

# A Basis for the $GL_n$ Tensor Product Algebra

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## Abstract

This paper focuses on the  $GL_n$  tensor product algebra, which encapsulates the decomposition of tensor products of arbitrary irreducible representations of  $GL_n$ . We will describe an explicit basis for this algebra. This construction relates directly with the combinatorial description of Littlewood-Richardson coefficients in terms of Littlewood-Richardson tableaux. Philosophically, one may view this construction as a recasting of the Littlewood-Richardson rule in the context of classical invariant theory.

*Key words:* Berenstein-Zelevinsky diagrams, Littlewood-Richardson coefficients, reciprocity algebra, skew tableau, tensor product algebra

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## 1 Introduction

This paper continues the authors' study of the branching rules for classical symmetric pairs by means of reciprocity algebras. In the papers [HTW1] and [HTW2], it was shown that in some sense, the most basic of these algebras is the tensor product algebra for the general linear group  $GL_n$ . In particular, the paper [HTW1] demonstrated that all stable branching multiplicities for classical symmetric pairs could be written in terms of Littlewood-Richardson coefficients, which describe the decomposition of tensor products of representations of  $GL_n$ . This paper will focus on the  $GL_n$  tensor product algebra. Our main goal is to describe an explicit basis for this algebra. As will be seen, our basis connects directly with the combinatorial description of Littlewood-Richardson coefficients in terms of Littlewood-Richardson tableaux [Ful], [Pro], [Sta], [Sun].

For any  $m \in \mathbb{N}$ , let  $B_m = A_m U_m$  be the standard Borel subgroup of upper triangular

matrices in  $GL_m = GL_m(\mathbb{C})$ , where  $A_m$  is the diagonal torus in  $GL_m$ , and  $U_m$  is the maximal unipotent subgroup, i.e., the upper triangular matrices with 1's on the diagonal. Let  $\rho_m^D$  be the irreducible representation of  $GL_m$  with highest weight represented by Young diagram  $D$  as in [How]. Let  $\widehat{A}_m^+$  be the semigroup of dominant polynomial characters of  $A_m$ , i.e., those characters which can be represented by Young diagrams.

Let  $M_{n,(k+\ell)}$  be the space of complex  $n \times (k+\ell)$  matrices. Let  $GL_n$  act by row operations (multiplication on the left) on  $M_{n,(k+\ell)}$  and  $GL_k \times GL_\ell$  to act on  $M_{n,(k+\ell)}$  by column operations, with  $GL_k$  acting on the first  $k$  columns, and  $GL_\ell$  acting on the last  $\ell$  columns.

Let us define a family of algebras  $TA_{n,k,\ell}$ , which we shall call the  $GL_n$  tensor product algebra:

$$TA_{n,k,\ell} = P(M_{n,k+\ell})^{U_k \times U_\ell \times U_n}.$$

It is straightforward (see §2.1) to show that the above algebra consists of the highest weight vectors for  $GL_n$  acting on the sum of one copy of each tensor product  $\rho_n^D \otimes \rho_n^E$ , where  $D$  and  $E$  are diagrams, having at most  $\min\{n, k\}$  and  $\min\{n, \ell\}$  rows respectively. This algebra is an  $\widehat{A}_n^+ \times \widehat{A}_k^+ \times \widehat{A}_\ell^+$ -graded algebra. In general, let  $\psi^D$  denote the  $A_m$  character corresponding to the highest weight of a  $GL_m$  representation  $\rho_m^D$ . Then, the dimension of the  $\psi^F \times \psi^D \times \psi^E$  homogeneous subspace in  $TA_{n,k,\ell}$  equals the multiplicity of the representation  $\rho_n^F$  in the tensor product  $\rho_n^D \otimes \rho_n^E$ . These multiplicities are the Littlewood-Richardson coefficients.

Let  $x_{ab}$  be the entries of a typical  $n \times k$  matrix  $X$ , and let  $y_{ac}$  be the entries of a typical  $n \times \ell$  matrix  $Y$ . Then

$$Z = [X \ Y] = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1k} & y_{11} & y_{12} & \cdots & y_{1\ell} \\ x_{21} & x_{22} & \cdots & x_{2k} & y_{21} & y_{22} & \cdots & y_{2\ell} \\ x_{31} & x_{32} & \cdots & x_{3k} & y_{31} & y_{32} & \cdots & y_{3\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nk} & y_{n1} & y_{n2} & \cdots & y_{n\ell} \end{bmatrix}$$

is a typical  $n \times (k + \ell)$  matrix. Let  $X_{a,b}$  denote the upper left hand  $a \times b$  submatrix of  $x$ 's, that is,

$$X_{a,b} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1b} \\ x_{21} & x_{22} & \cdots & x_{2b} \\ x_{31} & x_{32} & \cdots & x_{3b} \\ \vdots & \vdots & \vdots & \vdots \\ x_{a1} & x_{a2} & \cdots & x_{ab} \end{bmatrix}.$$

Similarly, let  $Y_{c,d}$  denote the upper left hand  $c \times d$  submatrix of  $y$ 's.

Select partitions

$$\begin{aligned} D &= \{d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_r > 0 = d_{r+1}\}, \\ E &= \{e_1 \geq e_2 \geq e_3 \geq \cdots \geq e_s > 0 = e_{s+1}\}, \text{ and} \\ F &= \{f_1 \geq f_2 \geq f_3 \geq \cdots \geq f_t > 0 = f_{t+1}\}, \end{aligned}$$

of depths  $r \leq k$ ,  $s \leq \ell$  and  $t \leq \min\{n, k + \ell\}$  respectively. Denoting  $|D| = \sum_{j \geq 0} d_j$  for the *size* of  $D$ , we will assume that  $|D| + |E| = |F|$ . We also let  $D^t$ ,  $E^t$  and  $F^t$  denote the transpose diagrams. Thus, the numbers  $d_j$  are the lengths of the columns of  $D^t$ , etc.

Let  $A = [\alpha_{ij}]$  and  $B = [\beta_{ij}]$  be  $t \times r$  and  $t \times s$  matrices of indeterminates in  $\alpha_{ij}$ 's and  $\beta_{ij}$ 's respectively. We define  $\tilde{\Delta}_{(D,E,F),(A,B)}$  to be the determinant of a square  $|F| \times (|D| + |E|)$  matrix as follows: Note that the entries of this matrix are block matrices in  $x$ 's or  $y$ 's with sizes determined by the partitions  $D$ ,  $E$  and  $F$ . For instance, the block entry  $\alpha_{jk} X_{f_j, d_k}$  is an  $f_j \times d_k$  matrix and likewise, the block entry  $\beta_{jk} Y_{f_j, e_k}$  is an  $f_j \times e_k$  matrix.

$$\tilde{\Delta}_{(D,E,F),(A,B)} = \begin{vmatrix} \alpha_{11} X_{f_1, d_1} & \alpha_{12} X_{f_1, d_2} & \cdots & \alpha_{1r} X_{f_1, d_r} & \beta_{11} Y_{f_1, e_1} & \beta_{12} Y_{f_1, e_2} & \cdots & \beta_{1s} Y_{f_1, e_s} \\ \alpha_{21} X_{f_2, d_1} & \alpha_{22} X_{f_2, d_2} & \cdots & \alpha_{2r} X_{f_2, d_r} & \beta_{21} Y_{f_2, e_1} & \beta_{22} Y_{f_2, e_2} & \cdots & \beta_{2s} Y_{f_2, e_s} \\ \alpha_{31} X_{f_3, d_1} & \cdots & \cdots & \vdots & \beta_{31} Y_{f_3, e_1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{t1} X_{f_t, d_1} & \alpha_{t2} X_{f_t, d_2} & \cdots & \alpha_{tr} X_{f_t, d_r} & \beta_{t1} Y_{f_t, e_1} & \beta_{t2} Y_{f_t, e_2} & \cdots & \beta_{ts} Y_{f_t, e_s} \end{vmatrix}$$

This is a polynomial function in the  $x_{ab}$ 's, the  $y_{ac}$ 's, the  $\alpha_{jk}$ 's and the  $\beta_{jk}$ 's. Lemma 2.1 shows that for any choice of coefficient matrices  $A$  and  $B$ , i.e., setting  $\alpha_{jk} \in \mathbb{C}$  and  $\beta_{jk} \in \mathbb{C}$ , the polynomial  $\tilde{\Delta}_{(D,E,F),(A,B)}$  is a  $GL_n \times GL_k \times GL_\ell$  highest weight vector, that is, it is invariant under  $U_n \times U_k \times U_\ell$ .

For generic values of  $A$ , we may reduce  $A$  by row and column operations to the  $t \times r$  “identity” matrix,  $J_{t,r}$ , without changing the value of  $\tilde{\Delta}_{(D,E,F),(A,B)}$  (see §2.2.2). Here,  $J = J_{t,r}$  is the  $t \times r$  matrix with ones down the diagonal and zeroes elsewhere. Thus we consider the polynomials  $\tilde{\Delta}_{(D,E,F),(J,B)}$ . We can expand these as polynomials in the  $\beta_{ih}$ :

$$\tilde{\Delta}_{(D,E,F),(J,B)} = \sum_M \tilde{\Delta}_{(D,E,F),M} \left( \prod_{i,h} \beta_{ih}^{m_{ih}} \right),$$

where  $M = [m_{ih}]$  is a matrix of non-negative integers, defining the exponents to which the  $\beta_{ih}$  appear in a given monomial. Each coefficient  $\tilde{\Delta}_{(D,E,F),M}$  is a polynomial in  $x_{ab}$  and  $y_{ac}$ .

As is well known, the Littlewood-Richardson coefficients count the number of *Littlewood-Richardson tableaux* (see §2.3). We focus on the LR<sup>1</sup> tableaux, which count the multiplicities of  $\rho_n^{F^t}$  in the tensor product  $\rho_n^{D^t} \otimes \rho_n^{E^t}$ . We will define a mapping from Littlewood-Richardson tableaux (for appropriate fixed  $D$ ,  $E$ ,  $F$ ) to certain of the coefficients  $\tilde{\Delta}_{(D,E,F),M}$ . More

<sup>1</sup> LR = Littlewood Richardson

precisely, we will show how to associate to each Littlewood-Richardson tableau  $T$ , a unique monomial  $M(T)$  (see §3.1) in the  $\beta_{ih}$ .

In essence, for fixed  $D$ ,  $E$  and  $F$ , a Young tableau  $T$  is a filling of a skew diagram (see §2.3). The monomial  $M(T)$  in the variables  $\beta_{ih}$  is determined directly from the entries in the boxes of the skew diagram  $T = F^t - D^t$ . The combinatorics of Young tableaux leading to the definition/construction of  $M(T)$  will be discussed in §3.1. Not surprising, one can easily and uniquely recover  $T$  from the monomial  $M(T)$ .

Now set  $\tilde{\Delta}_{(D,E,F),M(T)}$  to be the coefficient of  $M(T)$  in  $\tilde{\Delta}_{(D,E,F),(J,B)}$ . Note that  $\tilde{\Delta}_{(D,E,F),M(T)}$  is a polynomial in  $x_{ab}$  and  $y_{ac}$ . By considering a standard coefficient in the  $x$ -variables of  $\tilde{\Delta}_{(D,E,F),M(T)}$ , we have a polynomial  $\delta_{T,Y}$  (see §3.2) in the variables  $y_{ac}$ . This polynomial can also be directly related to the entries in the boxes of  $T = F^t - D^t$  (see Lemma 3.2).

The key result in this paper is a linear basis for the  $GL_n$  tensor product algebra, whose proof is explained in §3:

**Theorem.** *As the coefficient matrices  $A$  and  $B$  vary through all possible choices of constants, the polynomials  $\tilde{\Delta}_{(D,E,F),(A,B)}$  span the tensor product algebra  $TA_{n,k,\ell}$ . More precisely, the polynomials  $\tilde{\Delta}_{(D,E,F),M(T)}$  with the monomial  $M(T)$  associated to each Littlewood-Richardson skew tableau  $T$ , form a basis for  $TA_{n,k,\ell}$ .*

The final section looks at our construction for the generators of the  $SL_4$  tensor product algebra. We correlate our monomials with the diagrammatic description of multiplicities given by Berenstein and Zelevinsky (see [BZ]).

## 2 Preliminaries

### 2.1 $GL_n$ Tensor Product Algebra

We begin by reviewing from [How] and [HTW2] our realization of the tensor product algebra for  $GL_n = GL_n(\mathbb{C})$ . All groups are defined over  $\mathbb{C}$  unless otherwise stated. Let  $GL_n \times GL_k$  act on the space  $M_{n,k}$  of  $n \times k$  complex matrices, by the formula

$$(g, g')(T) = (g^t)^{-1}T(g')^{-1}, \quad g \in GL_n, g' \in GL_k, T \in M_{n,k}. \quad (2.1)$$

Extend this action to an action by algebra automorphisms on the ring  $P(M_{n,k})$  of polynomials on  $M_{n,k}$  in the standard way. Then we have the decomposition [How]

$$P(M_{n,k}) \simeq \sum_D \rho_n^D \otimes \rho_k^D, \quad (2.2)$$

into irreducible representations for  $GL_n \times GL_k$ . Here  $\rho_n^D$  is the irreducible representation of  $GL_n$  with highest weight represented by Young or Ferrers diagram  $D$  and the index  $D$  of summation ranges over all Young diagrams of depth not more than  $\min\{n, k\}$ .

Let  $B_n$  be the standard Borel subgroup of upper triangular matrices in  $GL_n$ , and let  $U_n$  be its maximal unipotent subgroup, i.e., the upper triangular matrices with 1's on the diagonal. Consider the algebra  $P(M_{n,k})^{U_k}$  of polynomials on  $M_{n,k}$  invariant under the action of  $U_k$ . In terms of the decomposition (2.2), we may write

$$P(M_{n,k})^{U_k} \simeq \left( \sum_D \rho_n^D \otimes \rho_k^D \right)^{U_k} \simeq \sum_D \rho_n^D \otimes (\rho_k^D)^{U_k}. \quad (2.3)$$

By the theory of the highest weight, the space  $(\rho_k^D)^{U_k}$  is one-dimensional. Thus  $P(M_{n,k})^{U_k}$  consists of one copy of each irreducible polynomial representation of  $GL_n$  corresponding to a diagram with not more than  $k$  rows. Note that the space  $\rho_n^D \otimes (\rho_k^D)^{U_k}$  is an eigenspace for the diagonal torus  $A_k$  in  $GL_k$ , consisting of the matrices

$$\mathbf{a} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_k \end{bmatrix}.$$

The eigenvalue of  $\mathbf{a}$  depends multiplicatively on  $\mathbf{a}$ : it is a character of  $A_k$ . The eigencharacter of  $A_k$  acting on  $\rho_n^D \otimes (\rho_k^D)^{U_k}$  is equal to the highest weight of  $\rho_k^D$ . This is the character

$$\psi^D(\mathbf{a}) = a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k},$$

where  $\{a_j\}$  are the standard coordinates on the diagonal torus  $A_k$  of  $GL_k$ , and  $\{d_k\}$  are the lengths of the rows of  $D$  (see [How] for more details). Thus, the algebra  $P(M_{n,k})^{U_k}$  is a graded algebra, with grading in the semigroup  $\widehat{A}_k^+$  of dominant characters of  $A_k$ . The graded components are irreducible representations for  $GL_n$ .

Let  $M_{n,(k+\ell)}$  be the space of  $n \times (k+\ell)$  matrices. We understand  $GL_n$  to act by row operations (multiplication on the left) on  $M_{n,(k+\ell)}$ , as in formula (2.1). We understand  $GL_k \times GL_\ell$  to act on  $M_{n,(k+\ell)}$  by column operations, with  $GL_k$  acting on the first  $k$  columns, and  $GL_\ell$  acting on the last  $\ell$  columns. We may also think of  $GL_k \times GL_\ell$  embedded in  $GL_{k+\ell}$  as block diagonal matrices:

$$(g'_1, g'_2) \sim \begin{bmatrix} g'_1 & 0 \\ 0 & g'_2 \end{bmatrix}$$

for  $g'_1 \in GL_k$  and  $g'_2 \in GL_\ell$ . The action of  $GL_k \times GL_\ell$  is then just the restriction of the action (2.1) of  $GL_{k+\ell}$ .

With these conventions, it is more or less straightforward from equation (2.3) to show (see [HTW2] for details) that the algebra

$$TA = TA_{n,k,\ell} = P(M_{n,k+\ell})^{U_k \times U_\ell \times U_n} \quad (2.4)$$

consists of the highest weight vectors for  $GL_n$  acting on the sum of one copy of each tensor product  $\rho_n^D \otimes \rho_n^E$ , where  $D$  and  $E$  are diagrams, having at most  $\min\{n, k\}$  and  $\min\{n, \ell\}$  rows respectively. This algebra is an  $\widehat{A}_n^+ \times \widehat{A}_k^+ \times \widehat{A}_\ell^+$ -graded algebra, and the dimension of the  $\psi^F \times \psi^D \times \psi^E$  homogeneous subspace equals the multiplicity of the representation  $\rho_n^F$  in the tensor product  $\rho_n^D \otimes \rho_n^E$ . We refer to the family of algebras  $TA_{n,k,\ell}$  of equation (2.4) as “the  $GL_n$  tensor product algebra”. A more careful justification for this terminology is given in [HTW2].

## 2.2 Canonical Highest Weight Vectors

Our goal is to construct a basis for  $TA_{n,k,\ell} = TA$ . To this end, let  $x_{ab}$ , for  $1 \leq a \leq n$ ,  $1 \leq b \leq k$ , be standard matrix coordinates on  $M_{n,k}$ , and let  $y_{ac}$ , for  $1 \leq a \leq n$ ,  $1 \leq c \leq \ell$ , be standard matrix coordinates on  $M_{n,\ell}$ . In other words, let

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ x_{31} & x_{32} & \cdots & x_{3k} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1\ell} \\ y_{21} & y_{22} & \cdots & y_{2\ell} \\ y_{31} & y_{32} & \cdots & y_{3\ell} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{n\ell} \end{bmatrix}$$

be typical elements of  $M_{n,k}$  and  $M_{n,\ell}$  respectively.

We combine the  $x_{ab}$  and the  $y_{ac}$  to get a set of coordinates on  $M_{n,(k+\ell)}$ . Specifically, let

$$Z = [X \ Y]$$

be a typical element of  $M_{n,(k+\ell)}$ . Then  $GL_k$  acts by column operations on  $X$ , while  $GL_\ell$  acts by column operations on  $Y$ , and  $GL_n$  acts by row operations on  $X$  and  $Y$  simultaneously.

Let  $X_{a,b}$  denote the upper left hand  $a \times b$  submatrix of  $X$ . That is,

$$X_{a,b} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1b} \\ x_{21} & x_{22} & \cdots & x_{2b} \\ x_{31} & x_{32} & \cdots & x_{3b} \\ \vdots & \vdots & \vdots & \vdots \\ x_{a1} & x_{a2} & \cdots & x_{ab} \end{bmatrix}. \quad (2.5)$$

Similarly, let  $Y_{c,d}$  denote the upper left hand  $c \times d$  submatrix of  $Y$ .

Following standard yoga, we think of a diagram alternatively as a partition, and/or as a decreasing sequence of non-negative integers, with two sequences whose positive elements coincide being considered equivalent. Select partitions

$$\begin{aligned} D &= \{d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_r > 0 = d_{r+1}\}, \\ E &= \{e_1 \geq e_2 \geq e_3 \geq \cdots \geq e_s > 0 = e_{s+1}\}, \end{aligned} \quad (2.6)$$

and

$$F = \{f_1 \geq f_2 \geq f_3 \geq \cdots \geq f_t > 0 = f_{t+1}\},$$

of depths  $r \leq k$ ,  $s \leq \ell$  and  $t \leq \min\{n, k + \ell\}$  respectively. Let  $|D| = \sum_{j \geq 0} d_j$  be the *size* of  $D$ . We will assume that  $|D| + |E| = |F|$ .

Let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2r} \\ \alpha_{31} & \alpha_{32} & \cdots & \alpha_{3r} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{t1} & \alpha_{t2} & \cdots & \alpha_{tr} \end{bmatrix}. \quad (2.7)$$

be a  $t \times r$  matrix of indeterminates  $\alpha_{jk}$ . Similarly, let

$$B = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1s} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2s} \\ \beta_{31} & \beta_{32} & \cdots & \beta_{3s} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{t1} & \beta_{t2} & \cdots & \beta_{ts} \end{bmatrix}.$$

be a  $t \times s$  matrix of other indeterminates  $\beta_{jk}$ .

Define a matrix  $\widetilde{X}_{(D,F,A)} = \widetilde{X}$  as follows.  $\widetilde{X}$  is a  $|F| \times |D|$  matrix, which is partitioned into submatrices according to the partition  $F$  of the rows of  $\widetilde{X}$ , and the partition  $D$  of the columns of  $\widetilde{X}$ . Thus  $\widetilde{X}$  consists of submatrices  $\widetilde{X}_{j,k}$  for  $1 \leq j \leq t$  and  $1 \leq k \leq r$ :

$$\widetilde{X} = \begin{bmatrix} \widetilde{X}_{1,1} & \widetilde{X}_{1,2} & \cdots & \widetilde{X}_{1,r} \\ \widetilde{X}_{2,1} & \widetilde{X}_{2,2} & \cdots & \widetilde{X}_{2,r} \\ \widetilde{X}_{3,1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{X}_{t,1} & \widetilde{X}_{t,2} & \cdots & \widetilde{X}_{t,r} \end{bmatrix} = \begin{bmatrix} \alpha_{11}X_{f_1,d_1} & \alpha_{12}X_{f_1,d_2} & \cdots & \alpha_{1r}X_{f_1,d_r} \\ \alpha_{21}X_{f_2,d_1} & \alpha_{22}X_{f_2,d_2} & \cdots & \alpha_{2r}X_{f_2,d_r} \\ \alpha_{31}X_{f_3,d_1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{t1}X_{f_t,d_1} & \alpha_{t2}X_{f_t,d_2} & \cdots & \alpha_{tr}X_{f_t,d_r} \end{bmatrix} \quad (2.8)$$

The submatrix  $\widetilde{X}_{j,k}$  is an  $f_j \times d_k$  matrix. Specifically,

$$\widetilde{X}_{j,k} = \alpha_{jk} X_{f_j, d_k} \quad (2.9)$$

with  $\alpha_{jk}$  as in formula (2.7), and  $X_{f_j, d_k}$  as in formula (2.5).

We also define a matrix  $\widetilde{Y}_{(E,F,B)} = \widetilde{Y}$  in parallel fashion, but using the matrix  $Y$  instead of  $X$ , the partition  $E$  instead of  $D$ , and the matrix  $B$  instead of  $A$ . Finally, we define a matrix

$$\widetilde{Z} = \begin{bmatrix} \widetilde{X} & \widetilde{Y} \end{bmatrix}. \quad (2.10)$$

We note that  $\widetilde{Z}$  is an  $|F| \times (|D| + |E|) = |F| \times |F|$  matrix. It is a square matrix, so we may take its determinant:

$$\widetilde{\Delta}_{(D,E,F),(A,B)} = \widetilde{\Delta}_{(D,E,F),(A,B)}(X, Y) = \det \widetilde{Z}. \quad (2.11)$$

This is a polynomial function in the  $x_{ab}$ 's, the  $y_{ac}$ 's, the  $\alpha_{jk}$ 's and the  $\beta_{jk}$ 's.

- Lemma 2.1.** (a) For any choice of coefficient matrices  $A$  and  $B$ , the polynomial  $\widetilde{\Delta}_{(D,E,F),(A,B)}$  is a  $GL_n \times GL_k \times GL_\ell$  highest weight vector, that is, it is invariant under  $U_n \times U_k \times U_\ell$ .
- (b)  $\widetilde{\Delta}_{(D,E,F),(A,B)}$  is a weight vector for the product  $A_n \times A_k \times A_\ell$  of diagonal tori of the groups  $GL_n$ ,  $GL_k$  and  $GL_\ell$ . The eigencharacter of  $A_n \times A_k \times A_\ell$  defined by  $\widetilde{\Delta}_{(D,E,F),(A,B)}$  is  $\psi^{F^t} \times \psi^{D^t} \times \psi^{E^t}$ .

**Proof:** If we can show that  $\widetilde{\Delta} = \widetilde{\Delta}_{(D,E,F),(A,B)}$  is annihilated by the infinitesimal generators of the groups  $U_n$ ,  $U_k$  and  $U_\ell$ , the lemma will follow. The infinitesimal generators of  $U_n$  act on  $\widetilde{\Delta}$  by row operations. Specifically, we have a basis of the infinitesimal generators for  $U_n$ , consisting of operators

$$E_{ad} = \sum_{b=1}^k x_{ab} \frac{\partial}{\partial x_{db}} + \sum_{c=1}^{\ell} y_{ac} \frac{\partial}{\partial y_{dc}},$$

for  $1 \leq a < d \leq n$ . Thus,  $E_{ad} \widetilde{\Delta}$  is a sum of terms, each of which is a determinant of a matrix in which one of the rows of  $\widetilde{\Delta}$  consisting of entries  $\alpha_{ij} x_{db}$  or  $\beta_{ih} y_{dc}$  gets replaced by a row with entries  $\alpha_{ij} x_{ab}$  or  $\beta_{ih} y_{ac}$ . Since the block matrices  $X_{f_j, d_k}$  of formula (2.5) contain all rows of  $X$  up to the  $f_j$ -th, we see that each term in the sum defining  $E_{ad} \widetilde{\Delta}$  will be the determinant of a matrix with two repeated rows, and therefore vanishes identically. We conclude that  $\widetilde{\Delta}$  is invariant under  $U_n$ . A similar argument, but using column operations, shows that  $\widetilde{\Delta}$  is invariant under  $U_k$  and  $U_\ell$ . This proves part (a) of the Lemma. Part (b) of the lemma is a straightforward calculation.  $\square$

### 2.2.1 Row and Column Operations on $\widetilde{\Delta}_{(D,E,F),(A,B)}$

The polynomials  $\widetilde{\Delta}_{(D,E,F),(A,B)}$  give us a large collection of elements of  $TA_{n,k,\ell}$ . We want to show that they in fact span  $TA_{n,k,\ell}$ . More precisely, we want to show that, by making



appropriate choices of the coefficient matrices  $A$  and  $B$ , we can extract from the polynomials  $\tilde{\Delta}_{(D,E,F),(A,B)}$  a basis for the  $\psi^{F^t} \times \psi^{D^t} \times \psi^{E^t}$ -eigenspace of  $TA_{n,k,\ell}$ .

Although the  $\tilde{\Delta}_{(D,E,F),(A,B)}$  were constructed as polynomials in the  $x_{ab}$ 's and  $y_{ac}$ 's, they in fact also depend polynomially on the auxiliary variables  $\alpha_{ij}$  and  $\beta_{ih}$ . Furthermore, these auxiliary variables are organized into matrices, and thus support an action of a product of general linear groups. Precisely, there is a natural action of  $GL_t \times GL_r$  on the  $\alpha_{ij}$  and an action of  $GL_t \times GL_s$  on the  $\beta_{ih}$ . (Here,  $r$ ,  $s$  and  $t$  are the depths of the partitions  $D$ ,  $E$  and  $F$  respectively (see equation (2.6)). The  $\tilde{\Delta}_{(D,E,F),(A,B)}$  have a very nice property with respect to these actions.

Let  $\tilde{U}_r$  denote the group of strict left-to-right column operations on the matrix  $A$ . (The tilde on  $\tilde{U}_r$  is simply to distinguish the action on the auxiliary variables  $\alpha_{ij}$  from the action on the variables  $x_{ij}$  and  $y_{ij}$ .) That is,  $\tilde{U}_r$  allows one to add a multiple of a column of the matrix  $A$  to any column strictly to its right, and to combine such operations. It is a maximal unipotent subgroup of  $GL_r$ . Similarly, let  $\tilde{U}_s$  be the strict left-to-right column operations on  $B$ . Finally, let  $\tilde{U}_t$  be the diagonal group of strict top-to-bottom row operations on  $A$  and  $B$  simultaneously. This is generated by the operations which add a multiple of one row of  $A$  to a lower row of  $A$ , and at the same time, do the same thing to  $B$ . Let  $\tilde{A}_r$  be the diagonal torus of  $GL_r$ , which multiplies each column of  $A$  by some scalar. Define  $\tilde{A}_s$  and  $\tilde{A}_t$  analogously.

**Lemma 2.2.** *As functions of the  $\alpha_{ij}$  and the  $\beta_{ih}$ , the polynomials  $\tilde{\Delta}_{(D,E,F),(A,B)}$  are invariant under the action of  $\tilde{U}_t \times \tilde{U}_r \times \tilde{U}_s$ . They are eigenvectors for the product  $\tilde{A}_t \times \tilde{A}_r \times \tilde{A}_s$ , with eigencharacter  $\psi^{F^t} \times \psi^{D^t} \times \psi^{E^t}$ .*

**Proof:** Consider the column operations which add  $\eta$  times the  $i$ -th column of  $A$  to the  $h$ -th column, where  $i < h$ . That is, this operation replaces the entries  $\alpha_{jh}$ ,  $1 \leq j \leq t$ , with entries  $\alpha_{jh} + \eta\alpha_{ji}$ . Here  $\eta$  is a complex number. From the form of the entries of  $\tilde{X}$  (see formula (2.9)), and the fact that the numbers  $d_i$  decrease as  $i$  increases, we can see that the effect of this operation on  $\tilde{X}$  can also be achieved by means of column operations directly on  $\tilde{X}$ . Precisely, for each of the first  $d_h$  columns of the  $i$ -th block of columns of  $\tilde{X}$ , we should add  $\eta$  times a column to the corresponding column of the  $h$ -th block of columns. Since these operations are all strict left-to-right column operation is, they do not change the value of  $\tilde{\Delta}_{(D,E,F),(A,B)}$ . The invariance statement of the lemma follows. The statement about the behavior under diagonal operations is again a straightforward computation.  $\square$

### 2.2.2 The Reduced Polynomials $\tilde{\Delta}_{(D,E,F),(J,B)}$

By the Lemma 2.2, we see that, for generic values of  $A$ , we may reduce  $A$  to the  $t \times r$  “identity” matrix,  $J_{t,r}$ , without changing the value of  $\tilde{\Delta}_{(D,E,F),(A,B)}$ . Here,  $J = J_{t,r}$  is the  $t \times r$  matrix with ones down the diagonal and zeroes elsewhere. In other words, if  $A = LV$ , where  $L$  is a lower triangular  $t \times r$  matrix, and  $V$  is an upper triangular  $r \times r$  unipotent

matrix, then we have

$$\tilde{\Delta}_{(D,E,F),(A,B)} = \left( \prod_{i=1}^t \alpha_{ii}^{f_i} \right) \tilde{\Delta}_{(D,E,F),(J,B)}. \quad (2.12)$$

In particular, the polynomials of the form  $\tilde{\Delta}_{(D,E,F),(J,B)}$  will span the same space as the collection of all  $\tilde{\Delta}_{(D,E,F),(A,B)}$ .

Thus we consider the polynomials  $\tilde{\Delta}_{(D,E,F),(J,B)}$ . We can expand these as polynomials in the  $\beta_{ih}$ :

$$\tilde{\Delta}_{(D,E,F),(J,B)} = \sum_M \tilde{\Delta}_{(D,E,F),M} \left( \prod_{i,h} \beta_{ih}^{m_{ih}} \right), \quad (2.13)$$

where  $M = [m_{ih}]$  is a matrix of non-negative integers, defining the exponents to which the  $\beta_{ih}$  appear in a given monomial. Each coefficient  $\tilde{\Delta}_{(D,E,F),M}$  is a polynomial in  $x_{ab}$  and  $y_{ac}$ .

### 2.3 The Structure of Littlewood-Richardson Tableaux

As is well known, the Littlewood-Richardson (which we abbreviate using LR) coefficients count the number of *Littlewood-Richardson tableaux* [CGR], [RW], [Ful], [Sta]. With fixed and appropriate  $D$ ,  $E$  and  $F$ , we focus on LR tableaux which capture the multiplicity of  $\rho_n^{F^t}$  in the tensor product  $\rho_n^{D^t} \otimes \rho_n^{E^t}$  (see [Ful]). We will define a mapping from Littlewood-Richardson tableaux (for appropriate fixed  $D$ ,  $E$  and  $F$ ) to certain of the coefficients  $\tilde{\Delta}_{(D,E,F),M}$ . More precisely, we will show how to associate to each Littlewood-Richardson tableau  $T$ , a monomial  $M(T)$  in the  $\beta_{ih}$ .

A Young tableau  $T$  is a filling of a skew diagram. Fix  $D$ ,  $E$  and  $F$ . This monomial  $M(T)$  in the variables  $\beta_{ij}$  is directly related to the entries of the skew diagram  $T = F^t - D^t$ . The related combinatorics of Young tableaux leading to the definition/construction of  $M(T)$  will be discussed in §3.1. Not surprising, one could easily and uniquely recover  $T$  from the monomial  $M(T)$ . We will also associate to  $T$  the polynomial  $\tilde{\Delta}_{(D,E,F),M(T)}$ , i.e., the coefficient of  $M(T)$  in the expansion (2.13) of  $\tilde{\Delta}_{(D,E,F),(J,B)}$ .

Note that  $\tilde{\Delta}_{(D,E,F),M(T)}$  is a polynomial in  $x_{ab}$  and  $y_{ac}$ . By considering a standard coefficient in the  $x$ -variables of  $\tilde{\Delta}_{(D,E,F),M(T)}$ , we have a polynomial  $\delta_{T,Y}$  (see §3.3) in the variables  $y_{ac}$ . This polynomial can also be directly related to the fillings of  $T = F^t - D^t$  (see Lemma 3.1 and Lemma 3.2). Lemma 3.2 is, in fact, the key to the proof of our linear basis result.

Constructing the correspondence as proposed above will involve some study of the structure of LR tableaux. We recall that a Littlewood-Richardson tableau (= LR tableau) is a skew diagram filled with positive integers 1 through  $k$ . The criteria for such a tableau  $T$  to be LR may be stated as follows [CGR], [Ful]:

LR1:  $T$  is semistandard, which means that the numbers in each row of  $T$  weakly increase

from left to right, and the numbers in each column strictly increase from top to bottom.

LR2: For every pair of positive numbers  $m$  and  $p$ , the number of times the number  $m$  occurs in the first  $p$  rows of  $T$  is not larger than the number of times that  $m - 1$  appears in the first  $p - 1$  rows of the tableau. This condition is interpreted as being vacuous when  $m = 1$ .

**Example:** As far as possible, we will try to use the same example throughout this paper, and this example has to be complicated enough for us to point out several features which are important. Our example will be as follows:

$$D = (3, 3, 2, 1, 1), \text{ i.e., } r = 5 \quad \text{and} \quad D^t = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

$$E = (3, 3, 2, 1), \text{ i.e., } s = 4 \quad \text{and} \quad E^t = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

$$F = (5, 5, 4, 3, 1, 1), \text{ i.e., } t = 6 \quad \text{and}$$

$$T = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \boxed{1} \\ \hline & & & & & \\ \hline & & & & \boxed{1} & \\ \hline & & & \boxed{1} & \boxed{2} & \\ \hline \boxed{1} & \boxed{2} & \boxed{3} & & & \\ \hline \boxed{2} & \boxed{3} & & & & \\ \hline \end{array}, \quad T_1 = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \boxed{1} \\ \hline & & & & & \\ \hline & & & & \boxed{1} & \\ \hline & & & \boxed{1} & \boxed{2} & \\ \hline \boxed{1} & \boxed{2} & \boxed{2} & & & \\ \hline \boxed{3} & \boxed{3} & & & & \\ \hline \end{array}$$

$$T_2 = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \boxed{1} \\ \hline & & & & & \\ \hline & & & & \boxed{1} & \\ \hline & & & \boxed{2} & \boxed{2} & \\ \hline \boxed{1} & \boxed{1} & \boxed{3} & & & \\ \hline \boxed{2} & \boxed{3} & & & & \\ \hline \end{array} \quad \text{and} \quad T_3 = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \boxed{1} \\ \hline & & & & & \\ \hline & & & & \boxed{2} & \\ \hline & & & \boxed{1} & \boxed{3} & \\ \hline \boxed{1} & \boxed{1} & \boxed{2} & & & \\ \hline \boxed{2} & \boxed{3} & & & & \\ \hline \end{array}$$

are four tableaux for  $F^t - D^t$ . Observe that the skew tableaux  $T, T_1, T_2$  and  $T_3$  are fillings of the tableau  $F^t - D^t$ , i.e., the difference between diagrams  $F^t$  and  $D^t$ , as depicted by the cells whose boundaries are highlighted. We can verify that the fillings are the only possible fillings satisfying conditions (LR1) and (LR2), and thus the LR coefficient  $c_{D^t, E^t}^{F^t} = 4$ . This of course means that the irreducible representation with highest weight parameterized by  $F^t$  appears four times in the tensor product of irreducible representations parameterized by  $D^t$  and  $E^t$ .

### 2.3.1 Banal Tableau

By a *banal* LR tableau, we will mean an ordinary (i.e., not skew) diagram in which the cells of the  $i$ -th row are all filled with the number  $i$ . Equivalently, a column of length  $k$  is filled with the numbers from 1 to  $k$ , in increasing order as you move down the column. We will describe a content preserving mapping between the cells of a general LR tableau and a banal LR tableau. By “*content preserving*”, we mean that each cell of one tableau is mapped to a cell of the other tableau with the same value (i.e., same number inside).

**Example:** The banal tableau for  $E^t$  in the above example is

$$E = (3, 3, 2, 1) \quad \text{and banal tableau for } E^t = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & & \\ \hline \end{array}$$

### 2.3.2 Peeling a Tableau $T$

In any diagram or skew-diagram, a cell which lies above and to the right of a given cell will be said to lie *northeast* of the given cell. Note that northeast also includes things in the same row (resp. column), but to the right (resp. above).

Consider a general LR tableau  $T$ . Let  $\ell_0$  be the largest number which occurs in any cell of  $T$ . For each number  $h$ ,  $1 \leq h \leq \ell_0$ , let  $C_1(h)$  be the cell of  $T$  which contains  $h$ , and which lies farthest to the northeast among all cells containing  $h$ .

**Example:** For our example in §2.3,  $\ell_0 = 3$ , and we have highlighted  $C_1(1)$ ,  $C_1(2)$  and  $C_1(3)$  for both tableaux  $T$  and  $T_3$ : Note that  $F = (5, 5, 4, 3, 1, 1)$  and

$$T = \begin{array}{|c|c|c|c|c|} \hline & & & & \boxed{1} \\ \hline & & & 1 & \\ \hline & & 1 & \boxed{2} & \\ \hline 1 & 2 & \boxed{3} & & \\ \hline 2 & 3 & & & \\ \hline \end{array} \quad \text{and} \quad T_3 = \begin{array}{|c|c|c|c|c|} \hline & & & & \boxed{1} \\ \hline & & & \boxed{2} & \\ \hline & & 1 & \boxed{3} & \\ \hline 1 & 1 & 2 & & \\ \hline 2 & 3 & & & \\ \hline \end{array}$$

We should check that  $C_1(h)$  is well-defined. Start with  $C_1(\ell_0)$ . Let  $C$  be a cell containing  $\ell_0$ , such that there is no cell containing  $\ell_0$  and lying to the northeast of  $C$ . Then  $C$  must lie on the end of a row, since rows are weakly increasing to the right, and  $\ell_0$  is the largest possible entry a cell can have.

Also, no cell in a row above the row of  $C$  can contain  $\ell_0$ . For, if  $C'$  is such a cell, then also the cell  $C''$  at the right end of the row of  $C'$ , must contain  $\ell_0$ , since the entries must weakly increase to the right, hence be at least equal to  $\ell_0$ , but also cannot be larger. But the end of each row of a skew diagram is to the northeast of the ends of lower rows. Thus, by choice

of  $C$ , the cell  $C''$  cannot exist, and therefore also,  $C'$  cannot exist. Therefore, we must have that  $C = C_1(\ell_0)$ , so  $C_1(\ell_0)$  is indeed well-defined.

Let  $T^+$  be the tableau consisting of the rows of  $T$  above the row of  $C_1(\ell_0)$ . By the argument of the previous paragraph, the entries of  $T^+$  are all less than  $\ell_0$ . On the other hand, the condition (LR2) for LR tableaux guarantees that some cells of  $T^+$  do contain  $\ell_0 - 1$ , which is thus the largest entry in  $T^+$ . Therefore,  $C_1(\ell_0 - 1)$  will be contained in  $T^+$ , and the argument just given implies that  $C_1(\ell_0 - 1)$  does indeed exist. Continuing by downward induction, we see that  $C_1(h)$  is well-defined for all  $h$ . Furthermore, as byproducts of the existence argument, we have found that each  $C_1(h)$  lies on the right end of its row, and that  $C_1(h - 1)$  lies to the northeast of  $C_1(h)$ , more precisely,  $C_1(h - 1)$  lies strictly above and weakly to the right of  $C_1(h)$ . In our example  $T_3$ , the boxes  $C_1(2)$  and  $C_1(3)$  lie in the same column. Thus, the cells  $C_1(h)$  constitute what is sometimes called a *vertical skew strip*. In our example above, this vertical strip has been highlighted.

We now remove from  $T$  the cells  $C_1(h)$  for  $1 \leq h \leq \ell_0$ , and reassemble them into a column of length  $\ell_0$ , in consecutive order from top to bottom. This column will be the first column of the banal tableau to which we will map  $T$ . We will denote this banal tableau by  $BT$ .

**Example:** For our example in §2.3,  $\ell_0 = 3$ , and we have highlighted  $C_1(1)$ ,  $C_1(2)$  and  $C_1(3)$ : Here  $F = (5, 5, 4, 3, 1, 1)$  and

$$T = \begin{array}{cccccc} & & & & & \boxed{1} \\ & & & & 1 & \\ & & & 1 & \boxed{2} & \\ 1 & 2 & \boxed{3} & & & \\ 2 & 3 & & & & \end{array} \longrightarrow \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array} \text{ to be attached to } BT'.$$

**Lemma 2.3.** *Let  $T'$  denote the configuration which is left after removal of the cells  $C_1(h)$  from  $T$ . We claim that  $T'$  is an LR tableau.*

**Example:** For our example in §2.3,  $\ell_0 = 3$ , after removal of the vertical strip formed by  $C_1(1)$ ,  $C_1(2)$  and  $C_1(3)$ : Recall that  $F = (5, 5, 4, 3, 1, 1)$  and

$$T' = \begin{array}{cccccc} & & & & & \\ & & & & 1 & \\ & & & 1 & & \\ 1 & 2 & & & & \\ 2 & 3 & & & & \end{array} \quad \text{and} \quad T'_3 = \begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & 1 & & \\ 1 & 1 & 2 & & & \\ 2 & 3 & & & & \end{array}$$

are again LR tableaux.

**Proof:** First, we argue that the cells of  $T'$  form a skew diagram. We have seen that each cell  $C_1(h)$  lies at the end of its row. Furthermore,  $C_1(\ell_0)$  must lie also at the bottom of its

column, since it contains the largest number in  $T$ , and columns are strictly increasing as you go down. Hence  $T - C_1(\ell_0)$  (i.e., removal of the cell  $C_1(\ell_0)$  from  $T$ ) is a skew diagram.

Suppose that  $T$  with the cells  $C_1(h)$  removed for  $h > g$  is a skew diagram. Call it  $T_{(g)}$  (not to be confused with the tableaux  $T_1, T_2$  and  $T_3$  in our example). Now let us remove the cell  $C_1(g)$ . As with all the cells  $C_1(h)$ , the cell  $C_1(g)$  lies at the end of its row, even in  $T$ , hence a fortiori in  $T_{(g)}$ . If there is a cell in  $T_{(g)}$  which lies below  $C_1(g)$ , the number in this cell must be greater than  $g$ . Say it is  $g' > g$ . Since  $C_1(g)$  is the farthest northeast occurrence of  $g$ , the cell immediately below must

be northeast of  $C_1(g')$ , contradicting the definition of  $C_1(g')$ . Hence, there is no cell in  $T_{(g)}$  lying directly below  $C_1(g)$ , so  $T_{(g)} - C_1(g)$  is a skew diagram. By downward induction from  $\ell_0$ , we conclude that the cells of  $T'$

do form a diagram, as desired.

Since  $T'$  is a subdiagram of  $T$ , it is clear by reference to condition (LR1) that  $T'$  is semistandard. To show that  $T'$  is an LR tableau, we have still to verify that condition (LR2) is valid. Hence consider positive integers  $m$  and  $p$ . We must argue that the number of times that  $m$  occurs in the first  $p$  rows of  $T'$  is at least as large as the number of times that  $m + 1$  occurs in the first  $p + 1$  rows of  $T'$ . If  $m + 1$  does not occur in the first  $p + 1$  rows of  $T'$ , then this is clearly true. If however  $m + 1$  does occur in the first  $p + 1$  rows of  $T'$ , then it also certainly occurs in the first  $p + 1$  rows of  $T$ , and therefore the cell  $C_1(m + 1)$  must have belonged to the first  $p + 1$  rows of  $T$ . It follows that the number of occurrences of  $m + 1$  in the first  $p + 1$  rows of  $T'$  is one less than the number of occurrences of  $m + 1$  in the first  $p + 1$  rows of  $T$ . On the other hand, the number of occurrences of  $m$  in the first  $p$  rows of  $T'$  is at least one less than the number of occurrences of  $m$  in the first  $p$  rows of  $T$ , and this in turn is at least equal to the number of occurrences of  $m + 1$  in the first  $p + 1$  rows of  $T$ , since  $T$  is LR. It follows that condition (LR2) is also satisfied by  $T'$ . Therefore,  $T'$  is LR.  $\square$

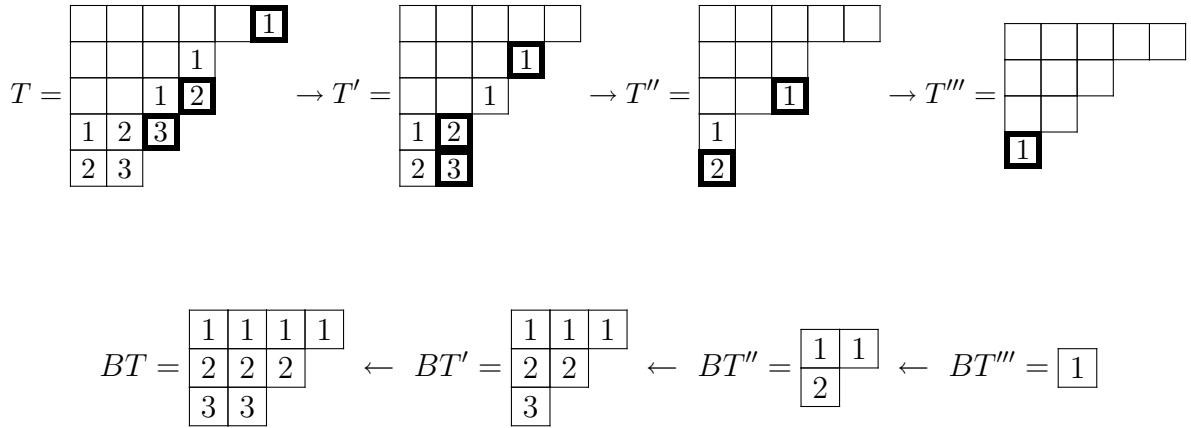
### 2.3.3 Standard Peeling of a Tableau $T$

We can now finish our description of the mapping from  $T$  to the banal tableau  $BT$ . We have removed from  $T$  the boxes  $C_1(h)$ , for  $1 \leq h \leq \ell_0$ , and reassembled them into the first column of the prospective  $BT$ . This has left us with a configuration  $T'$  of cells, and we have shown that  $T'$  is an LR tableau. By induction on the number of boxes in  $T$ , we may assume that the process we have outlined above allows us to define the banal tableau  $BT'$ , and the mapping from  $T'$  to  $BT'$ . To get  $BT$ , we append to the left of  $BT'$ , the column of length  $\ell_0$  which we have just constructed. More precisely, the first column of  $BT$  is this column, and for  $i > 1$ , the  $i$ -th column of  $BT$  is the  $(i - 1)$ -th column of  $BT'$ . Since the largest entry in any cell of  $T'$  is at most  $\ell_0$ , the columns of  $BT'$  cannot be longer than  $\ell_0$ , and therefore the shape of  $BT'$  is a diagram, and the entries in its  $i$ -th row will clearly all be equal to  $i$ . Thus,  $BT$  is a banal tableau. As for the mapping from  $T$  to  $BT$ , we take the union of the mappings from  $T'$  to  $BT'$  and the mapping of the cells  $C_1(h)$  to the first column of  $BT$ . This evidently is

content preserving. Hence, we have constructed the desired mapping.

Since the construction of  $BT$  from  $T$  involves successive removal of vertical skew strips from  $T$ , as described in §2.3.2, we will refer to the mapping from  $T$  to  $BT$  constructed above, in §2.3.2 to §2.3.3, as the *standard peeling* of  $T$ .

**Example:** The following gives an illustration of the standard peeling of  $T$  for our example in §2.3. The vertical strips are being highlighted at each stage of peeling:



### 3 Proof of Theorem

#### 3.1 The Monomial $M(T)$ , in the Variables $\beta_{ih}$ , Associated to $T$

With the discussions of §2.2 and §2.3 in mind, we can now describe how to associate to an LR tableau  $T$ , a monomial  $M(T)$  in the auxiliary variables  $\beta_{ih}$ .

The LR tableau lives in a skew-diagram, but we assume that this skew-diagram is embedded in a larger diagram, associated to the partition  $F^t$ . The skew diagram will be the difference  $F^t - D^t$ , between  $F^t$  and a smaller diagram  $D^t$ . The LR tableau  $T$  will result from a filling of  $F^t - D^t$  by the entries from a banal tableau  $E^t$ . The inverse of this filling will be the standard peeling from  $F^t - D^t$  to  $E^t = B(F^t - D^t)$ . The partitions  $D$ ,  $E$  and  $F$  are as in (2.6).

Consider the standard peeling of the LR tableau  $T$ , or rather, its inverse, the filling of the shape of  $F^t - D^t$  by the contents of  $E^t$ . The elements of the  $h$ -th column of  $E^t$  will be distributed among various columns of  $F^t - D^t$ . We number the columns of each diagram consecutively, from left to right. Thus, the  $h$ -th column of  $E^t$  has length  $e_h$ , and there are  $s$  columns all together (so  $s$  is the length of the first row of  $E^t$ ). Suppose that  $m_{ih}$  of the elements from the  $h$ -th column of  $E^t$  get put into the  $i$ -th column of  $F^t - D^t$ . Then we

associate to the tableau  $T$  the monomial

$$M(T) = \prod_{i,h} \beta_{ih}^{m_{ih}}. \quad (3.1)$$

As indicated above in §2.2.2, we now associate to  $T$  the polynomial  $\tilde{\Delta}_{(D,E,F),M(T)}$  in the  $x_{ab}$  and  $y_{ac}$ , which is the coefficient of  $M(T)$  in the expansion (2.13):

$$\tilde{\Delta}_{(D,E,F),(J,B)} = \sum_M \tilde{\Delta}_{(D,E,F),M} \left( \prod_{i,h} \beta_{ih}^{m_{ih}} \right).$$

**Example:** In our example from §2.3, we have

$$\tilde{\Delta}_{(D,E,F),(J,B)} = \begin{vmatrix} X_{5,3} & 0 & 0 & 0 & 0 & \beta_{11}Y_{5,3} & \beta_{12}Y_{5,3} & \beta_{13}Y_{5,2} & \beta_{14}Y_{5,1} \\ 0 & X_{5,3} & 0 & 0 & 0 & \beta_{21}Y_{5,3} & \beta_{22}Y_{5,3} & \beta_{23}Y_{5,2} & \beta_{24}Y_{5,1} \\ 0 & 0 & X_{4,2} & 0 & 0 & \beta_{31}Y_{4,3} & \beta_{32}Y_{4,3} & \beta_{33}Y_{4,2} & \beta_{34}Y_{4,1} \\ 0 & 0 & 0 & X_{3,1} & 0 & \beta_{41}Y_{3,3} & \beta_{42}Y_{3,3} & \beta_{43}Y_{3,2} & \beta_{44}Y_{3,1} \\ 0 & 0 & 0 & 0 & X_{1,1} & \beta_{51}Y_{1,3} & \beta_{52}Y_{1,3} & \beta_{53}Y_{1,2} & \beta_{54}Y_{1,1} \\ 0 & 0 & 0 & 0 & 0 & \beta_{61}Y_{1,3} & \beta_{62}Y_{1,3} & \beta_{63}Y_{1,2} & \beta_{64}Y_{1,1} \end{vmatrix}$$

Note that  $F = (5, 5, 4, 3, 1, 1)$  gives the heights of the blocks, from top to bottom. The widths of the blocks (from left to right) are  $D = (3, 3, 2, 1, 1)$  for the  $x$ 's and  $E = (3, 3, 2, 1)$  for the  $y$ 's.

**Example:** For the tableau  $T$ , the following matrix gives the entries  $m_{ih}$  of the elements from the  $h$ -th column of  $E^t$  get put into the  $i$ -th column of  $F^t - D^t$ :

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and so  $M(T) = \beta_{31}\beta_{41}\beta_{61}\beta_{22}^2\beta_{42}\beta_{13}\beta_{33}\beta_{14}$ .



Compare with that of tableau  $T_1$ :

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

with  $M(T_1) = \beta_{21}\beta_{41}\beta_{61}\beta_{12}\beta_{32}\beta_{42}\beta_{23}\beta_{33}\beta_{14}$ .

Further, as can easily be seen by comparison with the two matrices above, we have

$$\tilde{\Delta}_{(D,E,F),M(T)} = \begin{vmatrix} X_{5,3} & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{13}Y_{5,2} & \beta_{14}Y_{5,1} \\ 0 & X_{5,3} & 0 & 0 & 0 & 0 & \beta_{22}Y_{5,3} & 0 & 0 \\ 0 & 0 & X_{4,2} & 0 & 0 & \beta_{31}Y_{4,3} & 0 & \beta_{32}Y_{4,2} & 0 \\ 0 & 0 & 0 & X_{3,1} & 0 & \beta_{41}Y_{3,3} & \beta_{42}Y_{3,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & X_{1,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_{61}Y_{1,3} & 0 & 0 & 0 \end{vmatrix}$$

and

$$\tilde{\Delta}_{(D,E,F),M(T_1)} = \begin{vmatrix} X_{5,3} & 0 & 0 & 0 & 0 & 0 & \beta_{12}Y_{5,3} & 0 & \beta_{14}Y_{5,1} \\ 0 & X_{5,3} & 0 & 0 & 0 & \beta_{21}Y_{5,3} & 0 & \beta_{23}Y_{5,2} & 0 \\ 0 & 0 & X_{4,2} & 0 & 0 & 0 & \beta_{32}Y_{4,3} & \beta_{33}Y_{4,2} & 0 \\ 0 & 0 & 0 & X_{3,1} & 0 & \beta_{41}Y_{3,3} & \beta_{42}Y_{3,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & X_{1,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_{61}Y_{1,3} & 0 & 0 & 0 \end{vmatrix}$$

Note that  $F = (5, 5, 4, 3, 1, 1)$  gives the heights of the blocks, from top to bottom. The widths of the blocks (from left to right) are  $D = (3, 3, 2, 1, 1)$  for the  $x$ 's and  $E = (3, 3, 2, 1)$  for the  $y$ 's.

### 3.2 Relationship Between $T$ and $\tilde{\Delta}_{(D,E,F),M(T)}$

The polynomials  $\tilde{\Delta}_{(D,E,F),M(T)}$  are coefficients of the polynomial  $\tilde{\Delta}_{(D,E,F),(J,B)}$  of equation (2.13) with respect to the entries  $\beta_{ih}$  of  $B$ . This is a rather indirect definition, and we want to establish a more direct connection between the polynomials  $\tilde{\Delta}_{(D,E,F),M(T)}$  and their LR tableaux.

The polynomial  $\tilde{\Delta}_{(D,E,F),(J,B)}$  is the determinant of the matrix  $\tilde{Z}_{(D,E,F)} = \tilde{Z}$ , described in equation (2.10). The first  $|D|$  columns of  $\tilde{Z}$  consist of the matrix  $\tilde{X}$ , which is partitioned into the blocks  $\tilde{X}_{jk} = \alpha_{jk} X_{f_j, d_k}$ . Since we have specialized the coefficients  $\alpha_{jk} = \delta_{jk}$ , where the  $\delta_{jk}$  are Kronecker's deltas, only one block in each column of  $\tilde{X}$  is non-zero. The condition that  $D \subseteq F$  says that  $d_k \leq f_k$ , so each block  $X_{f_k, d_k}$  is taller than it is wide. Therefore, each term in the expansion of  $\tilde{Z}$  must contain one entry from each column of  $X_{f_k, d_k}$  as a factor.

Consider the terms which have  $\prod_{j=1}^{d_k} x_{jj}$  as factor, for all  $k$  where  $1 \leq k \leq r$ . Recall that  $r$  is the depth of the Young diagram  $D$  (see (2.6)). This will be the determinant of a submatrix of  $\tilde{Z}$ , the submatrix obtained by eliminating all the rows and columns of the diagonal entries of each submatrix  $\tilde{X}_{jj}$ . The columns eliminated are exactly the columns of the submatrix  $\tilde{X}$  of  $\tilde{Z}$ . Therefore, the submatrix  $\tilde{Z}$  in which we are interested is actually a submatrix of  $\tilde{Y}$ , which we will denote  $\tilde{Y}_o$ . The submatrix  $\tilde{Y}_o$  has a block structure, which is subordinate to the block structure of  $\tilde{Y}$ , in the sense that each block of  $\tilde{Y}_o$  is a submatrix of a block of  $\tilde{Y}$ .

More precisely, the block  $(\tilde{Y}_o)_{j,k}$  consists of the bottom  $f_j - d_j$  rows of  $\tilde{Y}_{j,k} = \beta_{jk} Y_{f_j, e_k}$ . We will denote this submatrix of  $\tilde{Y}_{jk}$  by

$$\beta_{jk} Y_{(d_j, f_j], e_k}. \quad (3.2)$$

Here, the notation  $(d_j, f_j]$  is intended to connote the open interval from  $d_j$  to  $f_j$ , and indicates that we are including the rows  $d_j + 1$  through  $f_j$ .

Summarizing, we may say that the coefficient of  $\prod_{k=1}^r \left( \prod_{j=1}^{d_k} x_{jj} \right)$  in  $\tilde{\Delta}_{(D,E,F),(J,B)}$  is the determinant of the matrix  $\tilde{Y}_o$ , and the coefficient of  $\prod_{k=1}^r \left( \prod_{j=1}^{d_k} x_{jj} \right)$  in  $\tilde{\Delta}_{(D,E,F),M(T)}$  is the coefficient of the monomial  $M(T)$  of formula (3.1) in the determinant of  $\tilde{Y}_o$ , considered as a function of the  $\beta_{ih}$ . We will write

$$\det \tilde{Y}_o = \Delta_{(D,E,F),B,Y}.$$

We are interested in the expansion of  $\Delta_{(D,E,F),B,Y}$  with respect to the coefficients  $\beta_{ih}$ :

$$\Delta_{(D,E,F),B,Y} = \sum_M \tilde{\Delta}_{(D,E,F),M,Y} \left( \prod_{M=(m_{ih})} \beta_{ih}^{m_{ih}} \right), \quad (3.3)$$

and in particular, we define using (3.1) and (3.3),

$$\delta_{T,Y} = \tilde{\Delta}_{(D,E,F),M(T),Y}. \quad (3.4)$$

**Example:** In our example §2.3, for tableaux  $T, T_1, T_2$  and  $T_3$ , the coefficient of  $\prod_{k=1}^r \left( \prod_{j=1}^{d_k} x_{jj} \right)$

in  $\tilde{\Delta}_{(D,E,F),(J,B)}$  is given by

$$\det \tilde{Y}_o = \Delta_{(D,E,F),B,Y} = \begin{vmatrix} \beta_{11}Y_{(3,5],3} & \beta_{12}Y_{(3,5],3} & \beta_{13}Y_{(3,5],2} & \beta_{14}Y_{(3,5],1} \\ \beta_{21}Y_{(3,5],3} & \beta_{22}Y_{(3,5],3} & \beta_{23}Y_{(3,5],2} & \beta_{24}Y_{(3,5],1} \\ \beta_{31}Y_{(2,4],3} & \beta_{32}Y_{(2,4],3} & \beta_{33}Y_{(2,4],2} & \beta_{34}Y_{(2,4],1} \\ \beta_{41}Y_{(1,3],3} & \beta_{42}Y_{(1,3],3} & \beta_{43}Y_{(1,3],2} & \beta_{44}Y_{(1,3],1} \\ \beta_{61}Y_{(0,1],3} & \beta_{62}Y_{(0,1],3} & \beta_{63}Y_{(0,1],2} & \beta_{64}Y_{(0,1],1} \end{vmatrix}$$

We note that

$$F - D = (5 - 3, 5 - 3, 4 - 2, 3 - 1, 1 - 1, 1 - 0) = (2, 2, 2, 2, 0, 1)$$

which gives the heights of the blocks, from top to bottom. The widths of the blocks are  $E = (3, 3, 2, 1)$ , from left to right.

The coefficient of  $\prod_{k=1}^r \left( \prod_{j=1}^{d_k} x_{jj} \right)$  in  $\tilde{\Delta}_{(D,E,F),M(T)}$  is the same as the coefficient of  $M(T)$  in  $\det \tilde{Y}_o$  and is given by

$$\tilde{\Delta}_{(D,E,F),M(T),Y} = \delta_{T,Y} = \begin{vmatrix} 0 & 0 & Y_{(3,5],2} & Y_{(3,5],1} \\ 0 & Y_{(3,5],3} & 0 & 0 \\ Y_{(2,4],3} & 0 & Y_{(2,4],2} & 0 \\ Y_{(1,3],3} & Y_{(1,3],3} & 0 & 0 \\ Y_{(0,1],3} & 0 & 0 & 0 \end{vmatrix}$$

### 3.3 The Polynomials $\delta_{T,Y}$ (in the Variables $y_{ac}$ ) Associated to a Tableau $T$

For a tableau  $T$ , we have defined a monomial  $M(T)$  in the variables  $\beta_{ih}$ . We then considered the coefficient  $\tilde{\Delta}_{(D,E,F),M(T)}$  of  $M(T)$  in  $\tilde{\Delta}_{(D,E,F),(J,B)}$ . This is a polynomial in the  $x_{ab}$ 's and the  $y_{ac}$ 's. By considering a standard coefficient in the  $x$ -variables of  $\tilde{\Delta}_{(D,E,F),M(T)}$ , we have found a polynomial  $\tilde{\Delta}_{(D,E,F),M(T),Y} = \delta_{T,Y}$  in the variables  $y_{ac}$ 's. This polynomial is associated to the LR tableau  $T$ , but in a rather indirect way. We would like to relate it more directly to  $T$ .

This is what we will do. We will associate to  $T$  a monomial  $e(T)$  in the variables  $y_{ac}$ 's. Also, we will define a term order on monomials in the variables  $y_{ac}$ 's. Then we will show that, with respect to the given term order, the monomial  $e(T)$  is the highest order term in the polynomial  $\delta_{T,Y}$ .

The LR tableau  $T$  is specified by giving the entry of each box of  $F^t - D^t$ . To each box  $\mathbf{b}$  of  $T$ , we associate the variable  $y_{a(\mathbf{b})c(\mathbf{b})}$ , where  $a(\mathbf{b})$  is the row of  $F^t$  in which the box  $\mathbf{b}$  lies, and  $c(\mathbf{b})$  is the entry in  $\mathbf{b}$ . To the tableau  $T$ , we associate the product

$$e(T) = \prod_{\mathbf{b} \in T} y_{a(\mathbf{b})c(\mathbf{b})} = \prod_{a,c} y_{ac}^{\ell(a,c)}, \quad (3.5)$$

where  $\mathbf{b}$  runs over all boxes of  $T$ . Also define the monomial

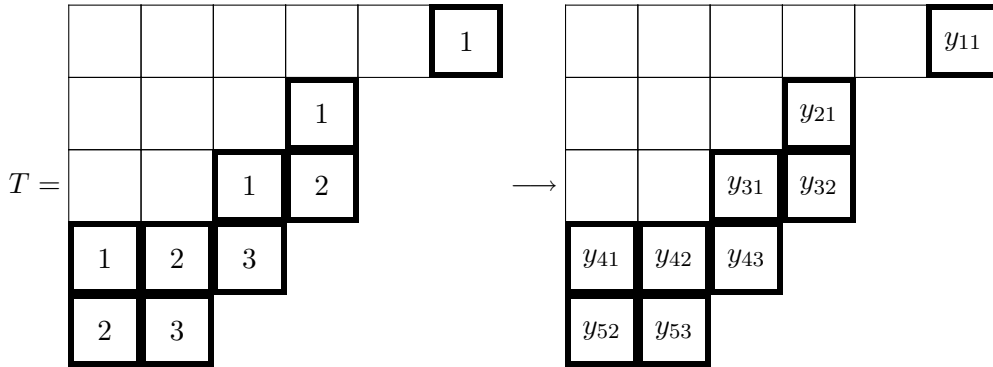
$$\mathcal{E}(T) = e(T) \prod_{k=1}^r \left( \prod_{j=1}^{d_k} x_{jj} \right) \quad (3.6)$$

as the monomial which contains  $e(T)$ , in the expansion of  $\tilde{\Delta}_{(D,E,F),M(T)}$  (see §3.1, (2.2) and (3.1)).

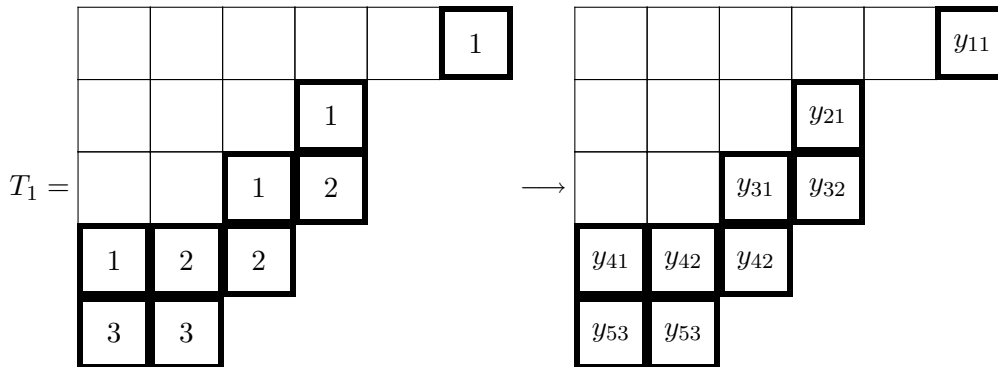
**Example:** Using the example in §2.3, we have

$$e(T) = y_{53}y_{43}y_{52}y_{42}y_{32}y_{41}y_{31}y_{21}y_{11}$$

using the following association by filling up the skew diagram  $F^t - D^t$  with  $y$ 's:



Compare with  $e(T_1) = y_{53}^2 y_{42}^2 y_{32} y_{41} y_{31} y_{21} y_{11}$ :



**Lemma 3.1.** *The LR tableau  $T$  is determined by the monomial  $e(T)$  together with the triple  $(D, E, F)$ . Further, the map  $T \mapsto \mathcal{E}(T)$  is one to one.*

**Proof:** Indeed, we can see from the definition (3.5) of  $e(T)$  that it may also be written

$$e(T) = \prod_{a,c} y_{ac}^{\ell(a,c)},$$

where  $\ell(a, c)$  is the number of entries equal to  $c$  in row  $a$ . In other words,  $e(T)$  records the number of boxes of each possible entry in each row. Since the entries are always arranged in weakly increasing order, and there are no gaps between consecutive boxes, this means that  $e(T)$  determines each row of  $T$ , up to translation in the horizontal direction. But, we also know that  $T$  occupies the boxes of  $F^t - D^t$ , so that its location is also fixed. This proves the first statement of the lemma.

For the second statement, note that the factor  $\prod_{k=1}^r \left( \prod_{j=1}^{d_k} x_{jj} \right)$  in the monomial  $\mathcal{E}(T)$  (see (3.6)) encodes the Young diagram  $D^t$ . Now, the banal tableau of shape  $E^t$  and the filling of  $F^t - D^t$  can be recovered from  $e(T)$ . The LR tableau  $T$  is thus effectively determined by  $\mathcal{E}(T)$ . Hence, the second statement.  $\square$

### 3.4 Proof of the Main Theorem

#### 3.4.1 Monomial Ordering in the $y$ Variables

We now fix a monomial ordering on monomials in the variables  $y_{ac}$ 's. First, we specify that

$$y_{ac} > y_{a'c'}$$

if and only if  $c' > c$ , or  $c = c'$  and  $a' > a$ . Note that this is a total ordering on the  $y_{ac}$ 's. It is essentially lexicographic order, except that we order on the second index first. Thus,

$$y_{11} > y_{21} > \dots > y_{n1} > y_{12} > y_{22} > \dots > y_{n\ell}.$$

We extend this to an ordering between all monomials on the  $y_{ac}$ 's by graded lexicographic order (see [CLO]). That is, if  $M_1$  and  $M_2$  are two monomials of unequal degree, then the one with larger degree is larger. If they have the same degree, we write them as products

$$M_1 = y_{a_1 c_1} y_{a_2 c_2} y_{a_3 c_3} \cdots \cdots ,$$

and

$$M_2 = y_{a'_1 c'_1} y_{a'_2 c'_2} y_{a'_3 c'_3} \cdots \cdots ,$$

where the factors of each monomial are written in decreasing order. Suppose that  $k$  is the first index such that  $y_{a_k c_k} \neq y_{a'_k c'_k}$ . Then  $M_1 > M_2$  if and only if  $y_{a_k c_k} > y_{a'_k c'_k}$ .

**Lemma 3.2.** *Given an LR tableau  $T$ , associate to it the polynomial  $\tilde{\Delta}_{(D,E,F),M(T),Y} = \delta_{T,Y}$  in the variables  $y_{ac}$ 's, as described in formula (3.4). Then the largest monomial, with respect to the monomial ordering described above, in the expansion of  $\delta_{T,Y}$  is the monomial  $e(T)$  defined in formula (3.5).*

The proof of Lemma 3.2 is by induction, and will be organized in several parts. It will occupy §3.4.2 to §3.4.6.

### 3.4.2 The Nature of $\widetilde{Y}_o$

To begin the proof of Lemma 3.2, we consider the nature of the matrix  $\widetilde{Y}_o$  of §3.2, and the role of the monomial  $e(T)$  in determining  $\delta_{T,Y}$ .

The matrix  $\widetilde{Y}_o$  is partitioned into blocks. The blocks of columns correspond to columns of  $E^t$ ; their sizes are  $e_k$ , the parts of the partition  $E$ . The blocks of rows correspond to the columns of  $F^t - D^t$ ; their sizes are  $f_j - d_j$ , the differences between the parts of  $F$  and the parts of  $D$ . The matrices which occupy the blocks are the matrices  $(\widetilde{Y}_o)_{j,k} = \beta_{jk} Y_{(d_j, f_j], e_k}$  of equation (3.2). In the discussion below, we will have to be referring both to individual rows or columns of the matrix  $\widetilde{Y}_o$ , and to rows or columns of blocks of  $\widetilde{Y}_o$ . To alleviate somewhat the awkwardness of distinguishing between the two, we will refer to the blocks of rows as “*superrows*” and to the blocks of columns as “*supercolumns*”. Thus, the matrix  $(\widetilde{Y}_o)_{j,k}$ , considered as a submatrix of  $\widetilde{Y}_o$ , is the intersection of the  $j$ -th superrow and the  $k$ -th supercolumn.

Consider the usual alternating sum expansion of  $\det \widetilde{Y}_o$ , with each term being a product of entries of  $\widetilde{Y}_o$ , one from each row and column. We see from the form (3.2) of the blocks of  $\widetilde{Y}_o$ , that a given term in this expansion will have a factor of  $\beta_{ih}$  for each entry chosen from the  $(i, h)$ -block. Thus, for a given monomial  $\prod_{i,h} \beta_{ih}^{m_{ih}}$ , the exponent  $m_{ih}$  determines the number of entries in the  $(i, h)$ -block that a term with this  $\beta$ -monomial will have. The total number of terms chosen from a given superrow of  $\widetilde{Y}_o$  will have to be the number of rows in that superrow; and similarly for columns. From this observation, it follows that, for a monomial  $\prod_{i,h} \beta_{ih}^{m_{ih}}$  to have a non-zero coefficient in the expansion of  $\det \widetilde{Y}_o$ , the conditions for row sums and column sums, respectively,

$$\sum_{h=1}^s m_{ih} = f_i - d_i \quad \text{and} \quad \sum_{i=1}^t m_{ih} = e_h \quad (3.7)$$

must hold.

Also, we may observe that, if some  $\beta_{ih}$  occurs does not occur in a monomial  $M(T)$  (that is, it occurs with exponent zero), then any term in the coefficient of  $M(T)$  will have to completely avoid the block  $(\widetilde{Y}_o)_{i,h}$ . This means that, as far as the coefficient of  $M(T)$  is concerned, we will get the same result if we replace  $(\widetilde{Y}_o)_{i,h}$  with the zero matrix.

**Example:** Reverting to our same tableau  $T$  as in §2.3, we look at the exponent matrix on the left. Note that  $m_{5j} = 0$  because  $f_5 - d_5 = 1 - 1 = 0$ . Because in the determinant expansion, we need to select one and only one element from each column as well as each row, we see that the row sums must be  $F - D = (2, 2, 2, 2, 0, 1)$  and the column sums must

be  $E = (3, 3, 2, 1)$  as in (3.7):

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \leftrightarrow \det \widetilde{Y}_o = \begin{vmatrix} \beta_{11}Y_{(3,5),3} & \beta_{12}Y_{(3,5),3} & \beta_{13}Y_{(3,5),2} & \beta_{14}Y_{(3,5),1} \\ \beta_{21}Y_{(3,5),3} & \beta_{22}Y_{(3,5),3} & \beta_{23}Y_{(3,5),2} & \beta_{24}Y_{(3,5),1} \\ \beta_{31}Y_{(2,4),3} & \beta_{32}Y_{(2,4),3} & \beta_{33}Y_{(2,4),2} & \beta_{34}Y_{(2,4),1} \\ \beta_{41}Y_{(1,3),3} & \beta_{42}Y_{(1,3),3} & \beta_{43}Y_{(1,3),2} & \beta_{44}Y_{(1,3),1} \\ \beta_{61}Y_{(0,1),3} & \beta_{62}Y_{(0,1),3} & \beta_{63}Y_{(0,1),2} & \beta_{64}Y_{(0,1),1} \end{vmatrix}$$

### 3.4.3 Further Constraints on $M(T)$

For a collection of exponents  $m_{ih}$  to be the exponents of a monomial  $M(T)$  coming from an LR tableau  $T$ , then the  $m_{ih}$  must satisfy further constraints besides the equations (3.7). Recall from §3.1 that the exponent  $m_{ih}$  of the monomial  $M(T)$  is the number of boxes from the  $h$ -th column of  $E^t$  which get put into the  $i$ -th column of  $F^t$  by the inverse of the standard peeling map. From the description of the standard peeling in §2.3, we see that in the filling of  $F^t$ , the  $m$ -th element of column  $i$  of  $E^t$  lies to the northeast, in fact, strictly right and weakly above, the  $m$ -th element of column  $i + 1$ . (Note that the  $m$ -th element of column  $i$  of  $E^t$  is the number  $m$ , since  $E^t$  is banal.) It follows that the number of elements of column  $i$  of  $E^t$  which are assigned to columns  $k + 1$  or greater of  $F^t$  is always at least as large as the number of elements from column  $i + 1$  which are assigned to columns  $k$  or greater of  $F^t$ . We can state these conditions in terms of the numbers  $m_{ih}$ :

$$\sum_{j=k+1}^s m_{ij} \geq \sum_{j=k}^s m_{(i+1)j}.$$

In particular, if  $i(h)$  is the column of  $F^t$  which gets the first entry of column  $h$  of  $E^t$ , that is, column  $i(h)$  is the rightmost column of  $F^t$  to receive elements from column  $h$ , then  $i(h)$  is strictly decreasing with  $h$ .

### 3.4.4 Acquiring Factors $y_{a1}$ 's in the Leading Term in the Coefficient $\delta_{T,Y}$ of $M(T)$

Now fix a tableau  $T$ , and consider the leading term in the coefficient  $\delta_{T,Y}$  of  $M(T)$  in the expansion (3.4) of  $\det \widetilde{Y}_o$ . Since the most important variables in determining the ordering of the term of  $\delta_{T,Y}$  are the  $y_{a1}$ 's, we will focus first on understanding how terms in  $\delta_{T,Y}$  can acquire these variables as factors.

As remarked at the end of §3.4.2, for any  $\beta_{ih}$  which occurs in  $M(T)$  with exponent zero, we may replace the block  $(\widetilde{Y}_o)_{i,h}$  with zeroes without changing  $\delta_{T,Y}$ . We will do this. Call the resulting matrix  $\widetilde{Y}_T$ . Since the first element from the  $(i + 1)$ -th column of  $E^t$  is always placed strictly to the left of the first element of the  $i$ -th column, we see that the matrix  $\widetilde{Y}_T$

will have a block triangular form. More precisely, in the  $h$ -th supercolumn of  $\widetilde{Y}_T$ , the last non-zero block of  $\widetilde{Y}_T$  will be in a strictly lower superrow than the last non-zero block of the  $(h+1)$ -th supercolumn. In the notation of §3.4.3, the last block of the  $h$ -th supercolumn is  $(\widetilde{Y}_T)_{i(h),h}$ . The triangularity of  $\widetilde{Y}_T$  amounts to the fact that  $i(h+1) < i(h)$ . We will call the last non-zero block in a given supercolumn of  $\widetilde{Y}_T$  the “final block” of that supercolumn.

What does this imply about the occurrence of the variables  $y_{a1}$ ’s in the terms in the expansion of  $\det \widetilde{Y}_T$ ? The variables  $y_{a1}$ ’s appear in the first (leftmost) column of a supercolumn. This makes  $s$  columns all together which contain entries  $y_{a1}$ ’s. We will call these columns the “ones columns”. Any term in the standard expansion of  $\det \widetilde{Y}_T$  will select one factor from each ones column, so the sum of all the exponents of the variables  $y_{a1}$ ’s in  $M(T)$  must be  $s$ .

For the leading term, we want to make the indices  $a$  in the factors  $y_{a1}$ ’s as small as possible. We will refer to the ones column of the  $h$ -th supercolumn as the  $h$ -th ones column. Let  $a_h$  be the smallest index among the indices  $a$  making up the  $y_{a1}$ ’s belonging to the  $h$ -th ones column. Then the largest conceivable product of elements chosen one from each ones columns is

$$e_1(T) = \prod_{h=1}^s y_{a_h 1}. \quad (3.8)$$

**Example:** We recall the matrix  $\widetilde{Y}_T$ , in the case for our  $T$  as in §2.3. Notice that the final blocks  $Y_{(0,1],3}$ ,  $Y_{(1,3],3}$ ,  $Y_{(2,4],2}$ ,  $Y_{(3,5],1}$  slopes upwards from left to right. Also since  $E = (3, 3, 2, 1)$ , i.e.,  $s = 4$ , we have four supercolumns with blocks containing  $y_{a1}$ ’s.

$$\det \widetilde{Y}_T = \begin{vmatrix} 0 & 0 & Y_{(3,5],2} & Y_{(3,5],1} \\ 0 & Y_{(3,5],3} & 0 & 0 \\ Y_{(2,4],3} & 0 & Y_{(2,4],2} & 0 \\ Y_{(1,3],3} & Y_{(1,3],3} & 0 & 0 \\ Y_{(0,1],3} & 0 & 0 & 0 \end{vmatrix}$$

To form  $e_1(T)$ , pick  $y_{11}$  from the final block  $Y_{(0,1],3}$  in the first supercolumn, pick  $y_{21}$  from the final block  $Y_{(1,3],3}$  in the second supercolumn, pick  $y_{31}$  from the final block  $Y_{(2,4],2}$  in the third supercolumn and finally pick  $y_{41}$  from the final block  $Y_{(3,5],1}$  in the last supercolumn. Thus  $e_1(T) = y_{11}y_{21}y_{31}y_{41}$ .

In fact we can realize the product  $e_1(T)$  as a factor in elements in the term expansion of  $\det \widetilde{Y}_T$ , and it is not hard to see explicitly how to do this. Indeed, we can explicitly locate one occurrence of  $y_{a_h 1}$  in the  $h$ -th ones column. It is in the top row of the final block of the supercolumn. This follows from the formula (3.2) defining the blocks of  $\widetilde{Y}_o$  (and thus defining the non-zero blocks of  $\det \widetilde{Y}_T$ ). It is clear from this formula that if the  $(i, h)$  block of  $\widetilde{Y}_T$  is non-zero, its ones column contains entries  $y_{a1}$  for  $a$  running from  $d_i + 1$  to  $f_i$ . Since the  $d_i$



decrease as  $i$  increase, it follows that the smallest possible first entry for the  $y_{a1}$  in a given ones column will occur in the final block of the corresponding supercolumn. (It may occur also in blocks above the last one.) Let us call this occurrence of  $y_{a_{h,1}}$  the “reference entry” of the  $h$ -th ones column.

Because of the triangular character of  $\widetilde{Y}_T$ , the final blocks of different supercolumns occur in different superrows. We conclude that the reference entries of different ones columns occur in different rows, and therefore, the product  $M_1(T)$  can occur in the terms of the standard expansion of  $\det \widetilde{Y}_T$ . To prove Lemma 3.2, we will investigate which terms of the expansion of  $\det \widetilde{Y}_T$  will have  $e_1(T)$  as a factor.

### 3.4.5 Factoring $\det \widetilde{Y}_T$ by $e_1(T)$

Several ones columns may have the same variable  $y_{a1}$  as their reference entry. For a given  $a$ , consider the set of  $h$  such that  $a_h = a$ , that is,  $y_{a1} = y_{a_{h,1}}$  is the reference entry of the  $h$ -th ones column. Suppose that there are  $m'_{a1}$  such ones columns. The supercolumns containing these ones columns correspond to columns of  $E^t$  whose first entry gets put in the  $a$ -th row of  $F^t$ . From the definition of the standard peeling and condition (LR1) defining an LR tableau, we see that this will be a consecutive set of columns of  $E^t$ , and that the ones entries of these columns will fill consecutive boxes in the  $a$ -th row of  $F^t$ . These boxes will be a consecutive set of  $m'_{a1}$  boxes in the  $a$ -th row of  $F^t$ ; in fact, they will be the leftmost  $m'_{a1}$  boxes in the  $a$ -th row of  $F^t - D^t$ . The next column of  $F^t - D^t$  to the left of the leftmost of the column of this set will start in a row below the  $a$ -th row of  $F^t$ . As an illustration, look at the entries  $y_{11}$ ,  $y_{21}$ ,  $y_{31}$  and  $y_{41}$  in the fillings of  $F^t - D^t$  in the Example of §3.3.

We will call the first element of the first column of a given block, the  $(1, 1)$  entry of the block. If the  $(1, 1)$  entry of a given block is  $y_{a1}$ , this means that the column of  $F^t - D^t$  corresponding to the superrow of the block starts in the  $a$ -th row of  $F^t$ .

We may conclude from this discussion that the following statements hold.

- (i) The ones columns which have a given variable  $y_{a1}$  as their reference entry form a consecutive set of ones columns, say from  $h_0(a)$  to  $h_1(a)$ . (Hence,  $h_1(a) - h_0(a) + 1 = m'_{a1}$ .)
- (ii) If the supercolumn  $h$ , with  $h_0(a) \leq h < h_1(a)$ , has final block in the  $i$ -th superrow, then supercolumn  $h + 1$  has final block in superrow  $(i - 1)$ .
- (iii) For column  $h$ , with  $h_0(a) \leq h \leq h_1(a)$ , the number of  $(1, 1)$  entries which are equal to  $y_{a1}$  is equal to  $(h_1(a) - h + 1)$ . These are the lowest  $(h_1(a) - h + 1)$   $(1, 1)$  entries on the  $h$ -th superrow.

On the basis of facts (i), (ii), (iii) above, we may make the following conclusions.

- (a) In a supercolumn  $h$  with reference element  $y_{a1}$ , we may add suitable multiples of the first row of the final block to the first rows of the  $(h_1(a) - h)$  blocks above the final block, and make these rows all equal to zero. (This procedure unfolds in the most

orderly fashion if one does the row operations from left to right, that is, beginning with  $h = h_o(a)$  and progressing consecutively to the larger  $h$ 's. Then the first rows of the blocks in the supercolumns with  $y_{a1}$  as reference element lying to the left of the first row of the final block of the  $h$ -row will have been eliminated before one does row operations in the  $h$ -th supercolumn, so these row operations will leave the earlier supercolumns unchanged. Alternatively, one could work backwards, moving to the right from the  $h_1(a)$ -th supercolumn. The row operations in supercolumn  $h$  will then alter the supercolumns to the left, but whatever effects it has can be cleaned up when it comes time to do row operations in those supercolumns.) In particular, these row operations, which will not affect  $\det \widetilde{Y}_T$ , will leave only one entry equal to  $y_{a1}$  in any of these columns, namely, the reference element itself.

- (b) Moreover, all these row operations affect only elements which share a row with one of the reference elements. Hence, they do not affect the cofactor of the reference elements.

After we have finished with the operations of statement (a) above for all values of  $a$  (as the first index of  $y_{a1}$ ), we are left with a matrix such that the only entry in any ones column equal to the reference entry is the reference entry itself. This leads us to the following statement.

**Lemma 3.3.** *With the monomial  $e_1(T)$  as in formula (3.8), we have*

$$\det \widetilde{Y}_T = e_1(T)\Gamma + R,$$

where

- (a)  $\Gamma$  is the cofactor in  $\det \widetilde{Y}_T$  of the reference elements – the determinant of the submatrix of the rows and columns **not** containing the reference elements; and  
(b) all terms of  $R$  are dominated by any term of  $e_1(T)\Gamma$ .

In particular, the leading term of  $\det \widetilde{Y}_T$  is  $e_1(T)$  times the leading term of  $\Gamma$ .

### 3.4.6 The Inductive Step

Let's compare the monomial  $e_1(T)$  of formula (3.8) with the monomial  $e(T)$  of formula (3.5). Observe that the first indices of the variables  $y_{ac}$  which occur in a given superrow label the rows of the column of  $F^t - D^t$  corresponding to that superrow. These first indices are the same for every block in the superrow. In particular, the index  $a_h$  of the reference element in the  $h$ -th ones column is the smallest row in the  $i(h)$ -th column of  $T = F^t - D^t$ . This in turn is where the 1 from the  $h$ -th column of  $E^t$  gets put by the inverse of the standard stripping of  $T$ . This means that the factor of the monomial  $e(T)$  corresponding to the first box of the  $i(h)$ -th column of  $F^t - D^t$  is just  $y_{a_h1}$ . We conclude that  $e_1(T)$  divides  $e(T)$ . On the other hand, we know that  $e_1(T)$  has degree  $s$  equal to the number of columns of  $E^t$ . From our description of the standard stripping, we know that this is the same as the number of boxes of  $T$  containing a 1. Therefore, the monomial  $e_1(T)$  contains exactly all the factors  $y_{a1}$  dividing  $e(T)$ : the quotient  $e(T)/e_1(T)$  is devoid of the variables  $y_{a1}$  – it contains only variables  $y_{ac}$  with  $c \geq 2$ . We may also express this as follows: the monomial  $e_1(T)$  is the

factor of the monomial  $e(T)$  attributable to the boxes of  $T$  filled with a 1, and the quotient  $e(T)/e_1(T)$  is the factor attributable to the boxes filled with the numbers 2 or larger.

Now consider the matrix whose determinant is the factor  $\Gamma$  in the Lemma 3.3. It is the matrix obtained from  $\widetilde{Y}_T$  by removing all the first columns of each supercolumn, and all the first rows of the superrows containing final blocks. We will call this matrix  $(\widetilde{Y}_T)_{/1}$ . It inherits a partitioned structure from  $\widetilde{Y}_T$ . The blocks of  $(\widetilde{Y}_T)_{/1}$  are just the blocks of  $\widetilde{Y}_T$  with the first column removed, and possibly also with the first row removed, if the block shares a row with a final block.

Let  $T_{/1}$  be the subtableau of  $T$  obtained by omitting all the cells of  $T$  which contain a 1. Checking the criteria of §2.3 for  $LR$  tableau, we see that, except for the fact that  $T_{/1}$  contains only entries 2 through  $\ell_0$ , it is an  $LR$  tableau. More precisely, if we subtract 1 from each entry of  $T_{/1}$ , it becomes an  $LR$  tableau.

Now, by comparing the tableau  $T_{/1}$ , the matrix  $(\widetilde{Y}_T)_{/1}$ , and the monomial  $e_{/1}(T) = e(T)/e_1(t)$ , we see that they are parallel to the original triple of tableau  $T$ , matrix  $\widetilde{Y}_T$  and monomial  $e(T)$ . The matrix  $(\widetilde{Y}_T)_{/1}$  and the monomial  $e_{/1}(T)$  are related to the tableau  $T_{/1}$  in the same way that the matrix  $\widetilde{Y}_T$  and the monomial  $e(T)$  are related to the original  $LR$  tableau  $T$ . Further all these relations are consistent with the use only of the numbers 2 through  $\ell_0$  to fill the boxes of  $T$ , rather than starting with the number 1.

From these observations, we conclude that repetition of the reasoning of §3.4.2 to §3.4.5 for  $2, 3, \dots, \ell_0$  proves Lemma 3.2.

### 3.4.7 Proof of Main Result

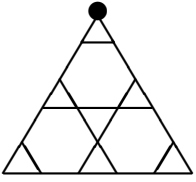
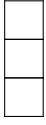

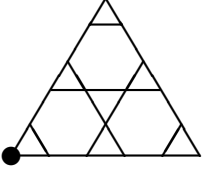

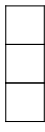

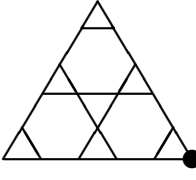


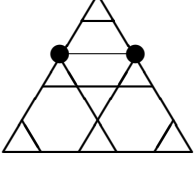


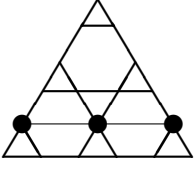


Lemma 3.2 makes it fairly easy to prove the main result of this paper:

**Theorem:** *As the coefficient matrices  $A$  and  $B$  of formulas (2.7) vary through all possible constants, the polynomials  $\widetilde{\Delta}_{(D,E,F),(A,B)}(X,Y)$  of formula (2.11) span the tensor product algebra  $TA_{n,k,\ell}$ . More precisely the polynomials  $\widetilde{\Delta}_{(D,E,F),M(T)}$  of formula (2.13), with the monomial  $M(T)$  given by (3.1), form a basis for  $TA_{n,k,\ell}$ .*

**Proof:** We know from Lemma 2.1 and the formula (2.13) that the polynomials  $\widetilde{\Delta}_{(D,E,F),M(T)}$  are  $GL_n$ -highest weight vectors with weight  $F^t$  in the tensor product of the representations  $\rho_n^{D^t}$  and  $\rho_n^{E^t}$  of  $GL_n$ . From Lemma 3.2, we know that the leading term in  $\widetilde{\Delta}_{(D,E,F),M(T)}$  is the monomial  $e(T)$  of formula (3.5). Lemma 3.1 tells us that  $e(T)$  together with the diagrams  $D$ ,  $E$ , and  $F$  determine  $T$ . It follows that the collection of polynomials  $\widetilde{\Delta}_{(D,E,F),M(T)}$  corresponding to some collection of tableaux  $T$ , is linearly independent. Since we know from [Ful] that  $LR$  tableaux attached to the diagrams  $D$ ,  $E$  and  $F$  count the multiplicity of  $\rho_n^{F^t}$  in  $\rho_n^{D^t} \otimes \rho_n^{E^t}$ , we see that the  $\widetilde{\Delta}_{(D,E,F),M(T)}$  will be a basis for the  $\psi^{F^t} \times \psi^{D^t} \times \psi^{E^t}$  weight space of  $TA_{n,k,\ell}$ . Letting the diagrams  $F$ ,  $E$  and  $D$  vary, the theorem follows.  $\square$



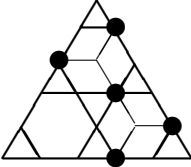


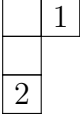
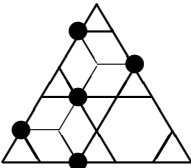
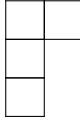


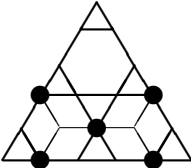

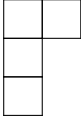

In the following table, we provide the generators of the tensor product algebra for  $SL_4$  using the primitive BZ diagrams. We also provide in the last two columns, key information such as the lead monomial  $e(T)$  in  $\delta_{T,Y}$  (see (3.5)) as well as the monomial  $\mathcal{E}(T) = e(T) \prod_{k=1}^r \left( \prod_{j=1}^{d_k} x_{jj} \right)$  containing the monomial  $e(T)$  in the polynomial  $\tilde{\Delta}_{(D,E,F),M(T)}$  of formula (2.13). Of course, this monomial  $\mathcal{E}(T)$  determines  $D^t$ ,  $E^t$  and  $F^t$  (see proof of Lemma 3.1). The exponents of  $\prod_{k=1}^r \left( \prod_{j=1}^{d_k} x_{jj} \right)$  determines  $D^t$ , while  $e(T)$  encodes the banal tableau of shape  $E^t$  and the filling of the skew tableau  $F^t - D^t$  (see Lemma 3.1).

Generators of $SL_4$ Tensor Product Algebra							
No.	BZ Diagrams	$D^t$	$E^t$	$F^t$	$\Delta_{(D,E,F),M(T)}$	$e(T)$	$\mathcal{E}(T)$
1			$\emptyset$		$ X_{3,3} $	1	$x_{11}x_{22}x_{33}e(T)$
2					$ X_{4,1}Y_{4,3} $	$y_{21}y_{32}y_{43}$	$x_{11}e(T)$
3		$\emptyset$			$ Y_{1,1} $	$y_{11}$	$e(T)$
4			$\emptyset$		$ X_{2,2} $	1	$x_{11}x_{22}e(T)$
5			$\emptyset$		$ X_{1,1} $	1	$x_{11}e(T)$

Generators of $SL_4$ Tensor Product Algebra							
No.	BZ Diagrams	$D^t$	$E^t$	$F^t$	$\Delta_{(D,E,F),M(T)}$	$e(T)$	$\mathcal{E}(T)$
6		$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$ X_{4,2}Y_{4,2} $	$y_{31}y_{42}$	$x_{11}x_{22}e(T)$
7		$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$ X_{4,3}Y_{4,1} $	$y_{41}$	$x_{11}x_{22}x_{33}e(T)$
8		$\emptyset$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$ Y_{2,2} $	$y_{11}y_{22}$	$e(T)$
9		$\emptyset$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$ Y_{3,3} $	$y_{11}y_{22}y_{33}$	$e(T)$
10		$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline X_{4,3} & Y_{4,3} \\ \hline 0 & Y_{23} \\ \hline \end{array}$	$y_{11}y_{22}y_{43}$	$x_{11}x_{22}x_{33}e(T)$

Generators of $SL_4$ Tensor Product Algebra							
No.	BZ Diagrams	$D^t$	$E^t$	$F^t$	$\Delta_{(D,E,F),M(T)}$	$e(T)$	$\mathcal{E}(T)$
11		$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline 1 \\ \hline \end{array}$	$ X_{3,2} Y_{3,1} $	$y_{31}$	$x_{11}x_{22}e(T)$
12		$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \begin{array}{ c } \hline 1 \\ \hline \end{array}$	$\begin{vmatrix} X_{4,2} & Y_{4,3} \\ 0 & Y_{13} \end{vmatrix}$	$y_{11}y_{32}y_{43}$	$x_{11}x_{22}e(T)$
13		$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline 1 \\ \hline 2 \\ \hline \end{array}$	$ X_{3,1} Y_{3,2} $	$y_{21}y_{32}$	$x_{11}e(T)$
14		$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline 2 \\ \hline \end{array} \begin{array}{ c } \hline 1 \\ \hline \end{array}$	$\begin{vmatrix} X_{4,3} & Y_{4,2} \\ 0 & Y_{1,2} \end{vmatrix}$	$y_{11}y_{42}$	$x_{11}x_{22}x_{33}e(T)$
15		$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline 1 \\ \hline \end{array}$	$ X_{2,1} Y_{2,1} $	$y_{21}$	$x_{11}e(T)$



Generators of $SL_4$ Tensor Product Algebra							
No.	BZ Diagrams	$D^t$	$E^t$	$F^t$	$\Delta_{(D,E,F),M(T)}$	$e(T)$	$\mathcal{E}(T)$
16					$\begin{vmatrix} X_{3,2} & Y_{3,2} \\ 0 & Y_{1,2} \end{vmatrix}$	$y_{11}y_{32}$	$x_{11}x_{22}e(T)$
17					$\begin{vmatrix} X_{4,3} & 0 & Y_{4,2} \\ 0 & X_{2,1} & Y_{2,2} \end{vmatrix}$	$y_{21}y_{42}$	$x_{11}^2 x_{22} x_{32} e(T)$
18					$\begin{vmatrix} X_{4,2} & Y_{4,3} & Y_{4,1} \\ 0 & Y_{2,3} & 0 \end{vmatrix}$	$y_{11}y_{22}y_{31}y_{43}$	$x_{11}x_{22}e(T)$

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