S = Set (firms, buyers, budget set, PPF, …)
x = Element (has a property of being in a set)

Define by Enumeration:  
\[ A = \{6, 8, 10\} \]
\[ B = \{0, 1, 2, 3, 4, \ldots\} = Z^+ = \text{Natural Numbers and Zero} \]
\[ = \text{Positive Integers and Zero} = Z^{++} \cup \{0\} \]

Define by Property:  
\[ C = \{x^3 : x \in Z^+\} \]
\[ D = \{x : 10 \leq x \leq 25 \land x/5 \in Z^+\} \]
\[ E = \{1/x : x \in Z^{++}\} \]

**Definition 2.0**

\( U \) (universe) is the set of all possible values (that variables can take in any particular problem.)
Definition 2.1 If all elements of a set $X$ are also elements of set $Y$, then $X$ is a **subset** of $Y$, and we write $X \subseteq Y$

where $\subseteq$ is the set-inclusion relation.

e.g. $A \subseteq B \quad B \subseteq \mathbb{Z}^+$

Definition 2.2 If all elements of a set $X$ are also elements of set $Y$, but not all elements of a set $Y$ are also elements of set $X$, then $X$ is a **proper subset** of $Y$, and we write $X \subset Y$

e.g. $A \subset B \quad B \not\subset \mathbb{Z}^+$
Definition 2.3: Two sets $X$ and $Y$ are equal if they contain exactly the same elements, and we write

$$X = Y$$

Note: $(B \subseteq \mathbb{Z}^+ \text{ and } B \not\subseteq \mathbb{Z}^+)$ imply $B = \mathbb{Z}^+$

Definition 2.4: The **intersection**, $W$ of two sets $X$ and $Y$ is the set of elements that are in both $X$ and $Y$. We write

$$W = X \cap Y$$

e.g. $A \cap C = \{8\}$

Definition 2.5: The **empty set** or the **null set** is the set with no elements. The empty set is always written $\emptyset$.

Note: An intersection of **disjoint** sets is an empty set. E.g. $A \cap E = \emptyset$

Definition 2.6: The **union**, $V$ of two sets $X$ and $Y$ is the set of elements in one or the other of the sets. We write

$$V = X \cup Y$$
Definition 2.7: The complement of set $X$ is the set of elements of the universal set $U$ that are not in $X$, we write it as $X$.

Note: $\emptyset = \overline{U}$

Definition 2.8: The relative difference of $X$ and $Y$, denoted $X - Y$, is the set of elements of $X$ that are not also in $Y$.

Definition 2.9: A partition of the set $X$ is a collection of disjoint subsets of $X$, the union of which is $X$.

Definition 2.10: The power set of a set $X$ is the set of all subsets of $X$. It is written $P(X)$.

Note: Null set is a subset of any power sets.
**Definition 2.10a**  An ordered pair is a set with two elements which occur in a definite order.

Note that for ordered sets we will use round (or oval) brackets. Take two elements 3 and 5. If they form a pair (3, 5) and the order of the numbers cannot be changed, these elements are said to form an ordered pair. Note that for ordered pairs (3, 5) \(\neq\) (5, 3).

Two ordered pairs \((a, b)\) and \((c, d)\) are equal, \((a, b) = (c, d)\) iff \(a = c\) and \(b = d\). This implies that two ordered pairs \((a, b)\) and \((b, a)\) are equal, \((a, b) = (b, a)\) iff \(a = b\).

Given two sets \(X\) and \(Y\). Then, all possible ordered pairs \((x, y)\) obtained such that \(x \in X\) and \(y \in Y\) is called the Cartesian product of the sets \(X\) and \(Y\):

**Definition 2.10b**  The **Cartesian Product** of two sets \(X\) and \(Y\), written \(X \times Y\), is the set of ordered pairs formed by taking in turn each element in \(X\) and associating with it each element in \(Y\).

For example, the Cartesian product of the sets \(\{1, 2, 3\}\) and \(\{a, 2\}\) is

\[
\{1,2,3\} \times \{a, 2\} = \{(1,a), (1,2), (2,a), (2,2), (3,a),(3,2)\}
\]
Recall definitions of sets A and D: 

\[ A = \{6, 8, 10\} \]

\[ D = \{x : 10 \leq x \leq 25 \text{ & } x/5 \in \mathbb{Z}^+\} \]

Assume that \( U = \mathbb{Z}^+ \) and answer the following:

\[ A \cap D = \] 

\[ A \cup D = \] 

\[ (\overline{A} \cap \overline{D}) = \] 

\[ (\overline{A} \cup \overline{D}) = \] 

\[ A - D = \]
Give two examples of “partition of $A$”

(i) : 

$P(A) = \phantom{0}$

(ii) :

$P(A) \otimes P(D) = \phantom{0}$
2.2 NUMBERS

$Z^{++} = \{1, 2, 3, \ldots\}$  \hspace{1cm} \textbf{Natural Numbers}

- **Properties:** Can be used to count
- **CARDINALITY** of set $S = \text{number of items in } S$
- $\Rightarrow$ cardinality of any non empty set is a subject of $Z^{++}$

Let $a, b \in Z^{++}$
- **Addition:** $a + b \in Z^{++}$ ... $Z^{++}$ is closed under these
- **Multiplication:** $a \ b \in Z^{++}$

$Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$  \hspace{1cm} \textbf{Integers}

- **Properties:** Closed under three operation: addition, multiplication and subtraction.
  - It is not closed under division

$Q = \{a/b : a \in Z, b \in Z^{++}\}$  \hspace{1cm} \textbf{Rational Numbers}

- **Properties:** Closed under addition, multiplication and subtraction.
  - Closed under division with exception of division by zero.
Milon of Croton
(late 6th century BC) was the most famous of Greek athletes in Antiquity.
He was born in the Greek colony of Croton in Southern Italy. He was a six time Olympic victor; once for Boys Wrestling in 540 BC at the 60th Olympics, and five time wrestling champion at the 62nd through 66th Olympiads.
Legend has it that he would train in the off years by carrying a newborn calf on his back every day until the Olympics took place. By the time the events were to take place, he was carrying a four year old cow on his back.
Another legend says that he offered to cut down a large tree for a woodsman, who promised to return with food later in the day. However, the woodsman never returned, and while Milon was working the tree collapsed on his hand, trapping him. The legend says that Milon was then eaten by wolves.
It is said that he was a follower of Pythagoras and that he commanded the army which defeated the Sybarites in 511 BC.
Pythagoras (circa 570 B.C. - c. 495 B.C.)

Pythagoras was born on the Greek island of Samos. He traveled to Egypt with the intention of learning the sacred rights and secrets of the region's religious sects. He was eventually accepted at Thebes, where he studied for at least a decade.

In 525 BC the Persians invaded Egypt and Pythagoras was taken captive to Babylon. There Pythagoras studied under Zaractas, from whom he learned astrology and the use of drugs for purifying the mind and body. He was also initiated into Zoroastrianism and his famous Phythagorian theorem probably had a Babylonian origin.

Pythagoras returned to Samos and tried to teach, but soon left forever and founded a religious community in Croton, Italy. He and his followers never cut their hair or beards and were forbidden to wear leather or wool.

Pythagoras believed the mystery of the universe revealed itself in numbers, to which he ascribed qualities like maleness (odd numbers) and femaleness (even numbers). He discovered that strings in a 2:1 ratio vibrate an octave apart, while those in a 3:2 ratio produce a musical fifth, etc.
Hippasus of Metapontum

One story claims that a young student by the name of Hippasus was idly toying with the number $\sqrt{2}$, attempting to find the equivalent fraction.

Eventually he came to realize that no such fraction existed, i.e. that $\sqrt{2}$ is an irrational number. Hippasus must have been overjoyed by his discovery, but his master was not. Pythagoras had defined the universe in terms of rational numbers, and the existence of irrational numbers brought his ideal into question.

The consequence of Hippasus’ insight should have been a period of discussion and contemplation during which Pythagoras ought to have come to terms with this new source of numbers. However, Pythagoras was unwilling to accept that he was wrong, but at the same time he was unable to destroy Hippasus’ argument by the power of logic. *To his eternal shame he sentenced Hippasus to death by drowning* (or exile).
Note: Not all numbers are rational. Prove that $\sqrt{2}$ is not rational (THEOREM 2.1)

**Proof by contradiction.** We want to prove $A (= \sqrt{2}$ is not rational).

We do so by showing that premise $\neg A$, some true premise $B$ and false conclusion $C$ is a valid argument. Recall that for an argument to be valid, a false conclusion must result of at least one false premise. Thus $\neg A$ is false & $A$ is TRUE.

1. Assume $\neg A$ (=that $\sqrt{2} = a/b$ where $a$ is integer and $b$ is a natural number)
2. Assume $B$ (=that $b$ is the smallest natural number s.t. $\sqrt{2} = a/b$. We choose it.)
3. $\sqrt{2} = a / b$ is equivalent to $2b^2 = a^2$
   As a multiple of odd numbers is always odd, therefore $a$ must be even.
4. Let $a = 2c$ and so $2b^2 = (2c)^2 = 4c^2$
   Therefore $b$ must be also even.
5. So we conclude $C$: $\sqrt{2} = (a/2)/(b/2)$ where $a/2$ is integer and $b/2$ is a natural number that is half of $b$.
6. $C$ is false. ($B$ assumes that $b$ is the smallest and nothing, including $b/2$ cannot be smaller than smallest $b$.)
7. Because $[\neg A \land B ] \rightarrow C$ (where $B$ is true and $C$ is false) is a valid argument then $\neg A$ must be false (Recall Exercise 16 with $\neg A$ being P.)
   … and thus $A$ must be true ($\sqrt{2}$ is not a rational number.)
\[ R = (-\infty, \infty) \]

**Real Numbers** (Rational & Irrational)

Consider three elements of R: \( a, b \) and \( c \). Properties:

1. **Closure:** If \( a, b \in R \) then \( a+b \in R \) and \( ab \in R \).

2. **Cummulative laws:** If \( a, b \in R \) then \( a+b = b+a \) and \( ab = ba \).

3. **Associative laws:** If \( a, b & c \in R \) then \( a+(b+c) = (a+b)+c \) and \( a(bc) = (ab)c \).

4. **Distributive law:** If \( a, b & c \in R \) then \( a(b+c) = ab + ac \).

5. **Zero:** The element \( 0 \in R \) is defined as having the property that for all \( a \in R \),
   \[
   a + 0 = a \quad \text{and} \quad a0 = 0.
   \]

6. **One:** The element \( 1 \in R \) is defined as having the property that for all \( a \in R \), \( 1a = a \).

7. **Negation:** If \( a \in R \) then there is an element \( -a \in R \) defined as having the property \( a + (-a) = 0 \).

8. **Reciprocals:** If \( a \in R - \{0\} \), then there is an element \( 1/a \in R \) defined as having the property \( a \, (1/a) = 1 \).

Note: For \( a = 0 \), the reciprocal is undefined.
Example: Show that if \( x + y = 0 \) then \( y = (-x) \).

1/ note that \((-x) + 0\) \(= (-x)\) from 5 where \( a = (-x) \)

2/ note that \((-x) + 0 = (-x) + (x+y)\) from assumption above

3/ note that \((-x) + (x+y) = ((-x)+x)+y\) from 3 where \( a = (-x), b = x, c = y \)

4/ note that \(((−x)+x)+y) = (x+(−x))+y\) from 2 where \( a = (-x), b = x \)

5/ note that \((x+(−x))+y = 0 + y\) from 7 where \( a = x \)

6/ note that \( 0 + y = y + 0\) from 2 where \( a = 0, b = y \)

7/ note that \( y + 0 = y\) from 2 where \( a = 0, b = y \)

Thus \( y = (-x) \)

**Definition 2.11**

The set \( \mathbb{R}^{++} \subset \mathbb{R} \) consists of *strictly positive numbers* with the characteristics that

(i) \( \mathbb{R}^{++} \) is closed under addition and multiplication.

(ii) For any \( a \in \mathbb{R} \), exactly one of the following is true:

\[ a \in \mathbb{R}^{++} \quad \text{or} \quad a = 0 \quad \text{or} \quad -a \in \mathbb{R}^{++} \]
Definition 2.12  The set $R_+ = R^{++} \cup \{0\}$ is the set of nonnegative real numbers.

Definition 2.13  Given any $a, b \in R$:
(i) if $a - b \in R^{++}$ then $a > b$
(ii) if $-(a - b) \in R^{++}$ then $b > a$
(iii) if $a - b \in R_+$ then $a \geq b$
(iv) if $-(a - b) \in R_+$ then $b \geq a$

We refer to “$>$” as the strict inequality and “$\geq$” as the weak inequality.

THEOREM 2.2  For any $a, b, c \in R$.

(i) **COMPLETENESS**  Exactly one of the following is true:

```
   a > b    or    a = b    or    b > a
```

(ii) **TRANSITIVITY**  If $a > b$ and $b > c$ then $a > c$
    and if $a \geq b$ and $b \geq c$ then $a \geq c$.

(iii) **REFLEXIVITY**  $a \geq a$

(iv) **EQUALITY**  If $a \geq b$ and $b \geq a$ then $a = b$
Definition 2.14 A set $S \subset \mathbb{R}$ is **bounded above** if there exists $b \in \mathbb{R}$ such that for all $x \in S$, $b \geq x$; $b$ is called an **upper bound** of $S$.

A set $S \subset \mathbb{R}$ is **bounded below** if there exists $a \in \mathbb{R}$ such that for all $x \in S$, $x \geq a$; $a$ is called an **lower bound** of $S$.

If a subset of $\mathbb{R}$ has an upper (lower) bound then it has an infinity of upper (lower) bounds. However, one upper and one lower bound are of a special interest. The smallest upper bound is called **supremum** and the largest lower bound is called **infimum**.

Definition 2.15 The **supremum** of a set $S$, $\sup S$, has the following properties:

(i) $\sup S \geq x$ for all $x \in S$.
(ii) If $b$ is an upper bound of $S$, then $b \geq \sup S$.

Definition 2.16 The **infimum** of a set $S$, $\inf S$, has the following properties:

(i) $x \geq \inf S$ for all $x \in S$.
(ii) If $a$ is a lower bound of $S$, then $\inf S \geq a$.

**Theorem 2.3** If the $\sup$ or the $\inf$ of subset of $\mathbb{R}$ exists, then it is unique.

**Theorem 2.4** Every nonempty subset of $\mathbb{R}$ that has an upper bound has a supremum in $\mathbb{R}$ and every nonempty subset of $\mathbb{R}$ that has a lower bound has an infimum in $\mathbb{R}$. 
**Definition 2.17**  
**Point** represents a location in (an $n$-dimensional) space. We can represent such location with an ordered set of $n$ real number coordinates.

**Definition 2.17b**  
Given points $a = (a_1, \ldots a_n)$ and $b = (b_1, \ldots b_n)$ in $\mathbb{R}^n$, $n \geq 1$, the **Euclidean distance** between them is

$$d(a,b) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

**Definition 2.18**  
An $\varepsilon$-**neighborhood** of a point $a_0 \in \mathbb{R}^n$ is given by the set $N_{\varepsilon}(a_0) = \{ a \in \mathbb{R}^n : d(a_0, a) < \varepsilon \}$. Simply, $N_{\varepsilon}(a_0)$ is the set of points lying within a distance $\varepsilon$ of $a_0$.

- $N_{\varepsilon}(2) = \{ x \in \mathbb{R} : \sqrt{(x-2)^2} < \varepsilon \}$
- $N_{\varepsilon}(3,2) = \{ x \in \mathbb{R} : \sqrt{[(x1-3)^2 + (x2-2)^2]} < \varepsilon \}$
- $N_{\varepsilon}(3,2,5) =$ sphere

**Definition 2.19**  
A **boundary point** of a set $A \subset \mathbb{R}^n$ is a point $b$, such that every $N_{\varepsilon}(b)$ contains a point that is in $A$ and a point that is not in $A$. 
**Definition 2.20**

A set $A \subseteq \mathbb{R}^n$ is **open** if, for every $a \in A$, there exists an $\varepsilon$ such that $N_\varepsilon(a) \subseteq A$.

Note that an open set does not contain any boundary points.

Round, oval brackets: $(0,1)$

**Definition 2.21**

A set $B \subseteq \mathbb{R}^n$ is **closed** if its complement $\overline{B} \subseteq \mathbb{R}^n$ is an open set.

Note that a closed set contains all its boundary points.
Also note that as set may contain only some of its boundary points than **some sets are neither open nor closed**.

Square, box brackets $[0,1]$ or curved $\{5\}$

**Definition 2.21a**

**Intervals** are special sets of points in one dimensional space $\mathbb{R}^1$.

Given points $a, b \in \mathbb{R}^1$, with $b > a$, we define:

- Open Interval $(a, b) = \{ x \in \mathbb{R}^1 : b > x > a \}$
- Half-Open Interval $(a, b] = \{ x \in \mathbb{R}^1 : b \geq x > a \}$
- Half-Open Interval $[a, b) = \{ x \in \mathbb{R}^1 : b > x \geq a \}$
- Closed Interval: $[a, b] = \{ x \in \mathbb{R}^1 : b \geq x \geq a \}$
**Definition 2.22**

A set $C \subset \mathbb{R}^n$ is **bounded** if, for every $c \in C$, there exists an $\varepsilon$ such that $C \subset N_\varepsilon(c)$.

>Can be enclosed by sufficiently big neighborhood.<

**Definition 2.22a**

A set that is not bounded is an **unbounded set**.

**Definition 2.22b**

Intervals can be unbounded above or/and below:

- above: $(a, +\infty) = \{ x \in \mathbb{R}^1 : x > a \}$
- $[a, +\infty) = \{ x \in \mathbb{R}^1 : x \geq a \}$

- below: $(-\infty, b) = \{ x \in \mathbb{R}^1 : b > x \}$
- $(-\infty, b] = \{ x \in \mathbb{R}^1 : b \geq x \}$

Note that $+\infty$ and $-\infty \notin \mathbb{R}$ and thus they are not boundary points. It follows that unbounded set can be open (e.g. interval $(1, +\infty)$), closed (e.g. interval $[1, +\infty)$) or neither open or closed (e.g. set $(1, +\infty) \cup \{0\}$).

**Definition 2.22c**

A **compact set** is both closed and bounded.
Definition 2.23: Given points \( a, b \in \mathbb{R}^n \), their **convex combination** is the set of points

\[
C = \left\{ c \in \mathbb{R}^n, c = \lambda a + (1 - \lambda) b = \left( \lambda a_1 + (1 - \lambda) b_1, \lambda a_2 + (1 - \lambda) b_2, \ldots, \lambda a_n + (1 - \lambda) b_n \right), \lambda \in [0,1] \right\}
\]

Definition 2.24: A set \( C \subset \mathbb{R}^n \) is **convex** if for every pair of points \( a, b \in C \), and any \( \lambda \in [0,1] \), the point \( c = \lambda a + (1 - \lambda) b \) also belongs to the set \( X \).

Definition 2.25: An **interior point** of a set \( C \subset \mathbb{R}^n \) is a point \( c \in C \) for which exists an \( \epsilon \) such that \( N_\epsilon(c) \subset C \).

Definition 2.26: A set \( C \subset \mathbb{R}^n \) is **strictly convex**, if for every pair of points \( a, b \in C \), and every \( \lambda \in (0,1) \), the \( c = \lambda a + (1 - \lambda) b \) is an interior point of \( C \).
Exercises

Find $\sup(Z++)$ and prove that it is a supremum, or prove that $Z++$ is not bounded from above.

Proof by contradiction: Assume $a$ is an upper bound of $Z++$.
Because $\max\{1, \text{round}(a+1)\} \in Z++$ and is greater than $a$ then $a$ is not an upper bound. Thus the upper bound does not exist.

Find $\inf(Z++)$ and prove that it is a infimum, or prove that $Z++$ is not bounded from below.

$1 = \inf(Z++)$. Proof (from definition 2.16): (i) yes, $x \geq 1$ for all $x \in Z++ = \{1,2,3,...\}$
(ii) yes, any number above 1 cannot be lower bound of $Z++$. as $1 = \min\{Z++\}$

Find $\inf(R++)$ and prove that it is a infimum, or prove that $R++$ is not bounded from above.

$0 = \inf(R++)$. Proof (from definition 2.16): (i) yes, $x \geq 0$ for all $x \in R++$
(ii) yes, any $y>0$ is not a lower bound because $y > y/2 \in R++$. 

Find the Euclidean distance between points $(1, 2, 3)$ and $(3, 1, -3)$ in $R^3$.

$$\sqrt{(1-3)^2 + (2-1)^2 + (3+3)^2} = \sqrt{41}$$

Describe the $N_\varepsilon(1, 2)$.

Decide if points $(3, 2, 1), (8, 6, 4)$ and $(5, 3, 1)$ are convex combinations of points $(2, 1, 0)$ and $(6, 5, 4)$.

Point $(3, 2, 1)$ is a convex comb. [because $(3,2,1) = \frac{3}{4} (2,1,0) + \frac{1}{4} (6,5,4)$];
points $(8, 6, 4)$ and $(5, 3, 1)$ are not convex combinations of points $(2, 1, 0)$ and $(6, 5, 4)$. 

22
Which of the following sets are convex in $\mathbb{R}^2$? Which are strictly convex $\mathbb{R}^2$?

The consumption set is: $A = \{(x, y) \in \mathbb{R}^2: x, y > 0, xy > a\}$ where $a > 0$.
The consumer’s budget set is: $B = \{(x, y) \in \mathbb{R}^2: x, y \geq 0, p_1x + p_2y \leq m\}$, where $p_1, p_2, m > 0$.


How about the set $A \cap B$?

Note: if $a > m^2/(4p_1p_2)$ then it is an empty set.

$A \cap B$ open? NO  Closed? NO  Bounded? YES  Convex? YES  Strictly Convex? NO
2.4 FUNCTIONS

**Definition 2.27** Given two sets $X$ an $Y$, a **function** from $X$ to $Y$ is a rule that associates with each element of $X$, one and only one element of $Y$.

We write a function $f$ as $f : X \mapsto Y$ or as $y = f(x)$, $x \in X$

where $y$ is the **image** of $x$ or the **value** of function $f$ at $x$.

$X$ is called the **domain** and $Y$ is the **codomain**.

$f(X)$, a subset of $Y$ with elements that are associated with elements of $X$, is the **range**: $f(X) = \{ y \in Y: y = f(x), x \in X \}$

Concavity, Convexity, Quasi-concavity, Strict Concavity, …
Graph of $y = f(x)$ and points A, B, C and D.

Quasi-concavity $\rightarrow$ C is not below (A or B)

Strict quasi-concavity $\rightarrow$ C is above (A or B)

Concavity $\rightarrow$ C is not below D
Strict concavity \[\rightarrow\] C is above D \hspace{1cm} \text{(switch below/above for convexities)}

**Definition 2.28** Let \( c = \lambda a + (1 - \lambda) b \).

(i) The function \( f \) is **quasi concave** iff
\[
f(c) \geq \min\{f(a), f(b)\}
\]
for any \( a, b \in X \) and any \( \lambda \in (0, 1) \).

(ii) It is **strictly quasi concave** iff
\[
f(c) > \min\{f(a), f(b)\}
\]
for any \( a, b \in X \) and any \( \lambda \in (0, 1) \).

(iii) It is **concave** iff
\[
f(c) \geq \lambda f(a) + (1 - \lambda) f(b)
\]
for any \( a, b \in X \) and any \( \lambda \in (0, 1) \).

(iv) It is **strictly concave** iff
\[
f(c) > \lambda f(a) + (1 - \lambda) f(b)
\]
for any \( a, b \in X \) and any \( \lambda \in (0, 1) \).

Note:

\[(iv) \rightarrow (iii), \quad (iii) \rightarrow (i),\]
\[(iv) \rightarrow (ii), \quad (ii) \rightarrow (i)\]
Prove that \( f_1(x), f_2(x), f_3(x) \) and \( f_4(x) \) have the following properties and draw an example of function \( f_5(x) \):

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<thead>
<tr>
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<th>( Q)-concave</th>
<th>Strictly ( Q)-concave</th>
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<td>( f_3(x) )</td>
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<td>( f_4(x) )</td>
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<tr>
<td>( f_5(x) )</td>
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<td>NO</td>
<td>( \checkmark )</td>
<td>NO</td>
</tr>
</tbody>
</table>
**Definition 2.29** Let \( c = \lambda a + (1 - \lambda) b \).

(i) The function \( f \) is **quasi convex** iff
\[
f(c) \leq \max \{ f(a), f(b) \}
\]
for any \( a, b \in X \) and any \( \lambda \in (0, 1) \).

(ii) It is **strictly quasi convex** iff
\[
f(c) < \max \{ f(a), f(b) \}
\]
for any \( a, b \in X \) and any \( \lambda \in (0, 1) \).

(iii) It is **convex** iff
\[
f(c) \leq \lambda f(a) + (1 - \lambda) f(b)
\]
for any \( a, b \in X \) and any \( \lambda \in (0, 1) \).

(iv) It is **strictly convex** iff
\[
f(c) < \lambda f(a) + (1 - \lambda) f(b)
\]
for any \( a, b \in X \) and any \( \lambda \in (0, 1) \).

Note: \( \text{(iv)} \to \text{(iii)}, \quad \text{(iii)} \to \text{(i)} \)
\( \text{(iv)} \to \text{(ii)}, \quad \text{(ii)} \to \text{(i)} \).
Draw an example of a **strictly quasi-concave** function that is also (i)  strictly convex \([f_6(x)]\)
(ii) not quasi convex \([f_7(x)]\)
(iii) not concave \([f_8(x)]\)
(iv) concave but not strictly concave \([f_9(x)]\)
Sanssouci
Sanssouci
\[ \rho dA ds \cdot \frac{Dw}{dt} = \rho g dA ds \cos \alpha + dA \left( p - \left[ p + \frac{\partial p}{\partial s} ds \right] \right) \]

mass \times \text{acceleration} = \text{gravity force} + \text{pressure force}