \leq x \leq 1 \end{cases}.$$
Mutually singular invariant measures

The set \( \{0, 1\} \) is invariant under both \( f_1 \) and \( f_2 \). If 
\[
m(\{\lambda = 1\}) = m(\{\lambda = 2\}) = \frac{1}{2}
\] and \( X_{n+1} = f_{\lambda_n}(X_n) \), then
\[
\frac{1}{2} (\delta_0 + \delta_1)
\] is an invariant probability measure.
The sequence of occupation measures

For the Markov chain starting with $X_0 = x$, let

$$\mu_{x,n}(A) = \frac{1}{n} \sum_{j=0}^{n-1} P_x(X_j \in A)$$

be the sequence of occupation measures. Since the state space $\Gamma$ is compact and the process $X_n$ has the Feller property, any vague limit point $\nu$ of the sequence $(\mu_{x,n})_{n=1}^{\infty}$ is an invariant probability measure.
Existence of the vague limit

Theorem
There exists a vague limit $\nu$ that assigns a positive measure to some interval $J = [\kappa, 1 - \kappa]$ with $\kappa > 0$.

This follows from Foster’s Theorem with the help of a Lyapunov function. Let

$$h(x) = \begin{cases} 
1 - 2x & \text{if } 0 \leq x < \frac{1}{2} \\
2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1 
\end{cases}.$$  

With $\kappa = \frac{1 - a}{2b}$ let $J = [\kappa, 1 - \kappa]$. Assume $\nu(J) = 0$ for all vague limits of $(\mu_{x,n})_{n=1}^{\infty}$, then it would follow

$$\lim_{n \to \infty} \frac{1}{n} (E_x(h(X_n)) - h(x)) \leq -\delta < 0.$$ 

However, the limit is indeed

$$\lim_{n \to \infty} \frac{1}{n} E_x(h(X_n)) \geq 0$$ 

since $h(x) \geq 0$. This shows that there is a vague limit $\nu$ with $\nu(J) > 0$. 

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Random concave maps
A density of the “interior” invariant measure
The uniqueness of the invariant probability measure has been shown for

- a partial family of subintervals \(0 < \lambda_- < \lambda_+ \leq 1\) where \(m\) can be an arbitrary probability measure with support bounds \(\lambda_-\) and \(\lambda_+\), and
- any subinterval \(0 < \lambda_- < \lambda_+ \leq 1\) if \(m\) has an absolutely continuous part supported on \([\lambda_-, \lambda_+]\).

Numerical results suggest that the actual range of compressible intervals is bigger.
Many open questions remain for invariant probability measures.

- Do all parameter intervals $0 < \lambda_- < \lambda_+ \leq 1$ admit a unique invariant probability measure?
- What are necessary conditions for existence of universal conjugacies between families of maps?
- If $I$ is an interval and, say, $C \subset I$ is a Lebesgue null set, can we construct a family of maps that has two invariant measures, one supported on $I$, one on $C$?

If we look at skew products over the space of bi-infinite sequences $\Omega_{\pm\infty} = K^\mathbb{Z}$, can we define a pullback attractor $\mathcal{A}(\omega)$?
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Thank you for your attention