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A Stable Range for Dimensions of Homogeneous $O(n)$-Invariant Polynomials on the $n \times n$ Matrices[1]

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The complex orthogonal group $O(n)$ acts on the $n \times n$ matrices, M_n, by restricting the adjoint action of $GL(n, \mathbb{C})$. This action provides us with an action on the ring of complex valued polynomial functions on the $n \times n$ matrices, $\mathcal{P}(M_n)$. The polynomials of degree d, denoted $\mathcal{P}^d(M_n)$, form a finite dimensional representation of $O(n)$ and provide a graded module structure on $\mathcal{P}(M_n)$ as well as the subring of invariant polynomials, $\mathcal{P}(M_n)^{O(n)}$. For $0 \leq d \leq n$, it is shown that $\dim \mathcal{P}^d(M_n)^{O(n)}$ is equal to the coefficient of q^d in $\Pi_{k=1}^{\infty}(1/(1-q^k))^{c_k}$, where c_k is the number of k vertex cyclic graphs with directed edges counted up to dihedral symmetry. The above formula provides a combinatorial interpretation of an initial segment of the Hilbert series for this ring.

Key Words: Schur–Weyl duality; symmetric pairs; Littlewood–Richardson coefficients.

1. INTRODUCTION

We consider the action of the complex orthogonal group, $O(n)$, on the polynomial functions on the $n \times n$ matrices, M_n, by the action defined by conjugation. That is, for $g \in O(n)$, $X \in M_n$, and $f \in \mathcal{P}(M_n)$,

$$g \cdot f(X) = f(g^{-1}Xg).$$

The homogeneous polynomials of degree d, denoted $\mathcal{P}^d(M_n)$, are a finite dimensional representation of $O(n)$. This provides a graded module

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structure on $\mathscr{P}(M_n)$, as well as the subalgebra of invariant polynomials, $\mathscr{P}(M_n)^{O(n)}$. In this paper we investigate the Hilbert series of this ring. At the end of this section we provide a summary of the results in the paper, but first we introduce some terminology.

A *directed cycle* is a (unlabeled) cyclic graph with its edges oriented, that is, an unlabeled directed graph with cyclic underlying graph. We will allow the one or two vertex cases, so an edge joining a vertex to itself is a directed cycle, as well as the case of two edges joining a pair of vertices. Let $\mathscr{C}_k$ denote the set of k vertex directed cycles. Define $c_k \equiv |\mathscr{C}_k|$. An example of an element in $\mathscr{C}_4$ is shown in Fig. 1.

For $m \geq 0$, let d_m denote the coefficient of q^m in the expansion of the following formal product,

$$\tilde{\eta}(q) \equiv \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right)^{c_k}. \tag{1}$$

We will see that the above formula provides a combinatorial interpretation for an initial segment of the Hilbert series of $\mathscr{P}(M_n)^{O(n)}$. By the *stable range* mentioned in the title, we mean the set, $\{(n, m) | 0 \leq m \leq n, \text{ and } n \geq 1\}$. For (n, m) in the stable range, it is shown in Section 4 that $\dim \mathscr{P}^m(M_n)^{O(n)}$ is equal to d_m. This fact has consequences in classical invariant theory which we describe next.

The algebra $\mathscr{P}(M_n)^{O(n)}$ is addressed in [6], where an argument based on the fundamental theorems of invariant theory describes a set of generators for this example. The generating set from [6] for $\mathscr{P}(M_n)^{O(n)}$ consists of traces of monomials in X and X^T, with $X \in M_n$. That is to say, functions of the form

$$\text{Trace}\big(X^{a_1}(X^T)^{a_2}X^{a_3}(X^T)^{a_4}\cdots\big) \tag{2}$$

generate $\mathscr{P}(M_n)^{O(n)}$. We remark that although products of the above expressions necessarily span the invariants, they may not be linearly independent.

It is interesting to note that there is a correspondence between elements of $\mathscr{C}_k$ and polynomials in the form of Line (2). Here we describe this correspondence. Let $C \in \mathscr{C}_k$. Choose arbitrarily an edge A_1 in C. Let $\{A_2, A_3, \ldots, A_k\}$ denote the sequence of edges traversed clockwise

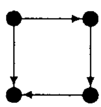

FIG. 1. An element in C_4.

from A_1. Let $f : M_n \to \mathbb{C}$ be defined by $f(X) = \text{Trace}(\prod_{i=1}^k f_i(X))$, where

$$f_i(X) = \begin{cases} X & \text{if } A_i \text{ is oriented clockwise,} \\ X^T & \text{if } A_i \text{ is oriented counterclockwise.} \end{cases} \tag{3}$$

$f(X)$ is an $O(n)$-invariant polynomial function in the form of Line (2). Observe that the definition is independent of the initial choice of A_1 because a cyclic permutation of the variables will not effect the value of the trace. Also, we could have interchanged the words "clockwise" and "counterclockwise" because the trace of a matrix is the same as the trace of its transpose. For example, Fig. 1 could correspond to either $\text{Trace}(X^3X^T)$, $\text{Trace}(X^2X^TX)$, or $\text{Trace}(X(X^T)^3)$.

Let $\mathcal{G}_{n,m}$ denote the set of all degree m polynomial functions on M_n which are products of polynomials in the form given in Line (2). Because the polynomials in Line (2) generate $[\mathcal{P}(M_n)]^{O(n)}$, the dimension of the space of homogeneous $O(n)$-invariant polynomial functions of degree m is bounded above by $|\mathcal{G}_{n,m}|$. In Section 4 we provide a more precise account of this fact by proving that in the stable range, d_m is the exact dimension of $[\mathcal{P}(M_n)]^{O(n)}$ and $|\mathcal{G}_{n,m}| = d_m$.

Next we will set up some notation. For $n \geq 1$, define the Hilbert series of $\mathcal{P}(M_n)^{O(n)}$ as $H_n(q) = \sum_{d=0}^{\infty} h_{n,d}q^d$, where

$$h_{n,d} \equiv \dim \mathcal{P}^d(M_n)^{O(n)}, \tag{4}$$

for all $d \geq 0$. Expressing the formal power series $H_n(q)$ in a simpler form will be the subject of further work. Specially, $H_n(q)$ should be written as a rational function of the form,

$$H_n(q) = \frac{a_0 + a_1q + a_2q^2 + \cdots + a_rq^r}{\prod_{i=1}^k (1 - q^{e_i})}, \tag{5}$$

where k, r, a_i, and e_i are non-negative integers depending only on n. Except for k, there is not a unique choice for these numbers. In general, k is the Krull dimension of the algebra of invariants. For the case addressed here k can be shown to be $\binom{n+1}{2}$ (see [10]). Calculating r, a_i, and e_i is still an open problem for $n \geq 5$. Some discussion of this problem is provided in [12].

Irreducible regular representations of the group $GL(n)$ (or for any connected reductive linear algebraic group for that matter) are indexed by highest weight vectors. The highest weight vectors for polynomial representations of $GL(n)$ are in one to one correspondence with non-negative integer partitions. For a detailed development see [2]. The irreducible regular representation of $GL(n)$ indexed by $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n)$ will be denoted by F^λ. The sum of the parts of a partition λ is denoted by $|\lambda|$. For non-negative integer partitions the number of non-zero parts will be called the *length* of the partition and denoted $l(\lambda)$.

DEFINITION 1.1. Let F^μ and F^ν be irreducible $GL(n)$ representations with highest weights μ and ν, respectively. Define the numbers $\{c_{\mu\,\nu}^{\lambda}\}$ by

$$F^\mu \otimes F^\nu = \bigoplus_\lambda c_{\mu\nu}^{\lambda}\, F^\lambda.$$

We will call the numbers $c_{\mu\,\nu}^{\lambda}$, the *Littlewood–Richardson coefficients*. These numbers can be equivalently defined as the structure constants for multiplication at the Schur basis of the ring of symmetric functions. In Section 3 we will encounter yet another characterization in terms of representations of the symmetric group which is a consequence of Schur–Weyl duality.

Remark 1.1. One can show from the Weyl character formula that if $c_{\mu\nu}^{\lambda} \neq 0$ then $\mu_i, \nu_i \leq \lambda_i$ for all i. In other words, the Young diagrams of μ and ν fit inside the Young diagram of λ. In the same light, one has that if $c_{\mu\nu}^{\lambda} \neq 0$ then $|\lambda| = |\mu| + |\nu|$.

For $n, d \leq 1$, define

$$\ell_{n,\,d} \equiv \sum_{\substack{\lambda,\,\mu:|\lambda|=|\mu|=d\\ l(\lambda),\,l(\mu)\leq n}} c_{\mu\mu}^{2\lambda}, \tag{6}$$

and

$$\tilde{\ell}_d \equiv \sum_{\lambda,\,\mu:|\lambda|=|\mu|=d} c_{\mu\mu}^{2\lambda}. \tag{7}$$

Here 2λ means multiply all the parts of λ by 2. It is convenient to define $\ell_{n,0} = 1$ (for $n \geq 0$), $\ell_{0,\,d} = 0$ (for $d \geq 1$), and $\tilde{\ell}_0 = 1$.

A consequence of Remark 1.1 is that if μ is a partition of d, and $c_{\mu\mu}^{2\,\lambda} \neq 0$ then λ must be a partition of d as well.

At times it will be convenient to work with the generating functions for the above numbers. With this is mind, set $L_n(q) \equiv \sum_{d=0}^{\infty} \ell_{n,\,d} q^d$, for $n \geq 0$, and set $\widetilde{L}(q) \equiv \sum_{d=0}^{\infty} \tilde{\ell}_d q^d$.

We now summarize the results in the rest of the paper. In Section 2 some well known results from the theory of symmetric pairs and multiplicity free spaces are presented. Using these tools it is established that $h_{n,\,d} = \ell_{n,\,d}$ for all $n \geq 1$ and $d \geq 0$, or equivalently,

$$H_n(q) = L_n(q).$$

In Section 3 the representation theory of the symmetric group is used to prove that for all $m \geq 1$, $d_m = \tilde{\ell}_m$, or equivalently,

$$\tilde{\eta}(q) = \widetilde{L}(q).$$

In Section 4 the ideas of Sections 2 and 3 are used to establish that for all $n \geq 1$ and $m \geq 0$,

$$h_{n,m} \leq d_m$$

with equality holding exactly when $m \geq n$. That is to say that the Hilbert series, $H_n(q)$, is dominated by $\tilde{\eta}(q)$ with the initial segments equal. Then it is explained how this fact proves that $\mathcal{G}_{n,m}$ is a basis for the space, $\mathcal{P}^m(M_n)^{O(n)}$, when $m \leq n$.

In Section 5 we state and prove another identity involving $\tilde{\eta}(q)$. This identity arises from the fact that the ring of polynomial functions on the symmetric and skew-symmetric matrices is multiplicity free as a representation of $GL(n)$.

In Section 6 we define a t-analog, $d_m(t)$, for the numbers d_m. An explicit combinatorial description of this t-analog would shed light on the problem of expressing $H_n(q)$ as a rational expression. Some data are provided at the end of Section 6. These data are computed from a generating function for the numbers c_k, which we developed in Section 6.1.

2. SYMMETRIC PAIRS

Let G denote a connected linearly reductive algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ and let θ denote a regular involution with differential (also denoted) $\theta : \mathfrak{g} \to \mathfrak{g}$. Let K be the set of fixed points of θ in G. (G, K) is said to be a symmetric pair. Let $\mathfrak{k}$ denote the Lie algebra of K. $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and under the adjoint representation of K, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} = \{X \in g | \theta(X) = -X\}$.

By $\mathcal{P}(V)$ we denote the algebra of polynomial functions on a vector space V. On this algebra we have the standard graduation, $\mathcal{P}(V) = \bigoplus_{d\geq 0} \mathcal{P}^d(V)$, where $\mathcal{P}^d(V)$ is the subspace consisting of the degree d homogeneous polynomials on V. For a subspace, $\mathcal{L} \subseteq \mathcal{P}(V)$, define the Hilbert series of $\mathcal{L}$ to be the formal power series in q, $\sum_{d=0}^{\infty} \dim(\mathcal{L} \bigcap \mathcal{P}^d(V))q^d$.

Consider the case where V is a regular representation of K. We have a linear action of K on the polynomial functions on V given by $g \cdot f(v) = f(g^{-1} \cdot v)$ for $g \in K, v \in V$. In our context, we consider the case when $\mathcal{L} = \mathcal{P}(V)^K$. K is a reductive group, and hence the ring $\mathcal{P}(V)^K$ is finitely generated, but the generators are not necessarily algebraically independent, so finding a "nice" expression for the Hilbert series is in general a hard problem.

We aim to make a statement about the ring of invariants when $V = \mathfrak{g}$, the adjoint representation of G restricted to K. For this work, we will focus our attention on the example $\mathfrak{g} = M_n(\mathbb{C})$ and $K = O(n, \mathbb{C})$. In this case, $\mathfrak{k}$

is the space of skew-symmetric matrices and $\mathfrak{p}$ will denote the symmetric matrices. All Lie algebras and (linear algebraic) group are over $\mathbb{C}$.

One approach is to view $\mathcal{P}(\mathfrak{g})$ as $\mathcal{P}(\mathfrak{k}) \otimes \mathcal{P}(\mathfrak{p})$ and then deduce the distribution of the invariants among the graded components from the pairing of arbitrary irreducible representations in $\mathcal{P}(\mathfrak{k})$ with their duals in $\mathcal{P}(\mathfrak{p})$. We will exploit this decomposition in Section 5. Also note that such an approach has been carried out for the pair $(SL(4), SO(4))$; see [11, Theorem 9, p. 13]. The Hilbert series for this case is

$$\frac{q^{15} + q^{11} + q^{10} + 3q^9 + 2q^8 + 2q^7 + 3q^6 + q^5 + q^4 + 1}{(1-q^6)(1-q^4)^3(1-q^3)^2(1-q^2)^3(1-q)}.$$

Using this approach in our context, it is not hard to deduce the following expressions for the Hilbert series when $n = 1, 2$, or 3,

$$\begin{aligned} H_1(q) &= \frac{1}{1-q} \\ H_2(q) &= \frac{1}{(1-q)(1-q^2)^2} \\ H_3(q) &= \frac{1+q^6}{(1-q)(1-q^2)^2(1-q^3)^2(1-q^4)}. \end{aligned} \tag{8}$$

2.1. *The Symmetric Pair* $(GL(n), O(n))$

As an $O(n)$ representation, the conjugation action on M_n is equivalent to the diagonal action of $O(n)$ on the space $\mathbb{C}^n \otimes \mathbb{C}^n$. This is a consequence of the fact that the standard representation of $O(n)$ is equivalent to its dual. In order to understand the space $\mathcal{P}(M_n)$ as a graded $O(n)$ representation we will investigate the restriction of the standard $GL(n) \times GL(n)$ action on $\mathbb{C}^n \otimes \mathbb{C}^n$ to the diagonal $GL(n)$ (that is, $\{(g, g) | g \in GL(n)\}$), and then restricted to the group $O(n)$. For this we will use the following special case of the Cartan–Helgason theorem.

THEOREM 2.1. *Let F^λ denote the irreducible finite dimensional representation of the group $GL(n)$ with highest weight λ. Then*

$$\dim(F^\lambda)^{O(n)} = \begin{cases} 1 & \text{if } \lambda \text{ has all even parts,} \\ 0 & \text{otherwise.} \end{cases}$$

This theorem follows from the fact that $(GL(n), O(n))$ is a symmetric pair; see [2, Chaps. 11 and 12]. So the dimension of the $O(n)$-invariant space in $\mathcal{P}^d(M_n)$ can be computed from a $GL(n)$ decomposition. We begin this program by asserting the following result sometimes referred to as Cauchy's identity or the Cauchy–Littlewood identity (see [1]), but which is also an instance of Roger Howe's theory of dual pairs (see [3]).

THEOREM 2.2. *The standard action of* $GL(n)\times GL(n)$ *on* $\mathbb{C}^n\widehat{\otimes}\mathbb{C}^n$ *defines an action on* $\mathscr{P}^d(\mathbb{C}^n\widehat{\otimes}\mathbb{C}^n)$ *by,* $(g,h)\cdot f(x\otimes y)=f(g^{-1}x\otimes h^{-1}y)$ *for* $(g,h)\in GL(n)\times GL(n)$ *and* f *a degree* d *homogeneous polynomial function on* $\mathbb{C}^n\widehat{\otimes}\mathbb{C}^n$*. Under this action the space of functions decomposes as*

$$\mathscr{P}^d(\mathbb{C}^n\widehat{\otimes}\mathbb{C}^n)=\bigoplus_{\substack{\lambda:l(\lambda)\le n\\ |\lambda|=d}}(F^\lambda)^*\widehat{\otimes}(F^\lambda)^*.$$

(Notation. $\widehat{\otimes}$ denotes the outer tensor product, while $*$ indicates the contragradient or dual representation.)

We now state and prove the goal of this section.

THEOREM 2.3. $H_n(q)=L_n(q)$, *for all* $n\ge 1$. *That is to say, for all* $d\ge 0$, $h_{n,d}=\ell_{n,d}$.

Proof. As mentioned at the beginning of this section, the overall plan is to restrict the $GL(n)\times GL(n)$ action from Theorem 2.2 to the diagonal $GL(n)$ subgroup and then to restrict further to $O(n)$. We then look for the invariants using Theorem 2.1. More precisely, $GL(n)\times GL(n)$ acts on M_n by $X\mapsto gX\ h^T$ for $X\in M_n$ and $g,h\in GL(n)$. Under this action, $M_n\cong\mathbb{C}^n\widehat{\otimes}\mathbb{C}^n$ as a $GL(n)\times GL(n)$ representation.

We will now restrict this action to the diagonal $GL(n)$ which acts on M_n by $X\mapsto g\,X\,g^T$ for $X\in M_n$ and $g\in GL(n)$. Under this action,

$$M_n\cong\mathbb{C}^n\otimes\mathbb{C}^n. \tag{9}$$

(We remark that under the above action, $M_n\cong SM_n\oplus AM_n$, where SM_n (resp. AM_n) is the space of symmetric (resp. skew-symmetric) matrices. We do not need this fact presently, but it will lead to an interesting identity which we develop in Section 5.)

Under the adjoint action of $GL(n)$ ($X\mapsto g\,X\,g^{-1}$ for $X\in M_n$ and $g\in GL(n)$) we have

$$M_n\cong(\mathbb{C}^n)^*\otimes\mathbb{C}^n. \tag{10}$$

We are concerned with the $O(n)$ decomposition of M_n under the adjoint action, but we are free to decompose with respect to Eq. (9), since as an $O(n)$ representation, $\mathbb{C}^n\cong(\mathbb{C}^n)^*$. Therefore,

$$h_{n,d}=\dim[\mathscr{P}^d(\mathbb{C}^n\otimes\mathbb{C}^n)]^{O(n)}.$$

Apply Theorem 2.2 to obtain the $GL(n)\times GL(n)$ decomposition. Observe that as far as $O(n)$ is concerned, $F^\lambda\cong(F^\lambda)^*$, so we can ignore the issue of duality. We obtain

$$h_{n,d}=\sum_{\substack{\mu:|\mu|=d\\ l(\mu)\le n}}\dim(F^\mu\widehat{\otimes}F^\mu)^{O(n)}.$$

Restrict the action of $GL(n) \times GL(n)$ to the diagonal $GL(n)$ subgroup and write out the decomposition using Littlewood–Richardson coefficients, to obtain

$$h_{n,d} = \sum_{\substack{\nu, \mu \\ |\nu|=2d, |\mu|=d \\ l(\nu)\le n}} c^{\nu}_{\mu\mu} \dim(F^{\nu})^{O(n)}.$$

Theorem 2.1 implies that $\dim(F^{\nu})^{O(n)}$ is non-zero exactly when $\nu = 2\lambda$ for some λ and in this case takes the value 1. The result follows. ∎

3. A COMBINATORIAL RESULT

Let $\mathscr{D}_m$ denote the set of (not necessarily connected) directed m vertex graphs in which each connected component is a directed cycle. An example of an element in $\mathscr{D}_{11}$ is shown in Fig. 2.

The elements of $\mathscr{D}_m$ are multisubsets of $\cup_{i=1}^{\infty}\mathscr{C}_i$ and therefore, $|\mathscr{D}_m| = d_m$. $\mathscr{L}\mathscr{D}_m$ will denote the set of graphs from $\mathscr{D}_m$ with each edge labeled by an element of the set $\{1, 2, \ldots, m\}$ such that each label is used exactly once. An example of an element in $\mathscr{L}\mathscr{D}_{10}$ is shown in Fig. 3.

Let I_r be the set of involutions on the set $\{1, 2, \ldots, r\}$, and let $\widetilde{I}_r$ be the subset of I_r consisting of involutions which do not have fixed points. Our strategy is to set up a bijective correspondence between $\mathscr{L}\mathscr{D}_m$ and $\widetilde{I}_{2m}$ for $m \geq 1$. This correspondence will then be used in the proof of Proposition 3.1.

The bijective correspondence will be stated precisely in the proof of Lemma 3.1, but first we illuminate the idea by an example. Consider the involution,

$$(1\,2)(3\,4)(5\,15)(6\,8)(7\,12)(9\,20)(10\,19)(11\,14)(13\,17)(16\,18).$$

(Here we are using the disjoint cycle representation.) Now we will describe how this involution corresponds to Fig. 3. Write the numbers from 1 to 10

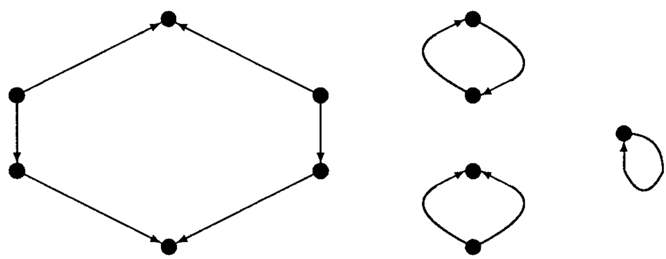

FIG. 2. An example of an element in $\mathscr{D}_{11}$.

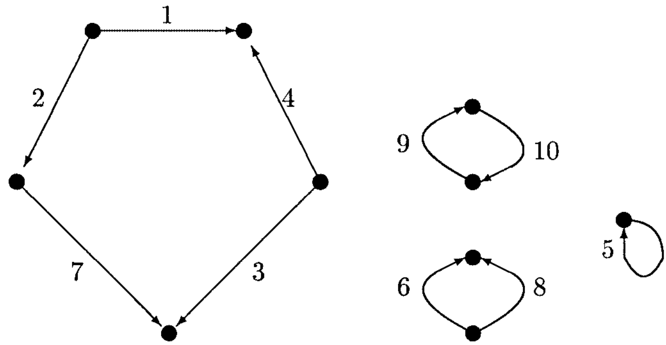

FIG. 3. An example of an element in $\mathscr{L}\mathscr{D}_{10}$.

in a row and above them write the numbers from 11 to 20. Next, connect each number with its image under the involution, as shown in Fig. 4.

It is helpful to introduce some easy terminology. The arrow end for an edge will be called the *head*, while the other end will be called the *tail*.

In Fig. 4, draw an arrow from each number to the number directly above it; see Fig. 5. Each number in the top row labels the head of an edge, while each number in the bottom row labels the tail of an edge. Next, identify the pairs of vertices in the diagram whenever a vertex is connected to another. The resulting picture will be an element of $\mathscr{D}_{10}$ which has each edge labeled by a pair $(i, i+10)$, where i labels the tail of the edge and $i+10$ labels the head of the edge. See Fig. 6.

Last, no information is lost if one relabels the edge $(i, i+10)$ with just i. This completes the correspondence for this example. The following is a precise description of what we have just done.

LEMMA 3.1. *There exists a bijective map* $\Theta : \mathscr{L}\mathscr{D}_m \longrightarrow \widetilde{I}_{2m}$.

Proof. Given $g \in \mathscr{L}\mathscr{D}_m$ we will make an involution $\sigma \in \widetilde{I}_{2m}$ as follows. Pick $k \in \{1, \dots, 2m\}$, and we will define the value of σ at k. There are two cases: $1 \leq k \leq m$ and $m+1 \leq k \leq 2m$.

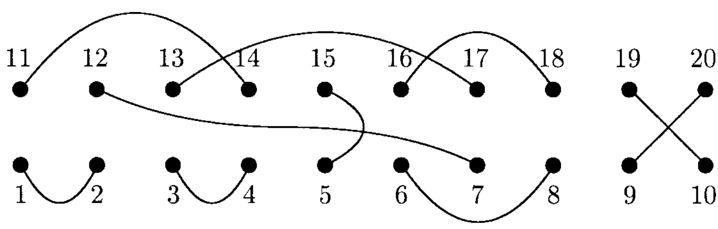

FIG. 4. Edges between the vertices paired by the involution.

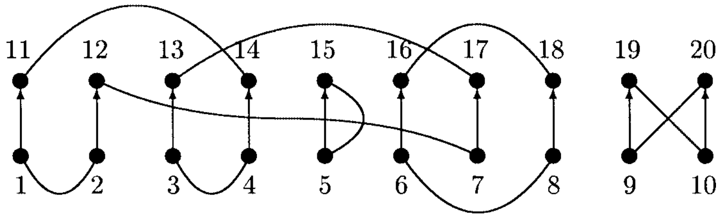

FIG. 5. An arrow from vertex i to vertex $i + n$.

Case 1. If $1 \leq k \leq m$ then find the edge labeled by k. If the *tail* of edge k is attached to the *tail* of the edge j then define $\sigma(k) = j$. On the other hand, if the *tail* of edge k is attached to the *head* of edge j then define $\sigma(k) = j + m$.

Case 2. If $m + 1 \leq k \leq 2m$ then find the edge labeled by $k - m$. If the *head* of edge $m - k$ is attached to the *tail* of edge j then define $\sigma(k) = j$. On the other hand, if the *head* of edge $k - m$ is attached to the *head* of edge j then define $\sigma(k) = j + m$.

It can be checked that σ is an involution in S_{2m}. We will now describe the inverse correspondence. Given an involution σ, create an element of $\mathscr{L}\mathscr{D}_m$ as follows: Start with m non-joined edges labeled with the numbers 1 through m. For each $1 \leq j, k \leq m$ use the values of $\sigma(k)$ and $\sigma(k + n)$ to identify the vertices at the head and tail of edge k and edge j according to the following four cases:

$\sigma(k) = j$	identify the tail of edge k with the tail of edge j,
$\sigma(k) = j + m$	identify the tail of edge k with the head of edge j,
$\sigma(k + m) = j$	identify the head of edge k with the tail of edge j,
$\sigma(k + m) = j + m$	identify the head of edge k with the head of edge j. ∎

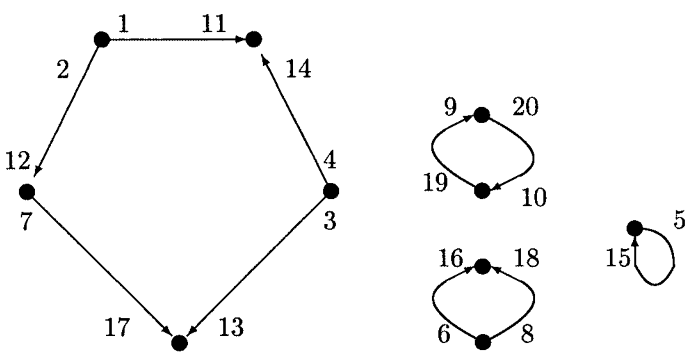

FIG. 6. Collapsed edges.

We will now define some subgroups of the symmetric group, S_{2m} which are of particular importance in our situation.

Let ΔS_m denote the diagonally embedded copy of S_m in $S_m \times S_m$. By relabeling the letters in the second copy of S_m we will embed $S_m \times S_m$ in S_{2m}. Hence, we can view both ΔS_m and $S_m \times S_m$ as subgroups of S_{2m}.

Let τ be the element of S_{2m} with disjoint cycle representation.

$$(1\,2)(3\,4)\cdots(i\,i+1)\cdots(2m-1\,2m).$$

Define H_m to be the centralizer group of τ in S_{2m}. H_m is isomorphic to the Weyl group B_m (or C_m). Let $(\Delta S_m)\backslash S_{2m}/H_m$ denote the set of double cosets,

$$\{(\Delta S_m)\sigma H_m | \sigma \in S_{2m}\}.$$

PROPOSITION 3.1. *$d_m = |(\Delta S_m)\backslash S_{2m}/H_m|$, for all $m \geq 1$.*

Proof. ΔS_m acts on $\widetilde{I}_{2m}$ by conjugation. This action is equivalent to the left action of ΔS_m on the cosets S_{2m}/H_m, by the correspondence

$$\psi : S_{2m}/H_m \longrightarrow \widetilde{I}_{2m}$$
$$\sigma H_m \longmapsto \sigma\tau\sigma^{-1}.$$

A quick check will establish that the above bijection is defined. The ΔS_m orbits in $\widetilde{I}_{2m}$, denoted $\widetilde{I}_{2m}/\Delta S_m$, are then the double cosets, $(\Delta S_m)\backslash S_{2m}/H_m$.

Next we establish a bijective correspondence between $\widetilde{I}_{2m}/\Delta S_m$ and the set $\mathscr{D}_m$. The result will follow from the fact that $d_m = |\mathscr{D}_m|$.

Observe that two involutions σ_1 and σ_2 are in the same orbit under the action of ΔS_m, if and only if they correspond (via Θ from Lemma 3.1) to two elements $g_1, g_2 \in \mathscr{L}\mathscr{D}_m$ which are different labelings of the same *unlabeled* directed graph. That is to say, the map Θ in Lemma 3.1 commutes with the S_m actions on $\mathscr{L}\mathscr{D}_m$ and $\widetilde{I}_{2m}$. Because Θ is bijective we have a one to one correspondence between the graph isomorphism classes in $\mathscr{L}\mathscr{D}_m$ and the ΔS_m orbits in $\widetilde{I}_{2m}$. The set of isomorphism classes in $\mathscr{L}\mathscr{D}_m$ is $\mathscr{D}_m$. ∎

The irreducible representations of the symmetric group are indexed by non-negative integer partitions of m. A precise indexing of the representations of S_m is given by the *Young symmetrizers* (see [1, 2, 4, 5, 7, 8], etc). So to each partition λ of m we can associate an irreducible representation, U_λ of S_m. The correspondence is implicitly stated in the following theorem.

THEOREM 3.1 (Schur–Weyl Duality, cf. [2, Theorem 9.1.2, p. 375]). *S_m acts on $\bigotimes^m \mathbb{C}^n$ by permutation of the tensor factors while $GL(n)$ acts on the same space diagonally. As a $GL(n)\times S_m$ representation we have*

$$\bigotimes\nolimits^m \mathbb{C}^n = \bigoplus_{\substack{\lambda:|\lambda|=m\\ l(\lambda)\leq n}} F^\lambda \widehat{\otimes} U_\lambda, \tag{11}$$

where the U_λ are irreducible representations of S_m.

A consequence of Schur–Weyl duality is the following rule for restricting an irreducible $S_{m_1+m_2}$ representation to the subgroup $S_{m_1} \times S_{m_2}$:

$$\mathrm{Res}^{S_{m_1+m_2}}_{S_{m_1}\times S_{m_2}} U_\lambda = \bigoplus_{\substack{\mu,\nu \\ |\mu|=m_1 \\ |\nu|=m_2}} c^\lambda_{\mu\nu} U_\mu \widehat{\otimes} U_\nu. \tag{12}$$

The following combinatorial result is the goal of this section.

THEOREM 3.2. *$\widetilde{L}(q) = \tilde{\eta}(q)$. That is to say, $\tilde{\ell}_m = d_m$ for all $m \geq 0$.*

Proof. By Frobenius reciprocity for finite groups we can restate Eq. (12) in terms of induced representations as

$$\mathrm{Ind}^{S_{m_1+m_2}}_{S_{m_1}\times S_{m_2}} U_\mu \widehat{\otimes} U_\nu = \bigoplus_{\lambda:|\lambda|=m_1+m_2} c^\lambda_{\mu\nu} U_\lambda. \tag{13}$$

The representation of S_{2m} induced from the trivial representation of ΔS_m can be decomposed into irreducible representations as follows. First, using the fact that induction of representations is transitive we see that

$$\mathrm{Ind}^{S_{2m}}_{\Delta S_m} 1 \cong \mathrm{Ind}^{S_{2m}}_{S_m\times S_m}\big(\mathrm{Ind}^{S_m\times S_m}_{\Delta S_m} 1\big). \tag{14}$$

Here 1 denotes the trivial representation. Then it is observed that representations of symmetric group are self dual (this is because every permutation is conjugate to its inverse), so by Schur's lemma the above is equivalent to

$$\mathrm{Ind}^{S_{2m}}_{S_m\times S_m}\Big(\bigoplus_\mu U_\mu \otimes U_\mu\Big). \tag{15}$$

Induction distributes across the direct sums, so (15) is equivalent to

$$\bigoplus_\mu \mathrm{Ind}^{S_{2m}}_{S_m\times S_m}(U_\mu \otimes U_\mu). \tag{16}$$

We use the (equivalent) definition of the Littlewood–Richardson coefficients in Eq. (13) to obtain

$$\mathrm{Ind}^{S_{2m}}_{\Delta S_m} 1 \cong \bigoplus_{\lambda,\,\mu} c^\lambda_{\mu\mu} U_\lambda. \tag{17}$$

An important fact explained in [5, Chap. VII, Sect. 2, p. 402] is

$$\mathrm{Ind}^{S_{2m}}_{H_m} 1 \cong \bigoplus_{\lambda:|\lambda|=m} U_{2\lambda}. \tag{18}$$

Equations (17) and (18) lead us to the following consequence of Schur's Lemma,

$$\tilde{\ell}_m = \dim \mathrm{Hom}_{S_{2m}}\Big(\mathrm{Ind}^{S_{2m}}_{H_m} 1, \mathrm{Ind}^{S_{2m}}_{\Delta S_m} 1\Big). \tag{19}$$

Now, by Frobenius reciprocity this number is seen to be

$$\dim \operatorname{Hom}_{\Delta S_m}\left(1, \operatorname{Res}^{S_{2m}}_{\Delta S_m} \operatorname{Ind}^{S_{2m}}_{H_m} 1\right), \tag{20}$$

which implies

$$\tilde{\ell}_m = \dim\left(\operatorname{Ind}^{S_{2m}}_{H_m} 1\right)^{\Delta S_m}. \tag{21}$$

As before, the group ΔS_m acts on the left cosets of H_m in S_{2m} by restricting the left coset action of S_{2m}. By Formula (21), the number of orbits of this action are seen to be $\tilde{\ell}_m$. This is because the representation of S_{2m} on the H_m cosets is equivalent to inducing the trivial representation of H_m to S_{2m}. The double cosets from Proposition 3.1 are the ΔS_m orbits in S_{2m}/H_m. The number of these orbits is the dimension of the space of ΔS_m-invariants in $\operatorname{Ind}^{S_{2m}}_{H_m} 1$. The result follows from Proposition 3.1. ∎

4. A STABILITY RESULT

In Section 1, we defined a generating function $L_n(q)$. Theorem 2.3 asserts that $L_n(q)$ is equal to $H_n(q)$. Also in Section 1, $\widetilde{L}(q)$ is defined and later in Section 3, Theorem 3.2 asserts that it is equal to $\tilde{\eta}(q)$. Now we compare the coefficients of $H_n(q)$, $L_n(q)$, $\widetilde{L}(q)$, and $\tilde{\eta}(q)$ in the following theorem. This serves two purposes: first, it summarizes the results of Section 3 and 2.1; second, it demonstrates that the power series $\tilde{\eta}(q)$ dominates the power series $H_n(q)$ for all $n \geq 1$ with equality in an initial segment.

THEOREM 4.1 (Stability Range for the Invariants). *For all $n \geq 1$, and $m \geq 0$,*

$$h_{n,m} = \ell_{n,m} \leq \tilde{\ell}_m = d_m$$

with equality holding exactly when $m \leq n$.

Proof. The domain of the sum defining $\ell_{n,m}$ is contained in the domain of the sum defining $\tilde{\ell}_m$. When $m \leq n$ these domains are the same. Therefore, $\ell_{n,m} \leq \tilde{\ell}_m$ with equality when $m \leq n$. By Theorem 2.3, $\ell_{n,m} = h_{n,m}$. By Theorem 3.2, $\tilde{\ell}_m = d_m$. The result follows. ∎

Explicitly calculating the values of $h_{n,m}$ for $m > n$ would lead to an expression of $H_n(q)$ as a rational function as described in Section 1.

We now return to the set $\mathscr{G}_{n,m}$ defined in Section 1. We have a map $\Omega : \mathscr{D}_m \to \mathscr{G}_{n,m}$ defined by applying the correspondence given in Eq. (3) to each connected component of an element of $\mathscr{D}_m$, and then multiplying the resulting polynomials. In the stable range, this map is bijective and the image is a basis for $[\mathscr{P}^m(M_n)]^{O(n)}$. We explain this fact presently.

The map Ω is clearly surjective, so $|\mathcal{D}_m| \geq |\mathcal{G}_{n,m}|$. $\mathcal{G}_{n,m}$ spans $[\mathcal{P}^m(M_n)]^{O(n)}$ so $|\mathcal{G}_{n,m}| \geq h_{n,m}$. In the stable range, $h_{n,m} = d_m$, by Theorem 4.1. Observe that

$$d_m = |\mathcal{D}_m| \geq |\mathcal{G}_{n,m}| \geq h_{n,m} = d_m$$

for $n \geq m$, so equality is forced. Therefore, Ω is bijective in the stable range. Furthermore, in the stable range, we see that $\mathcal{G}_{n,m}$ is a basis of $[\mathcal{P}^m(M_n)]^{O(n)}$ by a dimension count.

5. ANOTHER IDENTITY

$M_n \cong SM_n \oplus AM_n$, under the action of $GL(n)$ defined by $X \mapsto g\ X\ g^T$ for $X \in M_n$ and $g \in GL(n)$. Therefore,

$$\mathcal{P}(M_n) \cong \mathcal{P}(SM_n) \otimes \mathcal{P}(AM_n).$$

We will use this decomposition to prove another identity. First define

$$P_R = \{\lambda | \lambda \text{ has even parts}\},$$

and

$$P_C = \{\lambda | \lambda' \text{ has even parts}\}.$$

(Here λ denotes a partition, and λ' denotes the conjugate of λ.)

Remark 5.1. The condition that λ has even parts means that $\lambda = 2\mu$ for some partition μ and that the Young diagram of λ has even rows. The condition that λ' has even parts means that $\lambda = (2\mu)'$ for some partition μ and that the Young diagram of λ has even columns.

We now state two important multiplicity free results.

THEOREM 5.1. *For all $n \geq 1$ and $d \geq 0$,*

$$\mathcal{P}^d(SM_n^*) \cong \bigoplus_{\substack{\lambda \in P_R : l(\lambda) \leq n \\ |\lambda| = 2d}} F^\lambda.$$

Proof. See [2, p. 257, Sect. 5.2.5]. ∎

THEOREM 5.2. *For all $n \geq 1$ and $d \geq 0$,*

$$\mathcal{P}^d(AM_n^*) \cong \bigoplus_{\substack{\lambda \in P_C : l(\lambda) \leq n \\ |\lambda| = 2d}} F^\lambda.$$

Proof. See [2, p. 258, Sect. 5.2.6]. ∎

Combining the last two results together we obtain the following decomposition as a representation of $GL(n)$,

$$\mathscr{P}^d(M_n) \cong \bigoplus_{\substack{\mu\in P_R,\, \nu\in P_C \\ l(\mu),\, l(\nu)\le n \\ |\mu|=|\nu|=2d}} F^\mu \otimes F^\nu.$$

Therefore, for any partition λ with $l(\lambda) \le n$ the multiplicity of the irreducible $GL(n)$ representation, F^λ in $\mathscr{P}^d(M_n)$ is

$$\sum_{\substack{\mu\in P_R,\, \nu\in P_C \\ |\mu|=|\nu|=2d}} c^\lambda_{\mu\nu}.$$

Denote the above number by k^λ_d.

We then find the $O(n)$-invariants as before by taking one invariant for each even λ (as in Theorem 2.1). Define $K_n(q) \equiv \sum_{d=0}^\infty k_{n,\,d}\, q^d$, where

$$k_{n,\,d} \equiv \sum_{\substack{\lambda:|\lambda|=2d \\ l(\lambda)\le n}} k^{2\lambda}_{n,\,d}. \tag{22}$$

Consequently, $k_{n,\,d} = h_{n,\,d}$ for all $n \ge 1$ and $d \ge 0$. Here we also define $k_{n,\,0} = l_{n,\,0}$ and $k_{0,\,d} = l_{0,\,d}$(for $n, d \ge 0$). Next, define

$$\tilde{k}_d \equiv \sum_{\lambda:|\lambda|=2d} k^{2\lambda}_d,$$

and $\widetilde{K}(q) = \sum_{d=0}^\infty \tilde{k}_d\, q^d$ (set $\tilde{k}_0 = 1$).

THEOREM 5.3. (1) $K_n(q) = L_n(q)$ *for all* $n \ge 0$.

(2) $\tilde{\eta}(q) = \widetilde{K}(q) = \widetilde{L}(q)$.

Proof. By Theorem 2.3, $\ell_{n,\,m} = h_{n,\,m}$ for all $n \ge 1$ and $m \ge 0$. We have just seen that $h_{n,\,m} = k_{n,\,m}$. So we obtain, $k_{n,\,m} = \ell_{n,\,m}$ for all $n, m \ge 0$. Since by definition, $k_{0,\,m} = \ell_{0,\,m}$ for all $m \ge 0$. This proves assertion (1).

For each m and sufficiently large n ($n \ge 2m$), $k_{n,\,m} = \tilde{k}_m$. By Theorem 4.1, $\ell_{n,\,m} = \tilde{\ell}_m = d_m$, for $n \ge m$. This, combined with assertion (1), implies that for $n \ge 2m$, $d_m = \tilde{\ell}_m = \ell_{n,\,m} = k_{n,\,m} = \tilde{k}_m$. Therefore, $\tilde{k}_m = d_m = \tilde{\ell}_m$ for $m \ge 0$. Assertion (2) follows. ∎

6. SOME DATA

In the following we define a t-analog of the number d_m. Consider the formal power series $\widetilde{H}(q, t) \equiv \sum_{m,\,n=0}^\infty d_{n,\,m} q^m t^n$, where, for all $n, m \ge 0$,

$$d_{n,\,m} \equiv \begin{cases} \ell_{n,\,m} - \ell_{n-1,m} & \text{if } n \ge 1, \\ \ell_{0,\,m} & \text{if } n = 0. \end{cases}$$

(Note that $\ell_{n,m} \geq \ell_{n-1,m}$, and therefore, $d_{n,m} \geq 0$.) For $m \geq 0$, define $d_m(t)$ to be the coefficient of q^m in $\widetilde{H}(q,t)$. That is,

$$d_m(t) \equiv \sum_{n=0}^{\infty} d_{n,m} t^n.$$

For any partition λ, $l(\lambda) \leq |\lambda|$. Therefore the above sum is finite and $d_m(t)$ is a polynomial in t with non-negative integer coefficients. One can show that deg $d_m(t) = m$. Also, $d_m(t)$ has the property that $d_m(1) = d_m$ and so it is a t-analog of the number d_m.

An explicit formula for the coefficients of $d_m(t)$ (which could be effectively computed) would be of value. Such a formula might lead to a rational expression for $H_n(q)$ as described in Section 1. To see this relationship, extend the definition of $h_{n,m}$ to include $h_{0,0} = 1$, and $h_{0,m} = 0$ (for $m \geq 1$). Then, $H_0(q) = 1$. From Theorem 2.3, $H_n(q) = L_n(q)$ for $n \geq 0$ so,

$$H_n(q) = \sum_{m=0}^{\infty} \ell_{n,m}\, q^m = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{n} d_{k,m} \right) q^m. \tag{23}$$

We then sum $H_n(q)t^n$ over n, change the order of summation, and make the substitution, $n = k + r$ to obtain

$$\sum_{n=0}^{\infty} H_n(q) t^n = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} d_{k,m} q^m t^{k+r} = \frac{\widetilde{H}(q,t)}{1-t} \tag{24}$$

$$= \sum_{m=0}^{\infty} \left(\frac{d_m(t)}{1-t} \right) q^m. \tag{25}$$

Table I provides some initial data. In row m and column n, $d_{n,m}$ is displayed for n and m from 0 to 10. The sum of the first n columns in the

TABLE I

Initial value of $d_{n,m}$ (m = row, n = column.)

	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	1	2	0	0	0	0	0	0	0	0
3	0	1	2	2	0	0	0	0	0	0	0
4	0	1	5	3	3	0	0	0	0	0	0
5	0	1	5	7	4	3	0	0	0	0	0
6	0	1	9	13	12	5	4	0	0	0	0
7	0	1	9	21	21	14	6	4	0	0	0
8	0	1	14	33	48	30	19	7	5	0	0
9	0	1	14	51	75	67	39	21	8	5	0
10	0	1	20	73	145	133	98	48	26	9	6

table provides the initial 11 coefficients of the Hilbert series $H_n(q)$. The coefficients of $d_m(t)$ are the numbers in the mth row of the table. The data in this table were computed using Stembridge's MAPLE package, SF [9].

An initial segment of the numbers $d_m = d_m(1)$, for $m = 0 \dots 19$ is 1, 1, 3, 5, 12, 20, 44, 76, 157, 281, 559, 1021, 2005, 3721, 7237, 13631, 26433, 50297, 97543, 187129. We explain briefly how these data were computed.

An algorithm computing c_k leads to an algorithm for computing d_m by expanding the product defining $\tilde{\eta}(q)$. The numbers c_k can be effectively computed by elementary enumeration techniques, which we develop next.

6.1. *An Enumeration of* $\mathscr{C}_k$

A vertex in a directed cycle is called a *sink* vertex if the two edges joined to it are pointing into the vertex. A vertex is called a *source* vertex if the two edges joined to it are pointed away from the vertex. A vertex is called a *flow* vertex if one edge points into it and one edge points out of it. For $k \geq 1$ and $i \geq 0$ let

$$\mathscr{C}_k^{(i)} = \{c \in \mathscr{C}_k | c \text{ has } i \text{ sinks and } i \text{ sources}\}. \tag{26}$$

So $\mathscr{C}_k = \cup_i \mathscr{C}_k^{(i)}$. Observe that the source and sink vertices alternate with flow vertices scattered between them. For each directed cycle, construct a polygon whose corners correspond to the source vertices, with each side corresponding to a sink vertex. The flow vertices are then represented by ordered pairs of non-negative integers assigned to each side. In order to enumerate these, we will use Burnside's theorem. That is, we average the fix point set cardinalities of the action of the dihedral group on such n sided polygons. For $n \geq 0$, let $g_n(x)$ denote the generating function in which the coefficient of x^k is the number of directed cycles with exactly n sources, n sinks, and k flow vertices. Then $g_0(x) = 1/(1-x)$. For $n \geq 1$, $g_n(x)$ can be shown to be

$$g_n(x) = \frac{1}{2(1-x^2)^n} + \frac{1}{2n} \sum_{d|n} \frac{\phi(d)}{(1-x^d)^{\frac{2n}{d}}}. \tag{27}$$

(Here $\phi(d)$ denotes the Euler phi function.) The generating function in which the coefficient of x^k is the number of directed cycles with k vertices (i.e., c_k) is then

$$C(x) = \sum_{d=0}^{\infty} g_d(x) x^{2d}.$$

This formula for $C(x)$ can be used to compute an initial segment of the sequence c_k. This computation has been done with the aid of the software package, MAPLE, to produce $\{c_k\}_{k\geq 1}$: 1, 2, 2, 4, 4, 9, 10, 22, 30, 62, 94, 192, 316, 623, 1096, 2122, 3856, 7429, 13798, ...

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