Hilbert series, Howe duality and branching rules

Thomas J. Enright and Jeb F. Willenbring *

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Abstract

Let λ be a partition, with l parts, and let F^λ be the irreducible finite dimensional representation of $GL(m)$ associated to λ when $l \leq m$. The Littlewood Restriction Rule describes how F^λ decomposes when restricted to the orthogonal group $O(m)$ or to the symplectic group $Sp(\frac{m}{2})$ under the condition that $l \leq \frac{m}{2}$. In this paper, this result is extended to all partitions λ. Our method combines resolutions of unitary highest weight modules by generalized Verma modules with reciprocity laws from the theory of dual pairs in classical invariant theory.

Corollaries include determination of the Gelfand-Kirillov dimension of any unitary highest weight representation occurring in a dual pair setting, and the determination of their Hilbert series (as a graded module for $\mathfrak{p}^-$). Let L be a unitary highest weight representation of either $\mathfrak{sp}(n, R)$, $\mathfrak{so}^*(2n)$ or $\mathfrak{u}(p, q)$. When the highest weight of L plus ρ satisfies a partial dominance condition called quasi-dominance, we associate to L a reductive Lie algebra $\mathfrak{g}_L$ and a graded finite dimensional representation B_L of $\mathfrak{g}_L$. The representation B_L will have a Hilbert series $P(q)$ which is a polynomial in q with positive integer coefficients.Let $\delta(L) = \delta$ be the Gelfand-Kirillov dimension of L and set c_L equal to the ratio of the dimensions of the zeroth levels in the gradings of L and B_L. Then the Hilbert series of L may be expressed in the form

$$H_L(q) = c_L \frac{P(q)}{(1-q)^\delta} .$$

In the easiest example of the correspondence $L \to B_L$, the two components of the Weil representation of the symplectic group correspond to the two spin representations of an orthogonal group.

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Extending the Littlewood Restriction Rule (1) Let V be a complex vector space of dimension n with a nondegenerate symmetric or skew symmetric form. The group leaving the form invariant is the group G, the orthogonal group $O(n)$ or the symplectic group $Sp(\frac{n}{2})$ when n is even. The representations F^λ of $GL(V)$ are parameterized by the partitions λ with at most n parts. For any integer partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ with at most l parts, let $F^\lambda_{(l)}$ be the irreducible representation of $GL(l)$ indexed in the usual way by its highest weight (see [1] section 5.2.1). Similarly, for each nonnegative integer partition μ with at most l parts, let $V^\mu_{(l)}$ be the irreducible representation of $Sp(l)$ with highest weight μ. Let $E^\nu_{(l)}$ denote the irreducible representation of $O(l)$ associated to the non-negative integer partition ν with at most l parts and having Young diagram whose first two columns have lengths which sum to l or less. The parameters in the $O(l)$ case are somewhat more delicate than the earlier two cases due in part to the disconnectedness of the group. For more details consult ([1], Chapter 10).

In 1940 D.E. Littlewood gave a formula for the decomposition of F^λ as a representation of G by restriction.

Theorem 1 (Littlewood Restriction [2, 3]) *Suppose that λ is a partition having at most $\frac{n}{2}$ (positive) parts.*

(i) Suppose n is even and set $k = \frac{n}{2}$.
Then the multiplicity of the $Sp(k)$ representation V^μ in F^λ equals

$$\sum_\xi dimHom_{GL(n)}(F^\lambda, F^\mu \otimes F^\xi), \tag{1.1}$$

where the sum is over all nonnegative integer partitions ξ with columns of even length.

(ii) Then, for all n the multiplicity of the $O(n)$ representation E^ν in F^λ equals

$$\sum_\xi dimHom_{GL(n)}(F^\lambda, F^\nu \otimes F^\xi), \tag{1.2}$$

where the sum is over all nonnegative integer partitions ξ with rows of even length.

Recently Gavarini [4] used Brauer algebra methods to reprove and slightly extend Littlewood's formula. In this announcement we describe some new results in character theory and an interpretation of these results through Howe duality. This will yield yet another proof of Littlewood Restriction and more importantly a generalization valid for all parameters λ. Our formula is rather more complex in form (see (5) below) but reduces to the above formula when λ is a partition having at most $\frac{n}{2}$ (positive) parts.

Gelfand-Kirillov Dimension (2) The character theory mentioned above is for unitarizable highest weight representations of a real reductive group. Much is known about these representations. They were classified in 1980 [5] and

[6]. Their characters and cohomology formulas were determined a decade later [7, 8, 9]. Here we report some refinements to this character theory and focus on a set of coarser invariants which can be determined more precisely. These are the Hilbert series, Gelfand Kirillov dimension and Bernstein degree.

Let L denote a unitarizable highest weight representation for $\mathfrak{g}$, one of the classical Lie algebras $\mathfrak{su}(p,q)$, $\mathfrak{sp}(n,\mathbb{R})$ or $\mathfrak{so}^*(2n)$. These Lie algebras occur as part of the reductive dual pairs [10]:

$$\begin{array}{lll} \text{(i)} & Sp(k)\times\mathfrak{so}^*(2n) \text{ acting on } \mathcal{P}(M^*_{2k\times n}) \\ \text{(ii)} & O(k)\times\mathfrak{sp}(n) \text{ acting on } \mathcal{P}(M^*_{k\times n}) \\ \text{(iii)} & U(k)\times\mathfrak{u}(p,q) \text{ acting on } \mathcal{P}(M^*_{k\times n}), \end{array} \tag{2.1}$$

where $n = p + q$. Let $GKdim$ denote the Gelfand Kirillov dimension. Our first main result is:

Theorem 2 *Suppose that L is a unitarizable highest weight representation occurring in one of the dual pair settings (2.1).*

1. *If $\mathfrak{g}$ is $\mathfrak{so}^*(2n)$, then $GKdim(L)$ equals $k(2n-2k-1)$ for $1 \le k \le [\frac{n-2}{2}]$ and equals $\binom{n}{2}$ otherwise.*

2. *If $\mathfrak{g}$ is $\mathfrak{sp}(n)$, then $GKdim(L)$ equals $\frac{k}{2}(2n-k+1)$ for $1 \le k \le n-1$ and equals $\binom{n+1}{2}$ otherwise.*

3. *If $\mathfrak{g}$ is $\mathfrak{u}(p,q)$, then $GKdim(L)$ equals $k(n-k)$ for $1 \le k \le min\{p,q\}$ and equals pq otherwise.*

Note that in all cases the GK dimension depends only on the dual pair setting given by k and n and is independent of λ otherwise. It is of course convenient to compute GK dimension of L directly from the highest weight. Let β denote the maximal root of $\mathfrak{g}$.

Corollary 3 *Set $s = -\frac{2(\lambda,\beta)}{(\beta,\beta)}$. Then for $\mathfrak{so}^*(2n)$,*

$$GKdim\ L = \begin{cases} \frac{s}{2}(2n-s-1) & \textit{for } 2 \le s \le 2[\frac{n}{2}]-2 \\ \binom{n}{2} & \textit{otherwise}\ , \end{cases}$$

for $\mathfrak{sp}(n)$,

$$GKdim\ L = \begin{cases} s(2n-2s+1) & \textit{for } 1 \le 2s \le n \\ \binom{n+1}{2} & \textit{otherwise} \end{cases}$$

and for $\mathfrak{u}(p,q)$ with $n = p+q$,

$$GKdim\ L = \begin{cases} s(n-s) & \textit{for } 1 \le s \le min\{p,q\} \\ pq & \textit{otherwise}\ . \end{cases}$$

The referee has noted that Theorem 2 is known to experts.

Hilbert series and examples (3) For any Hermitian symmetric pair $\mathfrak{g}, \mathfrak{k}$, let $\mathfrak{h}$ be a Cartan subalgebra of both $\mathfrak{k}$ and $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ be a $\mathfrak{k}$-module decomposition of $\mathfrak{g}$. Choose Δ^+ a set of positive roots for the root system Δ such that $\mathfrak{p}^+$ is the span of the positive noncompact root spaces. Set ρ equal to half the sum of all the positive roots and let $\mathcal{W}$ (resp.$\mathcal{W}_\mathfrak{k}$) denote the Weyl group of $\mathfrak{g}$ (resp. $\mathfrak{k}$).

Suppose L is a unitarizable highest weight representation with highest weight λ. In [9] one associates to λ a subgroup $\mathcal{W}_\lambda$ of $\mathcal{W}$, a root system Δ_λ and a Hermitian symmetric pair $\mathfrak{g}_\lambda, \mathfrak{k}_\lambda$ as follows. Let Δ_n^+ denote the roots with root spaces in $\mathfrak{p}^+$. For any root α let $\alpha^\vee$ denote the coroot defined by $(\alpha^\vee, \xi) = \frac{2(\alpha,\xi)}{(\alpha,\alpha)}$. Suppose λ is $\mathfrak{k}$-dominant and define $\mathcal{W}_\lambda$ to be the subgroup of the Weyl group $\mathcal{W}$ generated by the identity and all the reflections r_α which satisfy the following three conditions:

$$\begin{array}{ll} (i) & \alpha \in \Delta_n^+ \text{ and } (\lambda+\rho, \alpha^\vee) \in \mathbb{N}^* \quad . \\ (ii) & \text{If } \delta \in \Delta \text{ and } (\lambda+\rho, \delta) = 0 \text{ then } (\alpha, \delta) = 0. \\ (iii) & \text{If } \delta \in \Delta \text{ is long and } (\lambda+\rho, \delta) = 0 \text{ then } \alpha \text{ is short} . \end{array} \tag{3.1}$$

When the root system Δ contains only one root length we call the roots short. Let Δ_λ equal the subset of Δ of elements δ for which $r_\delta \in \mathcal{W}_\lambda$ and let $\Delta_{\lambda,\mathfrak{k}} = \Delta_\lambda \cap \Delta_\mathfrak{k}$, $\Delta_\lambda^+ = \Delta_\lambda \cap \Delta^+$ and $\Delta_{\lambda,\mathfrak{k}}^+ = \Delta_{\lambda,\mathfrak{k}} \cap \Delta^+$. Then in our setting Δ_λ and $\Delta_{\lambda,\mathfrak{k}}$ are abstract root systems and we let $\mathfrak{g}_\lambda$ (resp. $\mathfrak{k}_\lambda$) denote the reductive Lie algebra with root system Δ_λ (resp. $\Delta_{\mathfrak{k},\lambda}$) and Cartan subalgebra equal to $\mathfrak{h}$. Then the pair $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$ is a Hermitian symmetric pair (although not necessarily of the same type as $(\mathfrak{g}, \mathfrak{k})$ as is indicated by the example of the Weil representation below in (4) where the first pair is (D_n, A_{n-1}) and the second is (C_n, A_{n-1})). Set $\mathcal{W}_{\lambda,\mathfrak{k}} = \mathcal{W}_\lambda \cap \mathcal{W}_\mathfrak{k}$ and define:

$$\begin{array}{l} \mathcal{W}_\lambda^\mathfrak{k} = \{x \in \mathcal{W}_\lambda | x\Delta_\lambda^+ \supset \Delta_{\lambda,\mathfrak{k}}^+\} \text{ and} \\ \mathcal{W}_\lambda^{\mathfrak{k},i} = \{x \in \mathcal{W}_\lambda^\mathfrak{k} | card(x\Delta_\lambda^+ \cap -\Delta_n^+) = i\} \ . \end{array} \tag{3.2}$$

Let ρ_λ denote half the sum of the positive roots of $\mathfrak{g}_\lambda$ and set $\delta_\lambda = \rho - \rho_\lambda$. For any $\mathfrak{k}$-integral $\xi \in \mathfrak{h}^*$, let ξ^{++} denote the unique element in the $\mathcal{W}_\mathfrak{k}$-orbit of ξ which is $\Delta_\mathfrak{k}^+$ dominant. For any $\mathfrak{k}$-dominant integral weight λ define the generalized Verma module with highest weight λ to be the induced module defined by: $N(\lambda+\rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}\oplus\mathfrak{p}^+)} E_\lambda$, with E_λ is the irreducible $\mathfrak{k}$ representation with highest weight λ. For any $\mathfrak{k}$-dominant integral weights ξ and ζ, let m_ζ^ξ denote the multiplicity of the $\mathfrak{k}$ representation E_ξ in the $\mathfrak{g}$ generalized Verma module $N(\zeta+\rho)$. If $S(\mathfrak{p}^-)$ denotes the symmetric tensor algebra of $\mathfrak{p}^-$, then:

$$m_\zeta^\xi = \dim Hom_\mathfrak{k}(E_\xi, E_\zeta \otimes S(\mathfrak{p}^-)). \tag{3.3}$$

We let $L(\lambda+\rho)$ denote the unique irreducible quotient of $N(\lambda+\rho)$, which is the irreducible highest weight representation of $\mathfrak{g}$ with highest weight λ.

Remark 1 *We have used three separate notations which overlap in the case of finite dimensional representations. The predominant notation for irreducible finite dimensional representations is by highest weight. However in some classical* *settings we switch to coordinate versions of highest weights which are then identified with integer partitions. Finally when we are considering highest weight* $\mathfrak{g}$*-modules we use the generalized Verma module parameters of highest weight plus* ρ*.*

Suppose λ is $\Delta_{\mathfrak{k}}^{+}$-dominant and let Σ_λ denote the set of all positive roots α where $(\alpha, \lambda+\rho) = 0$. Then Σ_λ is a set of strongly orthogonal noncompact roots. For such λ, we say $\lambda+\rho$ is *quasi-dominant* if $(\alpha, \lambda+\rho) > 0$ for all positive roots α which are orthogonal to all elements of Σ_λ.

For any Hermitian symmetric pair $\mathfrak{g}, \mathfrak{k}$ and irreducible highest weight $\mathfrak{g}$-module M, let M_0 denote the $\mathfrak{k}$-submodule generated by any highest weight vector. Set $M_j = \mathfrak{p}^- \cdot M_{j-1}$ for $j > 0$. Define the Hilbert series $H_M(q)$ of M by:

$$H_M(q) = \sum_{j \geq 0} \dim M_j \; q^j. \tag{3.4}$$

Since the enveloping algebra of $\mathfrak{p}^-$ is Noetherian there is a unique integer d and a unique polynomial $R_M(q)$ such that:

$$H_M(q) = \frac{R_M(q)}{(1-q)^d} \quad \text{where} \quad R_M(q) = \sum_{0 \leq j \leq e} a_j \; q^j \tag{3.5}$$

In this setting the integer d is the Gelfand Kirillov dimension, $d = GKdimM$. Expanding the denominator as a series we obtain:

$$H_M(q) = \sum_{s,j} a_j \binom{d-1+s}{d-1} \; q^{s+j} \; . \tag{3.6}$$

The coefficient of q^t can be expressed as a polynomial in t for large values of t. The leading coefficient of this polynomial is $\frac{1}{(d-1)!} \sum_{0 \leq j \leq n} a_j = \frac{R_m(1)}{(d-1)!}$ and is called the Bernstein degree of M and denoted $Bdeg(M)$.

For any $\mathfrak{g}_\lambda$-dominant integral μ we let B_μ denote the irreducible finite dimensional $\mathfrak{g}_\lambda$-module with highest weight μ. Let $\mathfrak{p}_\lambda^-$ denote the subalgebra of $\mathfrak{p}^-$ spanned by the root spaces for roots in $-\Delta_n^+ \cap \Delta_\lambda$ and let B_μ^i denote the grading of B_μ as a $\mathfrak{p}_\lambda^-$-module. Define the Hilbert series by :

$$P(q) = P_\lambda(q) = \sum dim \; B_\lambda^i \; q^i \; . \tag{3.7}$$

This is of course a polynomial. For any $\mathfrak{k}$-dominant integral weight μ, let E_μ (resp. $E_{\mathfrak{k}_\lambda, \mu}$) denote the irreducible finite dimensional $\mathfrak{k}$ (resp. $\mathfrak{k}_\lambda$) -module with highest weight μ.

Theorem 4 *Suppose that $L = L(\lambda + \rho)$ is a unitarizable representation occurring in one of the dual pair settings (2.1) and that $\lambda + \rho$ is quasi-dominant. Set d equal to the GK dimension of L as given by Theorem 2. Then the Hilbert series of L is:*

$$H_L(q) = \frac{dim\ E_\lambda}{\dim E_{\mathfrak{k}_\lambda,\lambda+\delta_\lambda}} \frac{P_{\lambda+\delta_\lambda}(q)}{(1-q)^d} \ . \tag{3.8}$$

In particular, the Bernstein degree of L is given by:

$$Bdeg(L) = \frac{dim\ E_\lambda}{dim\ E_{\mathfrak{k}_\lambda,\lambda+\delta_\lambda}}\ dim B_{\lambda+\delta_\lambda}. \tag{3.9}$$

To illustrate through an example all the parameters, roots, Hermitian symmetric pairs and finite dimensional representations involved we consider a special set of representations, the Wallach representations (see [12]). Let r equal the split rank of $\mathfrak{g}$, let ζ be the fundamental weight of $\mathfrak{g}$ which is orthogonal to all the roots of $\mathfrak{k}$. Suppose $\mathfrak{g}$ is isomorphic to either $\mathfrak{so}^*(2n)$, $\mathfrak{sp}(n)$ or $\mathfrak{su}(p,q)$ and set $c = 2$, $\frac{1}{2}$ or 1 depending on which of the three cases we are in. For $1 \le j < r$ define the j^{th} Wallach representation W_j to be the unitarizable highest weight representation with highest weight $-jc\zeta$.

Suppose L is a Wallach representation for $\mathfrak{sp}(n)$ having highest weight $\lambda = -\frac{j}{2}(1,1,\cdots,1)$ for some $j, 1 \le j \le n-1$. Then

$$\lambda + \rho = (n - \frac{j}{2}, n-1-\frac{j}{2}, \cdots, 1-\frac{j}{2}) \ ,$$

the set of strongly orthogonal roots

$$\Sigma_\lambda = \{e_{n+1-j+l} + e_{n+1-l} | 1 \le l \le [\frac{j}{2}]\}$$

and Δ_λ is the root system of type D_{n+1-j} with simple roots

$$\{e_1 - e_2, \cdots, e_{n-j} - e_{n+1-j}, , e_{n-j} + e_{n+1-j}\} \ ,$$

and the Hermitian symmetric pair $\mathfrak{g}_\lambda, \mathfrak{k}_\lambda$ is $\mathfrak{so}^*(2n+2-2j)$. Here Σ_λ contains a long root precisely when j is even and for all j, $\lambda+\rho$ is quasi-dominant. The first $n+1-j$ coordinates of $\lambda+\rho-\rho_\lambda$ all equal $\frac{j}{2}$ and so this weight $\lambda+\rho-\rho_\lambda$ equals j times the last fundamental weight of D_{n+1-j} on $[\mathfrak{g}_\lambda, \mathfrak{g}_\lambda] \cap \mathfrak{h}$ plus a central character of $\mathfrak{g}_\lambda$. Also in this example the dimensions of E_λ and $E_{\mathfrak{k}_\lambda,\lambda+\delta_\lambda}$ are both one.

We next apply Theorem 4 to some well-known representations to determine their $GKdim$, Hilbert series and Bernstein degree. We begin with the Wallach representations. For $\mathfrak{so}^*(2n)$ the Hilbert series for the first Wallach representation is:

$$H_L(q) = \frac{R(q)}{(1-q)^{2n-3}} = \tag{4.1}$$

$$\frac{1}{(1-q)^{2n-3}} \frac{1}{n-2} \sum_{0 \le s \le n-3} \binom{n-2}{s} \binom{n-2}{s+1} q^s \ . \tag{4.2}$$

For $\mathfrak{sp}(n)$ the Hilbert series for the first Wallach representation is:

$$H_L(q) = \frac{1}{(1-q)^n} \sum_{0 \le t \le [\frac{n}{2}]} \binom{n}{2t} q^t. \tag{4.3}$$

This is the Hilbert series for the half of the Weil representation generated by a one dimensional representation of $\mathfrak{k}$. The other part of the Weil representation has Hilbert series:

$$H_L(q) = \frac{1}{(1-q)^n} \sum_{0 \le t \le [\frac{n}{2}]} \binom{n}{2t+1} q^t. \tag{4.4}$$

For the components of the Weil representation of $\mathfrak{sp}(n)$ there is an elementary derivation of the Hilbert series which we now give for comparison. Let L_e and L_o denote the even and odd components of the Weil representation acting respectively on the even and odd degree polynomials in n variables. Then

$$\frac{(1+q)^n}{(1-q^2)^n} = \frac{1}{(1-q)^n} = H_{L_e}(q^2) + q \ H_{L_o}(q^2) \ . \tag{4.5}$$

Breaking the numerator $(1+q)^n$ into its even and odd degree parts in q gives the expressions (4.2) and (4.3).

For $U(p,q)$ the Hilbert series for the first Wallach representation is:

$$H_L(q) = \frac{1}{(1-q)^{n-1}} \sum_{0 \le t < min\{p,q\}} \binom{p-1}{t} \binom{q-1}{t} q^t \ . \tag{4.6}$$

These examples are obtained from Corollary 3 and Theorem 4 by writing out respectively the Hilbert series of the $n-3^{rd}$ exterior power of the standard representation of $\mathfrak{so}^*(2n-4)$, the two components of the spin representation of $\mathfrak{so}^*(2n)$ and the $p-1^{st}$ fundamental representation of $U(p-1,q-1)$. Combinatorial expressions for the Hilbert series of other Wallach representations can be given as well.

We say a highest weight representation L is positive if all the nonzero coefficients of the polynomial $R_L(q)$ in (3.5) are positive. All Cohen-Macaulay $S(\mathfrak{p}^-)$-modules including the Wallach representations are positive but many unitary highest weight representations are not. And so Theorem 4 introduces a large class of positive highest weight representations which include all the examples in the previous paragraph. This theorem asserts that if $\lambda+\rho$ is quasi-dominant then $L(\lambda+\rho)$ is positive.

We conjecture that among positive highest weight modules occurring in the dual pair setting which are non-trivial quotients of generalized Verma modules, the modules with quasi-dominant highest weights prevail. However, the conditions of positivity and quasi-dominance are not equivalent. The former properly contain the latter. There are irreducible positive generalized Verma modules with highest weights which are not quasi-dominant. We would like to know to what extent the conditions of positivity and quasi-dominance are related.

Reciprocity laws and resolutions (5) The refinements in the character theory for unitarizable highest weight representations discussed above lead to new finite dimensional branching formulas extending the work of Littlewood [2]. Let $\mathcal{S}=\mathcal{P}(M^*_{2k\times n})$ or $\mathcal{P}(M^*_{k\times n})$ as in (2.1). We consider the action of two dual pairs on $\mathcal{S}$. The first is $GL(m)\times GL(n)$ with $m=2k$ or k and the second is $G_1\times\mathfrak{g}_2$, one of the two pairs (i) or (ii) in (2.1). In this setting G_1 is contained in $GL(m)$ while $GL(n)$ is closely related to the maximal compact subgroup of $\mathfrak{g}_2$. We can calculate the multiplicity of an irreducible $G_1\times GL(n)$ representation in $\mathcal{S}$ in two ways. The resulting identity is the branching formula. These reciprocity laws have been studied by Howe ([11]) where the two dual pairs are called see-saw pairs in the sense of Kudla.

The theory of dual pairs gives a decompositions of $\mathcal{S}$ for each of the three actions.
As a $GL(m)\times GL(n)$ representation,

$$\mathcal{S}=\sum_{\lambda} F^{\lambda}_{(m)}\otimes F^{\lambda}_{(n)}, \tag{5.1}$$

where the sum is over all nonnegative integer partitions having $min\{m,n\}$ or fewer parts.
As a $Sp(k)\times\mathfrak{so}^*(2n)$ representation,

$$\mathcal{S}=\sum_{\mu} V^{\mu}_{(k)}\otimes V^{(n)}_{\mu}, \tag{5.2}$$

where the sum is over all nonnegative integer partitions μ having $min\{k,n\}$ or fewer parts.
And, as a $O(k)\times\mathfrak{sp}(n)$ representation,

$$\mathcal{S}=\sum_{\nu} E^{\nu}_{(k)}\otimes E^{(n)}_{\nu}, \tag{5.3}$$

where the sum is over all nonnegative integer partitions ν having $min\{k,n\}$ or fewer parts and having a Young diagram whose first two columns sum to k or less.

As remarked above computing the multiplicity of $V^{\mu}\otimes F^{\lambda}_{(n)}$ in $\mathcal{S}$ and $E^{\nu}\otimes F^{\lambda}_{(n)}$ in $\mathcal{S}$ we obtain:

Theorem 5 *(i) The multiplicity of the $Sp(k)$ representation $V^{\mu}_{(k)}$ in $F^{\lambda}_{(2k)}$ equals the multiplicity of $F^{\lambda}_{(n)}$ in the unitarizable highest weight representation $V^{(n)}_{\mu}$ of $\mathfrak{so}^*(2n)$.*

(ii) The multiplicity of the $O(k)$ representation $E^{\nu}_{(k)}$ in $F^{\lambda}_{(k)}$ equals the multiplicity of $F^{\lambda}_{(n)}$ in the unitarizable highest weight representation $E^{(n)}_{\nu}$ of $\mathfrak{sp}(n)$.

In the cases where the unitarizable highest weight representation is the full generalized Verma module we call the parameter a generic point. A short calculation shows that the Littlewood hypothesis is included in the generic set.

Corollary 6 *(i) Suppose μ is a generic point. Then the multiplicity of the $Sp(k)$ representation $V^{\mu}_{(k)}$ in $F^{\sigma}_{(2k)}$ equals*

$$\sum_{\xi} dimHom_{GL(n)}(F^{\sigma}_{(n)}, F^{\mu}_{(n)} \otimes F^{\xi}_{(n)}), \tag{5.4}$$

where the sum is over all nonnegative integer partitions ξ with columns of even length.

(ii) Suppose ν is a generic point. Then the multiplicity of the $O(k)$ representation $E^{\nu}_{(k)}$ in $F^{\sigma}_{(k)}$ equals

$$\sum_{\xi} dimHom_{GL(n)}(F^{\sigma}_{(n)}, F^{\nu}_{(n)} \otimes F^{\xi}_{(n)}), \tag{5.5}$$

where the sum is over all nonnegative integer partitions ξ with rows of even length.

The formulas (5.4) and (5.5) are those of the Littlewood Restriction Theorem ([2], p.240). Littlewood's hypotheses correspond to the unitarizable highest weight representations lying in the discrete series or in some cases the limits of the discrete series.

For a general unitarizable highest weight representation L from [9] we know the $\mathfrak{p}^+$-cohomology formulas for L. From this data we obtain explicit resolutions for L where the terms of this resolution are sums of generalized Verma modules.

Theorem 7 *Suppose $L = L(\lambda + \rho)$ is a unitarizable highest weight module. Then L admits a resolution in terms of generalized Verma modules. Specifically, for $1 \leq i \leq r_\lambda = card(\Delta_\lambda \cap \Delta_{n^+})$, set $\mathbf{C}^{\lambda}_{i} = \sum_{x \in \mathcal{W}^{\mathfrak{k},i}_{\lambda}} N((x(\lambda+\rho))^{++})$. Then there is a resolution of L:*

$$0 \to \mathbf{C}^{\lambda}_{r_\lambda} \to \cdots \to \mathbf{C}^{\lambda}_{1} \to \mathbf{C}^{\lambda}_{0} \to L \to 0\ . \tag{5.6}$$

Combining Theorems 5 and 7 we obtain:

Corollary 8 *Suppose that $F^\sigma \otimes F^\sigma$ occurs in $\mathcal{S}$ as in (5.1) and that $V^\mu_{(k)}$ and $E^\nu_{(k)}$ respectively occur as $Sp(k)$ and $O(k)$ subrepresentations of F^σ. Then*

(i) the multiplicity of the $Sp(k)$ representation $V^\mu_{(k)}$ in $F^\sigma_{(2k)}$ equals the multiplicity of $F^\sigma_{(n)}$ in $V^{(n)}_\mu$ which is given by:

$$\sum_i \sum_{x \in \mathcal{W}^{\mathfrak{k},i}_{\mu^\sharp}} (-1)^i \; m^{\sigma^\sharp}_{x(\mu^\sharp+\rho)^{++}-\rho} \quad , \tag{5.7}$$

where $\mu^\sharp$ (resp. $\sigma^\sharp$) is the highest weight of $V^{(n)}_\mu$ (resp. $F^\sigma_{(n)}$),

(ii) and the multiplicity of the $O(k)$ representation $E^\nu_{(k)}$ in $F^\sigma_{(k)}$ equals the multiplicity of $F^\sigma_{(n)}$ in $E^{(n)}_\nu$ which is given by:

$$\sum_i \sum_{x \in \mathcal{W}^{\mathfrak{k},i}_{\nu^\sharp}} (-1)^i \; m^{\sigma^\sharp}_{x(\nu^\sharp+\rho)^{++}-\rho} \; , \tag{5.8}$$

where $\nu^\sharp$ (resp. $\sigma^\sharp$) is the highest weight of $E^{(n)}_\nu$ (resp. $F^\sigma_{(n)}$).

The above corollary provides an algorithm for decomposing finite dimensional $GL(n)$ representations into irreducible orthogonal or symplectic representations. Such algorithms have been implemented using the MAPLE software package.

As an example of this multiplicity formula we list some cases for the orthogonal group.

Example 9 *Let ν be a partition such that,*

$$\nu = \underbrace{(d, \; \overbrace{2,\cdots,2,}^{a} \; \overbrace{1,\cdots,1,}^{b} \; \overbrace{0,\cdots,0}^{c} \;)}_{k}$$

with: $d \geq 2$ and $2+2a+b \leq k$.
Then for all σ,

$$dim \; Hom_{O(k)}(E^\nu, F^\sigma) = m^{\sigma^\sharp}_{\nu^\sharp} - m^{\sigma^\sharp}_{\nu^\sharp_1}$$

Where ν_1 is the k tuple given by:

$$\nu_1 = (d, \; \overbrace{2,\cdots,2,}^{c} \; \overbrace{1,\cdots,1,}^{b} \; \overbrace{0,\cdots,0}^{a} \;).$$

Final remarks (6) Theorem 7 is the technical backbone of the article. In addition to giving the extension to all parameters of Littlewood Restriction, the proof of Theorem 4 begins with two applications of Theorem 7, one for $\mathfrak{g}$ and one for $\mathfrak{g}_\lambda$. The argument then passes to the computation of a ratio using the hypothesis of quasi-dominance.

During the time this announcement has been refereed, there has been some related research which has appeared [13]. In this work the authors begin with a highest weight module L and then consider the associated variety $\mathcal{V}(L)$ as defined by Vogan. This variety is the union of $K_{\mathbb{C}}$–orbits and equals the closure of a single orbit. In [13] the GK dimension and the Bernstein degree of L are recovered from the corresponding objects for the variety $\mathcal{V}(L)$. As an example of their technique they obtain the GK dimension and the degree of the Wallach representations ([13] pp.149-150). Our results in this setting obtain these two invariants as well as the full Hilbert series since all the highest weights are quasi-dominant. The results of these two very different approaches have substantial overlap although neither subsumes the other.

References

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Contact information:
Thomas J. Enright
Dept. of Mathematics, 0112
University of California, San Diego
La Jolla, CA 92093-0112
E-mail: tenright@euclid.ucsd.edu

Jeb F. Willenbring
Mathematics Department
Yale University
New Haven, CT 06520-8283
E-mail: willenbring@math.yale.edu