TORIC DEGENERATION OF BRANCHING ALGEBRAS

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ABSTRACT. For each classical symmetric pair (G, H), there is a naturally defined multi-graded algebra $\mathfrak{A}_{(G,H)}$, called the branching algebra for (G, H), which encodes the branching rule from G to H. This algebra has a natural family of subalgebras, depending on integer parameters. For a certain range of the parameters, the subalgebras have a particularly simple structure and are called *stable branching algebras*.

In this paper, we show that the stable branching algebras for eight out of the ten families of classical symmetric pairs are flat deformations of the semigroup algebras of explicitly described lattice cones.

1. INTRODUCTION

This paper continues the program of [HTW1] to study branching rules of classical symmetric pairs using methods from commutative algebra.

Following Zhelobenko ([Zh]), the paper [HTW1] attaches to each pair (G, H) consisting of a reductive complex algebraic group G and a reductive subgroup H, a multi-graded algebra $\mathfrak{A}_{(G,H)}$ which encodes the branching rule from G to H - that is, it provides a description of the decomposition of any irreducible representation of G into irreducible representations of H. We call $\mathfrak{A}_{(G,H)}$ the *branching algebra* for (G, H). For the classical symmetric pairs (G, H), the paper [HTW1] also provides an explicit description of a natural family of subalgebras $\mathfrak{A}_{(G,H,\alpha)}$, where α is an appropriate discrete parameter, of $\mathfrak{A}_{(G,H)}$ in terms of classical invariant theory. It defines a notion of *stable* for these algebras, and gives a condition on α for the algebra $\mathfrak{A}_{(G,H,\alpha)}$ to be stable. The paper [HTW2] uses these constructions to give a unified, conceptual derivation of formulas for multiplicities of irreducible representations of H in irreducible representations of G in terms of Littlewood-Richardson coefficients. The papers [HTW3], [HL1], [HL2] and [HL3] provide explicit vector space bases for the stable branching algebras $\mathfrak{A}_{(G,H,\alpha)}$ for eight of the ten families of classical symmetric pairs.

The main result of this paper is that the stable branching algebras (at least, the ones for which bases have been constructed) are flat deformations of semigroup algebras ([EH], [BH] [Fu1]). More precisely, they are flat deformations of the semigroup algebras of *lattice cones*, by which we mean the intersection of a lattice in $\mathbb{R}^n$ with a convex polyhedral cone, rational with respect to the lattice.

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Alternatively, we say that semigroup algebras of lattice cones are *toric degenerations* of branching algebras. A *toric variety* is an algebraic variety which admits a torus action with a dense orbit ([Fu1]). If X is an affine toric variety, then by diagonalizing the action of the torus on the coordinate ring $\mathbb{C}[X]$, one sees that $\mathbb{C}[X]$ is an affine semigroup algebra. Conversely, if A is any affine semigroup algebra, then $\operatorname{Spec} A$ is an affine toric variety ([Fu1]).

Here is a slightly more detailed overview of the paper. This paper will deal with the following specific families of algebras. Their precise definitions are given in §2 and §3. They are all of the form $\mathfrak{A}_{(G,H,\alpha)}$ for G and H as specified. In these descriptions we use the standard notations: GL_n for the general linear group of invertible $n \times n$ complex matrices; O_n for the orthogonal group, the subgroup of GL_n which preserves a non-degenerate symmetric bilinear form on $\mathbb{C}^n$; and Sp_{2n}, the symplectic group, for the subgroup of GL_{2n} which preserves a non-degenerate skew-symmetric bilinear form on $\mathbb{C}^{2n}$.

(A1)(a) GL_n tensor product algebras: $\mathfrak{A}_{(G,H,[(k,0),(\ell,0)])}$ for $G = \mathrm{GL}_n \times \mathrm{GL}_n$, H equal to the diagonally embedded copy of GL_n and $1 \leq k, \ell \leq n$. These algebras describe the decomposition of the tensor product of two polynomial representations of GL_n which are indexed by Young diagrams with at most k rows and ℓ rows respectively. (The term "polynomial representation" follows standard usage in the literature [GW].) By a reciprocity phenomenon (which applies to all the branching algebras [HTW1]), these algebras also describe the branching rules for pairs $(\mathrm{GL}_{k+\ell}, \mathrm{GL}_k \times \mathrm{GL}_\ell)$.

(A1)(b) GL_n k-fold tensor product algebras. These algebras describe the decomposition of the tensor product of k polynomial representations of GL_n. They are of the form $\mathfrak{A}_{(G,H,[(l_1,0),...,(l_k,0)])}$ where $G = (\mathrm{GL}_n)^k$, the product of k copies of GL_n, H is again the diagonally embedded copy of GL_n and $1 \leq l_1, ..., l_k \leq n$. These pairs are not symmetric pairs when $k \geq 3$, but they can be treated by arguments similar to those used for case (A1)(a), and they are in turn used to analyze branching algebras for other symmetric pairs. By the reciprocity phenomenon mentioned in case (A1)(a), these algebras may also be thought of as branching algebras for pairs (GL_l, H), where $l = l_1 + \cdots + l_k$ and H is any subgroup of block diagonal matrices with block sizes $l_1, ..., l_k$.

(A1)(c) A variant of the GL_n tensor product algebra, describing the decomposition of tensor products of representations, in which one factor is a polynomial representation, and the other is the dual of a polynomial representation. It is of the form $\mathfrak{A}_{(G,H,[(k,0),(0,l)])}$ for $G = \mathrm{GL}_n \times \mathrm{GL}_n$, H equal to the diagonally embedded copy of GL_n, where $k, l \geq 1$ and $k + l \leq n$.

(A2) Stable branching algebras $\mathfrak{A}_{(G,H,k)}$ where (G, H) is one of the classical symmetric pairs $(\mathrm{GL}_n, \mathrm{O}_n)$, $(\mathrm{O}_{n+m}, \mathrm{O}_n \times \mathrm{O}_m)$, $(\mathrm{Sp}_{2n}, \mathrm{GL}_n)$, $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$, $(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$, or $(\mathrm{O}_{2n}, \mathrm{GL}_n)$, and k is a positive integer in the range $2k < n$, $2k < \min(n, m)$, $2k \leq n$, $k \leq n$, $k \leq \min(n, m)$ and $2k \leq n$ respectively.

(A3) O_n and Sp_{2n} stable tensor product algebras: these algebras describe the decomposition of tensor products of representations of O_n (respectively, Sp_{2n}). They are the stable branching algebras $\mathfrak{A}_{(H\times H,\Delta H,(k,l))}$ where H is O_n or Sp_{2n}, and ΔH indicates the copy of H embedded diagonally in $H \times H$, and k and l are positive integers which satisfy $2(k+l) < n$ for $H = \mathrm{O}_n$ and $k + l \leq n$ for $H = \mathrm{Sp}_{2n}$.

To each of these stable branching algebras $\mathfrak{A}_{(G,H,\alpha)} = \mathfrak{A}$, we will associate in §2 and §3 a rational polyhedral cone $\mathcal{C}_{\mathfrak{A}} = \mathcal{C}_{(G,H,\alpha)}$ in $\mathbb{R}^N$ for appropriate N, and a finite index sublattice $L_{\mathfrak{A}} = L_{(G,H,\alpha)}$ of $\mathbb{Z}^N$. By taking the intersection $\mathcal{C}_{\mathfrak{A}} \cap L_{\mathfrak{A}} = \Omega_{\mathfrak{A}} = \Omega_{(G,H,\alpha)}$, we obtain a lattice cone, which in particular is a semigroup under vector addition. The multi-graded structure on $\mathfrak{A}$ is reflected by certain linear functionals on $\mathbb{R}^N$. The level sets in $\mathcal{C}_{\mathfrak{A}}$ of these linear functionals define bounded convex polytopes, and the cardinalities of the points of $\Omega_{\mathfrak{A}}$ contained in these polytopes give the multiplicities for the relevant branching rule.

For the case of the full (polynomial representation) branching algebra for $(\mathrm{GL}_n \times \mathrm{GL}_n, \Delta(\mathrm{GL}_n))$ (i.e. $\mathfrak{A}_{(\mathrm{GL}_n\times\mathrm{GL}_n,\Delta(\mathrm{GL}_n),[(n,0),(n,0)])}$), the associated rational cone is the Littlewood-Richardson (LR) cone described in [PV]. In this case $N = n(n+3)/2$. For other branching algebras $\mathfrak{A}$, the associated lattice cone $\Omega_{\mathfrak{A}}$ is constructed in an explicit manner from the LR cone. These relations between lattice cones lead directly to the multiplicity formulas describing branching multiplicities in terms of Littlewood-Richardson coefficients (see [HTW2] and the references there). The multiplicities are the cardinalities of certain convex polytopes contained in the lattice cone $\Omega_{\mathfrak{A}}$.

Our main result is that for most classical symmetric pairs, the stable branching algebra $\mathfrak{A}$ is a flat deformation of the semigroup ring $\mathbb{C}[\Omega_{\mathfrak{A}}]$.

Main Theorem: *For any of the branching algebras $\mathfrak{A}$ listed in cases (A1) and (A2) above, the algebra $\mathfrak{A}$ is a flat deformation of the semigroup ring $\mathbb{C}[\Omega_{\mathfrak{A}}]$. That is, there is a flat one-parameter family $\mathfrak{A}_t$ of multi-graded affine algebras, such that $\mathfrak{A}_t \simeq \mathfrak{A}$ for $t \neq 0$, and $\mathfrak{A}_0 \simeq \mathbb{C}[\Omega_{\mathfrak{A}}]$.*

The proof of the Main Theorem uses the bases constructed in [HTW3], [HL1], [HL2] and [HL3]. The basis of $\mathfrak{A}$ is parametrized by the points of $\Omega_{\mathfrak{A}}$. This parametrization uses the ideas of SAGBI (Subalgebra Analogue of Grobner Basis for Ideals) bases ([RS], [St1], [St2], [MS]). The algebra $\mathfrak{A}$ is realized explicitly inside some polynomial ring. On this polynomial ring, we construct a term order in the sense of Grobner basis theory ([CLO], [St2], [MS]). We then show that the highest terms of our basis elements are all distinct, and form a semigroup isomorphic to $\Omega_{\mathfrak{A}}$. It then follows by general results about SAGBI bases ([CHV]) that $\mathfrak{A}$ is a flat deformation of $\mathbb{C}[\Omega_{\mathfrak{A}}]$.

Remarks: a) The Main Theorem immediately implies that the branching multiplicities for (G, H) are counted by lattice points in the level sets for the linear functionals on $\Omega_{\mathfrak{A}}$ defining the multi-grading on $\mathbb{C}[\Omega_{\mathfrak{A}}]$. This provides a rationale for, or interpretation of, the formulas for multiplicities in terms of lattice points in polyhedra (see [KT1], [KT2], [PV]).

b) Inspired by the paper [GL1], there has recently been a substantial amount of work showing that the homogeneous coordinate rings of flag varieties, and even of general multiplicity-free actions, are flat deformations of semigroup rings ([AB], [Ca], [GL1], [GL2], [Ka], [Ki], [KM]). This provides a pleasing conceptual framing of the long development of standard monomial theory, initiated by the work of Hodge ([Hd]) on the full flag variety for GL_n.

The branching algebra $\mathfrak{A}_{(G,H)}$ is a subalgebra of the coordinate ring of the flag variety for G. However, branching algebras appear to be substantially more complicated than the coordinate rings containing them. Whereas the generators and relations for the coordinate rings were known (and have an elegant description in terms of representation theory) before the connection with semigroup rings was made, generators and relations for the branching algebras are not yet known in general. We hope that the results of this paper, relating branching algebras to the more transparent semigroup algebras, can contribute to the understanding of branching algebras, and in particular, to the project of finding generators and relations for them.

c) The Main Theorem together with general results ([BH]) also implies that the stable branching algebras are normal and Cohen-Macauley.

d) We expect the main theorem also to be valid for the case (A3).

This paper is arranged as follows. We first discuss the case of the GL_n tensor product algebra in Section 2. In this case, the associated polyhedral cone is a "truncated" version of the Littlewood-Richardson cone ([PV]). As the proof of our main theorem depends on several previous works, we hope that by treating one case separately and with more details, we can better explain the background and the ideas of the proof. In Section 3, we consider the remaining 10 branching algebras. An important feature of Section 3 is the explicit description of lattice cone attached to each branching algebra. Finally we prove our main theorem in Section 4.

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2. The case of the GL_n tensor product algebras

In this section, we will show that the GL_n tensor product algebras are are flat deformations of semigroup algebras.

2.1. **The construction of the algebra.**

We first review the construction of a GL_n tensor product algebra. The first step of the construction is to realize the irreducible representations of GL_n on a space of polynomials. Let $k \leq n$ and let the groups GL_n and GL_k act on the space $\mathrm{M}_{n,k} = \mathrm{M}_{n,k}(\mathbb{C})$ of $n \times k$ complex matrices by

$$(g, h).(T) = (g^{-1})^t T h^{-1}, \quad g \in \mathrm{GL}_n, h \in \mathrm{GL}_k.$$

This induces an action of $\mathrm{GL}_n \times \mathrm{GL}_k$ on the algebra $\mathcal{P}(\mathrm{M}_{nk})$ of polynomial functions on M_{nk} in the usual way.

To describe the $\mathrm{GL}_n \times \mathrm{GL}_k$ module structure of $\mathcal{P}(\mathrm{M}_{nk})$, we use the well-known notation of Young diagrams and tableaux ([Wy],[GW],[Fu2]). Recall that a Young diagram D is an array of square boxes arranged in left-justified horizontal rows, with each row no longer than the one above it ([Fu2]). It is written as

$$D = (\lambda_1, ..., \lambda_m)$$

where for each i, λ_i is the number of boxes in the i-th row of D. We shall denote the number of rows in D by $r(D)$.

Let $B_n = A_n U_n$ be the standard Borel subgroup of upper triangular matrices in GL_n, where A_n is the diagonal torus in GL_n and U_n is the maximal unipotent subgroup consisting of all the upper triangular matrices with 1's on the diagonal. For a Young diagram $D = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n$, let $\psi_n^D : A_n \to \mathbb{C}^\times$ be the character given by

$$\psi_n^D[\mathrm{diag}(a_1, ..., a_n)] = a_1^{\lambda_1} \cdots a_n^{\lambda_n}. \tag{2.1}$$

Let A_n^+ be the semigroup of dominant weights for GL_n with respect to the Borel subgroup B_n. Then $\psi_n^D \in \hat{A}_n^+$ ([GW]), and we shall denote the irreducible representation of GL_n with highest weight ψ_n^D by ρ_n^D. We call ρ_n^D a *polynomial representation* of GL_n. By the theory of highest weight ([GW]), the space $\left(\rho^D\right)^{U_n}$ of U_n-invariants in ρ_n^D is one-dimensional, and the nonzero elements in $\left(\rho^D\right)^{U_n}$ are the GL_n highest weight vectors of weight ψ_n^D.

Under the action by $\mathrm{GL}_n \times \mathrm{GL}_k$ ([Ho]),

$$\mathcal{P}(\mathrm{M}_{nk}) \cong \sum_{r(D) \le k} \rho_n^D \otimes \rho_k^D. \tag{2.2}$$

Its subalgebra of U_k invariants is a module for $\mathrm{GL}_n \times A_k$:

$$\mathcal{P}(\mathrm{M}_{nk})^{U_k} \cong \sum_{r(D) \le k} \rho_n^D \otimes \left(\rho_k^D\right)^{U_k}.$$

For each Young diagram D with $r(D) \le k$, the subspace $\rho_n^D \otimes \left(\rho_k^D\right)^{U_k}$ of $\mathcal{P}(\mathrm{M}_{nk})$ is the ψ_k^D-eigenspace for A_k. On the other hand, since $\dim \left(\rho_k^D\right)^{U_k} = 1$,

$$\rho_n^D \otimes \left(\rho_k^D\right)^{U_k} \cong \rho_n^D.$$

It follows that we can realize the GL_n representation ρ_n^D as the ψ_k^D-eigenspace for A_k in the algebra $\mathcal{P}(\mathrm{M}_{nk})^{U_k}$.

To realize tensor products of GL_n representations, we take another algebra $\mathcal{P}(\mathrm{M}_{n\ell})$ where $\ell \le n$ and form the tensor product

$$\mathcal{P}(\mathrm{M}_{nk}) \otimes \mathcal{P}(\mathrm{M}_{n\ell}) \cong \mathcal{P}(\mathrm{M}_{n,k+\ell}).$$

This algebra has a $(\mathrm{GL}_n \times \mathrm{GL}_k) \times (\mathrm{GL}_n \times \mathrm{GL}_\ell) \cong (\mathrm{GL}_n \times \mathrm{GL}_n) \times \mathrm{GL}_k \times \mathrm{GL}_\ell$ module structure given by

$$\begin{aligned} \mathcal{P}(\mathrm{M}_{n,k+\ell}) &\cong \left\{ \sum_{r(D) \leq k} \rho_n^D \otimes \rho_k^D \right\} \otimes \left\{ \sum_{r(E) \leq \ell} \rho_n^E \otimes \rho_\ell^E \right\} \\ &\cong \sum_{r(D) \leq k, r(E) \leq \ell} \left(\rho_n^D \otimes \rho_n^E\right) \otimes \rho_k^D \otimes \rho_\ell^E. \end{aligned}$$

Its subalgebra of $U_k \times U_\ell$ invariants is a module for $(\mathrm{GL}_n \times \mathrm{GL}_n) \times A_k \times A_\ell$ and can be decomposed as

$$\mathcal{P}(\mathrm{M}_{n,k+\ell})^{U_k \times U_\ell} \cong \sum_{r(D) \leq k, r(E) \leq \ell} \left(\rho_n^D \otimes \rho_n^E\right) \otimes \left(\rho_k^D\right)^{U_k} \otimes \left(\rho_\ell^E\right)^{U_\ell}.$$

In particular, the tensor product of GL_n representations

$$\rho_n^D \otimes \rho_n^E \cong \left(\rho_n^D \otimes \rho_n^E\right) \otimes \left(\rho_k^D\right)^{U_k} \otimes \left(\rho_\ell^E\right)^{U_\ell}$$

can now be identified with the $\psi_k^D \times \psi_\ell^E$-eigenspace of $A_k \times A_\ell$ in $\mathcal{P}(\mathrm{M}_{n,k+\ell})^{U_k \times U_\ell}$. To decompose this tensor product, we restrict the action of $\mathrm{GL}_n \times \mathrm{GL}_n$ to its diagonal subgroup

$$\mathrm{GL}_n = \{(g,g) : \ g \in \mathrm{GL}_n\}.$$

Thus we consider the subalgebra of U_n-invariants

$$\left(\mathcal{P}(\mathrm{M}_{n,k+\ell})^{U_k \times U_\ell}\right)^{U_n} \cong \mathcal{P}(\mathrm{M}_{n,k+\ell})^{U_n \times U_k \times U_\ell}$$

in $\mathcal{P}(\mathrm{M}_{n,k+\ell})^{U_k \times U_\ell}$, where U_n is the maximal unipotent subgroup of the diagonal GL_n. This algebra is a module for $A_n \times A_k \times A_\ell$, so it is a triply-graded algebra by $A_n^+ \times A_k^+ \times A_\ell^+$ and can be decomposed as

$$\mathcal{P}(\mathrm{M}_{n,k+\ell})^{U_n \times U_k \times U_\ell} \cong \sum_{r(D) \leq k, r(E) \leq \ell} \left(\rho_n^D \otimes \rho_n^E\right)^{U_n} \otimes \left(\rho_k^D\right)^{U_k} \otimes \left(\rho_\ell^E\right)^{U_\ell} \cong \sum_{\substack{r(D) \leq k, r(E) \leq \ell \\ r(F) \leq k+\ell}} \mathcal{E}_{F,D,E},$$

where for Young diagrams F, D and E, $\mathcal{E}_{F,D,E}$ is the $\psi_n^F \times \psi_k^D \times \psi_\ell^E$-eigenspace of $A_n \times A_k \times A_\ell$ in $\mathcal{P}(\mathrm{M}_{n,k+\ell})^{U_n \times U_k \times U_\ell}$. Note that $\mathcal{E}_{F,D,E}$ can be identified with the space of GL_n highest weight vectors of weight ψ_n^F in the tensor product $\rho_n^D \times \rho_n^E$. Consequently, its dimension coincides with the multiplicity of ρ_n^F in $\rho_n^D \times \rho_n^E$. Thus information on how tensor products of GL_n representations decompose can be deduced from the structure of the algebra $\mathcal{P}(\mathrm{M}_{n,k+\ell})^{U_n \times U_k \times U_\ell}$. In view of this property, we call this algebra a GL_n *tensor product algebra*. It is a branching algebra for the symmetric pair $(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n)$, so we denote it by

$$\mathfrak{A}_0 = \mathfrak{A}_{(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n),[(k,0),(\ell,0)]}.$$

2.2. **The dimensions of the homogeneous components.** As described above, the GL_n tensor product algebra is a triply graded algebra and has the following decomposition into a sum of homogeneous components:

$$\mathfrak{A}_0 = \bigoplus_{F,D,E} \mathcal{E}_{F,D,E}.$$

Our next objective is to describe the dimension of the homogeneous component $\mathcal{E}_{D,E,F}$. We have observed that this dimension is equal to the multiplicity of ρ_n^F in the tensor product $\rho_n^D \otimes \rho_n^E$. This multiplicity is given by the Littlewood-Richardson rule. To state this rule, we need several basic definitions ([Fu2]).

(i) If the Young diagrams D is contained in the Young diagram F, then the *skew diagram* F/D is the diagram obtained by removing all the boxes of D from F.

(ii) A *Littlewood-Richardson tableau* (LR tableau) T is a skew diagram F/D filled with positive integers 1 through k, for some k, which satisfies the following conditions:

LR1: The numbers in a row are weakly increasing from left to right, and the numbers in a column are strictly increasing from top to bottom. This condition is also known as *semi-standardness*.

LR2: For every pair of positive integers $m \geq 2$ and $p \geq 1$, the number of times the number m occurs in the first p rows of T is not larger than the number of times the number of times $m-1$ occurs in the first $p-1$ rows of the T. This is also known as the *Yamanouchi word condition* or the *lattice permutation condition* ([Ma]).

(iii) The *content* of a LR tableau T is the Young diagram $E = (\mu_1, ..., \mu_k)$ where for each $1 \leq i \leq k$, μ_i is the number of boxes in T which are filled with the number i.

(iv) If D, E and F are Young diagrams such that D is contained in F and $|F| = |D| + |E|$, then the *Littlewood-Richardson coefficient* $c_{D,E}^F$ is the number of LR tableaux T of shape F/D and content E.

The Littlewood-Richardson Rule: *If the Young diagrams D, E and F have at most n rows, then $c_{D,E}^F$ is the multiplicity of the representation ρ_n^F in the tensor product $\rho_n^D \otimes \rho_n^E$.*

The Littleweood-Richardson rule implies that

$$\dim \mathcal{E}_{F,D,E} = c_{D,E}^F.$$

The paper [HTW3] constructs a basis for $\mathcal{E}_{F,D,E}$ whose elements are indexed by LR tableaux. We will mention some of the essential properties of this basis in a later subsection.

2.3. **Polyhedral cones and polytopes.** In this subsection, we will define a rational polyhedral cone $\mathbf{LR}_n$ called the *Littlewood-Richardson cone*. We shall see that each Littlewood-Richardson coefficient counts the the number of integral points in a polytope in $\mathbf{LR}_n$.

We first recall the definitions of polyhedral cone and polytope. Let $(.,.)$ denote the dot product in $\mathbb{R}^n$. For $\mathbf{a} \in \mathbb{R}^n$ with $\mathbf{a} \neq \mathbf{0}$ and $b \in \mathbb{R}$, the set

$$H^+(\mathbf{a}, b) = \{\mathbf{v} \in \mathbb{R}^n : \ (\mathbf{v}, \mathbf{a}) \geq b\}$$

is called a *closed half-space*. A finite intersection

$$P = \bigcap_{i=1}^{r} H^+(\mathbf{a}_i, b_i)$$

of closed half-spaces is called a *polyhedron*. A *polytope* is a polyhedron which is bounded.

If $b_1 = b_2 = \cdots = b_r = 0$, that is, if

$$P = \bigcap_{i=1}^{r} H^+(\mathbf{a}_i, 0),$$

then P is called a *polyhedral cone*. In this case, there exists $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \in \mathbb{R}^n$ such that

$$P = \{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k \mid \ r_i \geq 0 \ \text{ for all } 1 \leq i \leq k\} \tag{2.9}$$

([BH]). It is called a *rational polyhedral cone* if we can choose $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{Q}^n$, or equivalently if we can choose $\mathbf{a}_1, ..., \mathbf{a}_r \in \mathbb{Q}^n$.

Definition. A *lattice cone* Ω in $\mathbb{R}^n$ is the intersection of a rational polyhedral cone $\mathcal{C}$ in $\mathbb{R}^n$ and a finite index sublattice L of $\mathbb{Z}^n$.

We note that every lattice cone $\Omega = \mathcal{C} \cap L$ is an *affine semigroup*, that is, Ω is a finitely generated subsemigroup of $\mathbb{Z}^n$ containing 0. It is a semigroup because the polyhedral cone $\mathcal{C}$ is closed under vector addition. The following result which is a consequence of Gordan's Lemma ([BH]) guarantees that Ω is finitely generated.

Lemma 2.3.1. *Let $\mathcal{C}$ be a rational polyhedral cone in $\mathbb{R}^n$ and let G be a finitely generated additive subgroup of $\mathbb{Q}^n$. Then $G \cap \mathcal{C}$ is a finitely generated semigroup.*

Definition. A *Littlewood-Richardson triangle* (or simply LR triangle, see [PV]) of size n is a triangular array of real numbers

$$A = [a_{ij}]_{0 \leq i \leq j \leq n} = \begin{array}{ccccccccccc} & & & & & a_{00} & & & & & \\ & & & & a_{01} & & a_{11} & & & & \\ & & & a_{02} & & a_{12} & & a_{22} & & & \\ & & a_{03} & & a_{13} & & a_{23} & & a_{33} & & \\ & \vdots & & \cdots & & \cdots & & \cdots & & \ddots & \\ a_{0n} & & a_{1n} & & \cdots & & \cdots & & \cdots & & a_{nn} \end{array}$$

such that

[LRT1]: $a_{00} = 0$ and $a_{ij} \geq 0$ for all $0 \leq i \leq j \leq n$,
[LRT2]: $\sum_{p=0}^{i-1} a_{pj} \geq \sum_{p=0}^{i} a_{p,j+1}$ for all $1 \leq i \leq j+1 \leq n$,
[LRT3]: $\sum_{q=i}^{j} a_{iq} \geq \sum_{q=i+1}^{j+1} a_{i+1,q}$ for all $1 \leq i \leq j < n$.

We shall denote the set of all LR triangles of size n by $\mathbf{LR}_n$.

Note that if we identify the LR triangle $A = [a_{ij}]_{0\leq i\leq j\leq n}$ with the point

$$(a_{01}, a_{11}, a_{02}, a_{12}, a_{22}, ..., a_{nn})$$

of $\mathbb{R}^N$ where $N = n(n+3)/2$, then $\mathbf{LR}_n$ corresponds to a rational polyhedral cone in $\mathbb{R}^N$. So we call $\mathbf{LR}_n$ a *Littlewood-Richardson cone* (or simply a LR cone).

Let $D = (\mu_1, ..., \mu_n)$, $E = (\nu_1, ..., \nu_n)$ and $F = (\lambda_1, ...\lambda_n)$ be Young diagrams with at most n rows. We say that a LR triangle $A = [a_{ij}]_{0\leq i\leq j\leq n}$ is of type (D, E, F) if

$$\mu_j = a_{0j}, \quad \lambda_j = \sum_{p=0}^{j} a_{pj}, \quad \nu_j = \sum_{q=j}^{n} a_{jq}, \qquad 1 \leq j \leq n. \tag{2.10}$$

Note that the set of all LR triangles in $\mathbf{LR}_n$ of type (D, E, F) is a polytope in $\mathbb{R}^N$. We will denote it by $\mathbf{LR}_n(D, E, F)$.

Lemma 2.3.2. ([PV]) *There is a bijection between the set of all LR tableaux of shape F/D and content E and $\mathbf{LR}_n(D, E, F) \cap \mathbb{Z}^N$, the set of all LR triangles of type (D, E, F) and with integer entries.*

In fact, if T is an LR tableau of shape F/D and content E, then the LR triangle $A_T = [a_{ij}]_{0\leq i\leq j\leq n}$ corresponding to T is defined as follows:
(a) $a_{00} = 0$, $a_{0j} = \mu_j$ for $1 \leq j \leq n$, and
(b) a_{ij} is equal to the number of i's in row j of T for $1 \leq i \leq j \leq n$.

Lemma 2.3.2 raises the possibility that a basis for the homogeneous component $\mathcal{E}_{F,D,E}$ of $\mathfrak{A}$ can be indexed by integral points in the polytope $\mathbf{LR}_n(D, E, F)$. To define a polyhedral cone for the GL_n tensor product algebra $\mathfrak{A}_0$, we need to adapt the definition of $\mathbf{LR}_n$ by setting certain entries in an LR triangle to 0. Specifically, we let

$$\mathbf{LR}_{n,k,\ell} = \{A = [a_{ij}] \in \mathbf{LR}_n : \ a_{0,k+1} = a_{i,j} = 0 \ \forall i, j > \ell\}.$$

We note that if $A = [a_{ij}] \in \mathbf{LR}_{n,k,\ell}$, then by the condition LRT2,

$$0 = a_{0,k+1} \geq a_{0,k+2} + a_{1,k+2} \geq a_{0,k+3} + a_{1,k+3} + a_{2,k+3} \geq \cdots$$

we obtian

$$a_{ij} = 0 \quad \text{if } i < n-k \text{ and } j > k.$$

If we remove the entries in a LR triangle in $\mathbf{LR}_{n,k,\ell}$ which are necessarily 0, then $\mathbf{LR}_{n,k,\ell}$ can be regarded as a rational polyhedral cone in $\mathbb{R}^{N_0}$ where $N_0 = N_{(n,k,\ell)}$ and

$$\begin{aligned} N_{(n,k,\ell)} \ &= \ \tfrac{1}{2}\{n(n+3) - (n-k)(n-k+1) - (n-\ell)(n-\ell+1) \\ &\qquad + \max(|n-k-\ell|(n-k-\ell-1), 0)\}. \end{aligned} \tag{2.11}$$

We now let

$$\Omega_0 = \Omega_{\mathfrak{A}_0} = \mathbf{LR}_{n,k,\ell} \cap \mathbb{Z}^{N_0}.$$

Then Ω_0 is a lattice cone in $\mathbb{R}^{N_0}$. The paper [HTW3] constructs a distinguished basis for $\mathfrak{A}_0$ whose elements are indexed by Ω_0. We will describe some of the important properties of this basis in the next subsection.

Remark. There is another cone constructed by Berenstein and Zelevinsky ([BZ]) which can be used to describe tensor products of representations of SL_n. $\mathbf{LR}_n$ is more closely related to our construction than is the cone of [BZ]. The relationship between these two cones is discussed in [PV].

2.4. **A SAGBI basis for $\mathfrak{A}_0$.** Recall that $\mathfrak{A}_0 = \mathcal{P}(\mathrm{M}_{n,k+\ell})^{U_n \times U_k \times U_\ell}$. So elements of $\mathfrak{A}$ are polynomials on $\mathrm{M}_{n,k+\ell} = \mathrm{M}_{n,k} \oplus \mathrm{M}_{n,\ell}$. We now define a monomial ordering τ_0 on $\mathcal{P}(\mathrm{M}_{n,k+\ell})$ as follows: Write a typical element of M_{nk} and of $\mathrm{M}_{n,\ell}$ as $X = (x_{ij})$ and $Y = (y_{ih})$ respectively. Then τ_0 is the graded lexicographic order ([CLO]) such that

$$\begin{aligned} &x_{11} > x_{21} > \cdots > x_{n1} > x_{12} > x_{22} > \cdots > x_{nk} \\ &> y_{11} > y_{21} > \cdots > y_{n1} > y_{12} > y_{22} > \cdots > y_{nk}. \end{aligned} \tag{2.1}$$

For each $f \in \mathcal{P}(\mathrm{M}_{n,k+\ell})$, let $\mathrm{in}_{\tau_0}(f)$ be the leading monomial of f with respect to the monomial ordering τ_0.

Proposition 2.4.1. ([HTW3]) *The* GL_n *tensor product algebra* $\mathfrak{A}_0$ *has a basis* $\mathcal{B}_0 = \{\Delta_A^{(0)} : A \in \Omega_0\}$ *such that for each* $A = [a_{ij}] \in \Omega_0$*,*

$$\mathrm{in}_{\tau_0}(\Delta_A^{(0)}) = \left(\prod_{h=1}^{k} x_{hh}^{a_{0h}} \right) \left(\prod_{1 \le i \le j \le n} y_{ji}^{a_{ij}} \right).$$

Corollary 2.4.2. *The set*

$$\mathrm{in}_{\tau_0}(\mathcal{B}_0) = \{\mathrm{in}_{\tau_0}(\Delta_A^{(0)} : A \in \Omega_0\}$$

of monomials forms an affine semigroup isomorphic to Ω_0*.*

Proof. By Proposition 2.4.1, we have:

(i) The leading monomials $\mathrm{in}_{\tau_0}(\Delta_A^{(0)})$ are distinct, that is, if $A_1, A_2 \in \Omega_0$ and $A_1 \neq A_2$, then $\mathrm{in}_{\tau_0}(\Delta_{A_1}^{(0)}) \neq \mathrm{in}_{\tau_0}(\Delta_{A_2}^{(0)})$.

(ii) For $A_1, A_2 \in \Omega_0$,

$$\mathrm{in}_{\tau_0}(\Delta_{A_1+A_2}^{(0)}) = \mathrm{in}_{\tau_0}(\Delta_{A_1}^{(0)}) \cdot \mathrm{in}_{\tau_0}(\Delta_{A_2}^{(0)}).$$

We now define a map $\varphi : \Omega_0 \to \mathrm{in}_{\tau_0}(\mathcal{B}_0)$ by

$$\varphi(A) = \mathrm{in}_{\tau_0}(\Delta_A), \quad A \in \Omega_0.$$

Then by (i) and (ii), φ is an isomorphism of semigroups. □

We now give an example to illustrate the relationship between LR tableaux, LR triangle and the leading monomials of the basis elements. Let $D = (5, 3, 2)$, $E =$

$(4,3,2)$, $F = (6,4,4,3,2)$ and

$$T = \begin{array}{|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \; & 1 \\ \hline \; & \; & \; & 1 \\ \cline{1-4} \; & \; & 1 & 2 \\ \cline{1-4} 1 & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array} .$$

Then T is a LR tableau of shape F/D and content E, and it corresponds to the LR triangle

$$A_T = \begin{array}{ccccccccccc} & & & & & 0 & & & & & \\ & & & & 5 & & 1 & & & & \\ & & & 3 & & 1 & & 0 & & & \\ & & 2 & & 1 & & 1 & & 0 & & \\ & 0 & & 1 & & 2 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 2 & & 0 & & 0 \end{array} .$$

By Proposition 2.4.1, the basis element $\Delta^{(0)}_{A_T}$ has leading monomial given by

$$\mathrm{in}_{\tau_0}(\Delta^{(0)}_{A_T}) = x_{11}^5 x_{22}^3 x_{33}^2 y_{11} y_{21} y_{31} y_{32} y_{41} y_{42}^2 y_{53}^2.$$

It turns out that the basis $\mathcal{B}_0$ is a SAGBI basis for $\mathfrak{A}_0$. We recall the definition of a SAGBI basis . Let $R = k[x_1, ..., x_n]$ be a polynomial algebra over a field k, S a subalgebra of R and τ a monomial ordering on R. For each $a \in R$, let $\mathrm{in}_\tau(a)$ be the leading monomial of a with respect to τ. The *initial algebra of* S, denoted by $\mathrm{in}_\tau(S)$, is the subalgebra of R generated by the set $\{\mathrm{in}_\tau(a) : \ a \in S\}$ of leading monomials. A subset F of S is called a *SAGBI basis* for S if the set $\{\mathrm{in}_\tau(a) : \ a \in F\}$ generates the initial algebra $\mathrm{in}_\tau(S)$.

Proposition 2.4.3. *The basis* $\mathcal{B}_0$ *of* $\mathfrak{A}_0$ *given in Proposition 2.4.1 is a SAGBI basis with respect to the monomial ordering* τ_0 *defined in* (2.1).

Proof. Let $a \in \mathfrak{A}_0$. Then

$$a = c_1 \Delta^{(0)}_{A_1} + c_2 \Delta^{(0)}_{A_2} + \cdots + c_r \Delta^{(0)}_{A_r}$$

for some $c_1, c_2, ..., c_r \in \mathbb{C}$ and $A_1, A_2, ..., A_r \in \Omega_0$. Since the leading monomials $\mathrm{in}_{\tau_0}(\Delta^{(0)}_{A_1})$, $\mathrm{in}_{\tau_0}(\Delta^{(0)}_{A_2})$, ..., $\mathrm{in}_{\tau_0}(\Delta^{(0)}_{A_r})$ are distinct, $\mathrm{in}_\tau(a) = \mathrm{in}_{\tau_0}(\Delta^{(0)}_{A_j})$ for some $1 \le j \le r$. Hence $\mathrm{in}_{\tau_0}(\mathcal{B})$ generates the initial algebra $\mathrm{in}_{\tau_0}(\mathfrak{A}_0)$ of $\mathfrak{A}_0$. □

2.5. **Toric degeneration of $\mathfrak{A}_0$.** Let $\mathbb{C}[\Omega_0]$ denotes the complex algebra with Ω_0 as a basis and with relations given by the semigroup laws in Ω_0. We call $\mathbb{C}[\Omega_0]$ an affine semigroup algebra ([BH]). By Lemma 2.3.1, this algebra is finitely generated.

Corollary 2.5.1. *The initial algebra* $\mathrm{in}_{\tau_0}(\mathfrak{A}_0)$ *of* $\mathfrak{A}_0$ *is isomorphic to the affine semigroup algebra* $\mathbb{C}[\Omega_0]$*. Consequently* $\mathfrak{A}$ *has a finite SAGBI basis.*

Proof. By Corollary 2.4.2, the semigroups $\mathrm{in}_{\tau_0}(\mathcal{B})$ and Ω_0 are isomorphic. So the algebras they generate are also isomorphic. If Ω_0 is generated by $A_1, ..., A_r$, then $\{\Delta^{(0)}_{A_1}, ..., \Delta^{(0)}_{A_r}\}$ is a finite SAGBI basis for $\mathfrak{A}_0$. □

Our main result for the algebra $\mathfrak{A}_0$ can be deduced from the discussion above and the following general result.

Proposition 2.5.2. ([CHV]) *Let S be a subalgebra of a polynomial algebra $R = k[x_1, ..., x_n]$ over a field k and let τ be a monomial ordering on S. If the initial algebra $\mathrm{in}_\tau(S)$ is finitely generated, then there exists a flat 1-parameter family of k-algebras with general fibre S and special fibre $\mathrm{in}_\tau(S)$.*

Theorem 2.5.3. *There exists a flat one-parameter family of complex algebras with general fibre $\mathfrak{A}_0$ and special fibre $\mathbb{C}[\Omega_0]$.*

Proof. This follows from Corollary 2.5.1 and Proposition 2.5.2. □

3. Stable branching algebras and the associated lattice cones

Let G be a complex classical group and H a symmetric subgroup, that is, H is the subgroup of fixed points of an involution on G. We call (G, H) a classical symmetric pair. There is a naturally defined multi-graded algebra $\mathfrak{A}_{(G,H)}$, called the branching algebra for (G, H), which encodes the branching rule from G to H ([HTW1]). The algebra $\mathfrak{A}_{(G,H)}$ has a certain family of well behaved subalgebras which can be realized as subalgebras of polynomial algebras. We refer to these subalgebras as *stable branching algebras*. For a detailed discussion on stable branching algebras, see [HTW1].

The GL_n tensor product algebra $\mathfrak{A}_0 = \mathfrak{A}_{(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n),[(k,0),(\ell,0)]}$ which we considered in Section 2 is an example of a stable branching algebra and it corresponds to the symmetric pair $(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n)$. By Theorem 2.5.3, $\mathfrak{A}_0$ is a flat deformation of the semigroup algebra $\mathbb{C}[\Omega_0]$ on Ω_0 where Ω_0 is the lattice cone obtained by taking the intersection of the polyhedral cone $\mathbf{LR}_{n,k,\ell}$ in $\mathbb{R}^{N_0}$ and $\mathbb{Z}^{N_0}$. It turns out that similar results hold for most other stable branching algebras. In fact, using the results of [HL1, HL2, HL3], which describe bases for other branching algebras analogous to the basis for $\mathfrak{A}_0$ described on §2, we can show that all but two stable branching algebras are flat deformation of semigroup algebras on lattice cones. The main purpose of this section is to identify these lattice cones. Each subsection §3.i will be devoted to a specific stable branching algebra $\mathfrak{A}_i$. The subsections are organized as follows:

- In §3.i.1, we recall the definition of $\mathfrak{A}_i$.
- In §3.i.2, we describe the relevant branching multiplicities.
- In §3.i.3, we define a lattice cone Ω_i based on the multiplicity formula in §3.i.2.
- A basis $\mathcal{B}_i$ for $\mathfrak{A}_i$ has been constructed in [HL1, HL2, HL3]. In §3.i.4, we describe the connection between the lattice cone Ω_i and this basis $\mathcal{B}_i$. In

particular, the basis elements in $\mathcal{B}_i$ are indexed by the points in Ω_i, so we can write

$$\mathcal{B}_i = \{\Delta_A^{(i)} : \ A \in \Omega_i\}.$$

Moreover, $\mathfrak{A}_i$ is realized as a subalgebra of a polynomial algebra $\mathcal{P}(V_i)$. We define a monomial ordering τ_i on $\mathcal{P}(V_i)$ and give a formula for the leading monomial $\text{in}_{\tau_i}(\Delta_A^{(i)})$ of the element $\Delta_A^{(i)}$ of $\mathcal{B}_i$. From the formula, we see that:

(i) The leading monomials $\text{in}_{\tau_i}(\Delta_A^{(i)})$ are distinct.

(ii) For $A_1, A_2 \in \Omega_i$,

$$\text{in}_{\tau_i}(\Delta_{A_1+A_2}^{(i)}) = \text{in}_{\tau_i}(\Delta_{A_1}^{(0)}) \cdot \text{in}_{\tau_i}(\Delta_{A_2}^{(0)}).$$

It follows that (see Corollary 2.4.2) the set

$$\text{in}_i(\mathcal{B}_i) = \{\text{in}_{\tau_i}(\Delta_A^{(i)}) : \ A \in \Omega_i\}$$

of leading monomials forms a semigroup isomorphic to Ω_i. In §4, this fact will be used to prove our main theorem.

3.1. $\mathfrak{A}_1$: **k-fold tensor product algebras for** GL_n**.** This algebra generalizes the GL_n tensor product algebra. It corresponds to the pair (G, H) where $G = \text{GL}_n^k$, the product of k copies of GL_n, and $H = \text{GL}_n = \{(g, g, ..., g) : \ g \in \text{GL}_n\}$ and it describes the decomposition of the tensor product of k polynomial representations of GL_n. The pair (G, H) is not a symmetric pair but k-fold tensor product algebras can be treated by arguments similar to the 2-fold case and they are in turn used to analyze stable branching algebras for other symmetric pairs.

3.1.1. *The definition of* $\mathfrak{A}_1$. Let $k \geq 3$ and let $l_1, ..., l_k$ be positive integers less than or equal to n. We consider the space

$$V_{n,l_1,...,l_k} = \text{M}_{n,l_1} \oplus \text{M}_{n,l_2} \oplus \cdots \oplus \text{M}_{n,l_k}.$$

Let $\text{GL}_n \times \text{GL}_{l_1} \times \cdots \times \text{GL}_{l_k}$ act on $V_{n,l_1,l_2,...,l_k}$ by

$$[(g, h_1, ..., h_k), (X_1, ..., X_k)] \to ((g^t)^{-1} X_1 h_1^{-1}, (g^t)^{-1} X_2 h_2^{-1}, ..., (g^t)^{-1} X_k h_k^{-1})$$

where $g \in \text{GL}_n$, $h_j \in \text{GL}_{l_j}$ and $X_j \in \text{M}_{n,l_j}$ for $1 \leq j \leq k$. This action induces an action of $\text{GL}_n \times \text{GL}_{l_1} \times \cdots \times \text{GL}_{l_k}$ on the algebra $\mathcal{P}(V_{n,l_1,...,l_k})$ of polynomial functions on $V_{n,l_1,...,l_k}$. We call the subalgebra

$$\mathfrak{A}_1 = \mathfrak{A}_{((\text{GL}_n)^k, \Delta(\text{GL}_n), [(l_1,0),...,(l_k,0)])} = \mathcal{P}(V_{n,l_1,...,l_k})^{U_n \times U_{l_1} \times \cdots \times U_{l_k}}$$

of $U_n \times U_{l_1} \times \cdots \times U_{l_k}$ invariants in $\mathcal{P}(V_{n,l_1,...,l_k})$ a *k-fold tensor product algebra for* GL_n ([HL1]). It describes how a tensor product of the form $\rho_n^{D_1} \otimes \cdots \otimes \rho_n^{D_k}$, where for $1 \leq i \leq k$, D_i is a Young diagram with at most l_i rows, decomposes into a sum of irreducible GL_n representations.

3.1.2. *Branching multiplicity.* The algebra $\mathfrak{A}_1$ is a module for $A_n \times A_{l_1} \times \cdots \times A_{l_k}$, so it decomposes into a sum of joint $A_n \times A_{l_1} \times \cdots \times A_{l_k}$ eigenspaces. For Young diagrams $D_1, ..., D_k$ and F such that $r(D_i) \leq l_i$ for $1 \leq i \leq k$ and $r(F) \leq n$, the dimension of the $\psi_n^F \times \psi_{l_1}^{D_1} \times \cdots \times \psi_{l_k}^{D_k}$ eigenspace in $\mathfrak{A}_1$ coincides with the multiplicity of ρ_n^F in the tensor product $\rho_n^{D_1} \otimes \cdots \otimes \rho_n^{D_k}$ (see equation (2.1) for notation). By the Littlewood-Richardson rule, this multiplicity is given by

$$m_1(\rho_n^F; \rho_n^{D_1}, ..., \rho_n^{D_k}) = \sum_{F_1, F_2, ..., F_{k-2}} c_{D_1,D_2}^{F_1} c_{F_1,D_3}^{F_2} c_{F_2,D_4}^{F_3} \cdots c_{F_{k-2},D_k}^{F}.$$

3.1.3. *The lattice cone* Ω_1. For each $1 \leq i \leq k$, let $L_i = l_1 + \cdots + l_i$. Then the truncated LR cone $\mathbf{LR}_{n,L_{i-1},l_i}$ is a lattice cone in $\mathbb{R}^{N(n,L_{i-1},l_i)}$ (see Section 2.3 for notation). Let $\mathcal{S}_1$ be the subspace of $\mathbb{R}^{N(n,L_1,l_2)} \oplus \mathbb{R}^{N(n,L_2,l_3)} \oplus \cdots \oplus \mathbb{R}^{N(n,L_{k-1},l_k)}$ consisting of $(A_1, ..., A_{k-1})$ where $A_h = [a_{ij}^{(h)}]$ for $1 \leq h \leq k-1$, and

$$\sum_{p=0}^{j} a_{pj}^{(h)} = a_{0j}^{(h+1)}, \qquad 1 \leq h \leq k-2,\ 1 \leq j \leq L_{h+1}.$$

Then $\dim \mathcal{S}_1 = N_1$ where

$$N_1 = (N_{(n,L_1,l_2)} + N_{(n,L_2,l_3)} + \cdots + N_{(n,L_{k-1},l_k)}) - (L_2 + \cdots + L_{k-1}).$$

So $\mathcal{S}_1$ can be identified with $\mathbb{R}^{N_1}$. Let

$$\mathcal{C}_1 = \mathcal{S}_1 \cap \left(\mathbf{LR}_{n,L_1,l_2} \times \mathbf{LR}_{n,L_2,l_3} \times \cdots \times \mathbf{LR}_{n,L_{k-1},l_k}\right)$$

Then $\mathcal{C}_1$ is a rational polyhedral cone in $\mathcal{S}_1$, and the intersection

$$\Omega_1 = \mathcal{C}_1 \cap \mathbb{Z}^{N_1}$$

is a lattice cone in $\mathcal{S}_1$.

If $D_i = (\lambda_1^{(i)}, ..., \lambda_{l_i}^{(i)})$ for $1 \leq i \leq k$ and $F = (\mu_1, ..., \mu_{L_k})$, let $\mathcal{C}_1(F; D_1, ..., D_k)$ be the set of points $(A_1, ..., A_{k-1})$ in $\mathcal{C}_1$ such that

$$\begin{cases} A_h = [a_{ij}^{(h)}], & 1 \leq h \leq k-1, \\ a_{0j}^{(1)} = \lambda_j^{(1)}, & 1 \leq j \leq l_1, \\ \sum_{q=j}^{n} a_{jq}^{(h)} = \lambda_j^{(h)}, & 1 \leq j \leq l_h, \\ \sum_{p=0}^{j} a_{pj}^{(k-1)} = \mu_j, & 1 \leq j \leq L_k. \end{cases}$$

Note that $\mathcal{C}_1(F; D_1, ..., D_k)$ is a polytope contained in $\mathcal{C}_1$, and $m_1(\rho_n^F, \rho_n^{D_1}, ..., \rho_n^{D_k})$ is the number of points in the intersection $\mathcal{C}_1(F; D_1, ..., D_k) \cap \mathbb{Z}^{N_1}$. Since

$$\Omega_1 = \bigcup_{F; D_1, ..., D_k} \mathcal{C}_1(F; D_1, ..., D_k) \cap \mathbb{Z}^{N_1},$$

Ω_1 can be used to label a basis for $\mathfrak{A}_1$.

3.1.4. *A basis for* $\mathfrak{A}_1$. [HL1] constructs a basis $\mathcal{B}_1 = \{\Delta^{(1)}_{(A_1,...,A_{k-1})} : (A_1, ..., A_{k-1}) \in \Omega_1\}$ for $\mathfrak{A}_1$ with the following properties: Write a typical element of $V_{n,l_1,...,l_k}$ as $X = (X^{(1)}, ..., X^{(k)})$, and $X^{(h)} = (x^{(h)}_{ij})$ for $1 \leq h \leq k$. Let τ_1 be the following monomial ordering on $\mathcal{P}(V_{n,l_1,...,l_k})$: it is the graded lexicographic order [CLO] such that

$$x^{(h)}_{ab} > x^{(h+1)}_{cd} \quad \text{for } 1 \leq h \leq k-1,\ 1 \leq a, c \leq n,\ 1 \leq b \leq l_h,\ 1 \leq d \leq l_{h+1},$$

and

$$x^{(h)}_{11} > x^{(h)}_{21} > \cdots > x^{(h)}_{n1} > x^{(h)}_{12} > x^{(h)}_{22} > \cdots > x^{h)}_{n,l_h}$$

for $1 \leq h \leq k$. Then for each $(A_1, ..., A_{k-1}) \in \Omega_1$ where $A_h = [a^{(h)}_{ij}]$ for $1 \leq h \leq k-1$,

$$\text{in}_{\tau_1}\left(\Delta^{(1)}_{(A_1,...,A_{k-1})}\right) = \left[\prod_{i=1}^{l_1} x_{ii}^{a^{(1)}_{0i}}\right]\left[\prod_{h=1}^{k-1}\prod_{1\leq i\leq j\leq L_{h+1}}\left(x^{(h+1)}_{ji}\right)^{a^{(h)}_{ij}}\right].$$

3.2. $\mathfrak{A}_2$ **: a variant of the** $\mathbf{GL_n}$ **tensor product algebras.** This algebra describes the decomposition of the tensor product of one polynomial representation of GL_n with the dual of another polynomial representation.

3.2.1. *The definition of* $\mathfrak{A}_2$. Let $n, k, l \in \mathbb{Z}^+$ be such that $n \geq k + l$, and consider the space

$$V_{n,k,l} = \mathrm{M}_{nk} \oplus \mathrm{M}_{nl}.$$

Let the groups GL_n, GL_k and GL_ℓ act on $V_{n,k,l}$ by

$$(g, h_1, h_2)(X, \hat{Y}) = ((g^{-1})^t X h_1^{-1}, g\hat{Y}h_2^{-1}),$$

where $(X, \hat{Y}) \in V_{n,k,l}$, $g \in \mathrm{GL}_n$, $h_1 \in \mathrm{GL}_k$ and $h_2 \in \mathrm{GL}_\ell$. This action induces an action of these groups on the algebra $\mathcal{P}(V_{n,k,l})$ of polynomial functions on $V_{n,k,l}$. The subalgebra

$$\mathfrak{A}_2 = \mathfrak{A}_{(\mathrm{GL}_n\times\mathrm{GL}_n,\Delta(\mathrm{GL}_n),[(k,0),(0,l)])} = \mathcal{P}(V_{n,k,l})^{U_n\times U_k\times U_l}$$

of $U_n \times U_k \times U_l$ invariants in $\mathcal{P}(V_{n,k,l})$ is a variant of the GL_n tensor product algebra in the following sense: it describes the decomposition of a tensor product of the form $\rho^D_n \otimes (\rho^E_n)^*$, where D and E are Young diagrams with at most k rows and l rows respectively, and $(\rho^E_n)^*$ is the contragredient representation of ρ^E_n ([HL1]).

3.2.2. *Multiplicity formula.* The algebra $\mathfrak{A}_2$ is a module for $A_n \times A_k \times A_l$, so it decomposes into a sum of joint $A_n \times A_k \times A_l$ eigenspaces. For Young diagrams D, E, F and G such that $r(D), r(F) \leq k$ and $r(E), r(G) \leq l$, the dimension of $\psi^{D,E}_n \times \psi^F_k \times \psi^G_l$ eigenspace is the multiplicity of $\rho^{D,E}_n$ in the tensor product $\rho^F_n \otimes (\rho^G_n)^*$, which is given by

$$m_2(\rho^{D,E}_n, \rho^F_n \otimes (\rho^G_n)^*) = \sum_{r(H)\leq\min(k,l)} c^F_{H,D}c^G_{H,E}$$

(see [HL1]). Here $\psi^{D,E}_n : A_n \to \mathbb{C}^\times$ is the character given by

$$\psi^{D,E}_n[\mathrm{diag}(a_1, ..., a_n)] = [a_1^{\lambda_1} \cdots a_k^{\lambda_k}][a^{-\mu_l}_{n-l+1} \cdots a_n^{-\mu_1}]. \tag{3.1}$$

where $D = (\lambda_1, ..., \lambda_k)$ and $E = (\mu_1, ..., \mu_l)$, and $\rho_n^{D,E}$ is the irreducible representation of GL_n with highest weight $\psi_n^{D,E}$.

3.2.3. *The lattice cone* Ω_2. Let $m = \min(k, l)$, and let $\mathcal{S}_2$ be the subspace of $\mathbb{R}^{N_{(k,m,k)}} \oplus \mathbb{R}^{N_{(l,m,l)}}$ consisting of (A, B) such that $A = [a_{ij}]$, $B = [b_{ij}]$ and

$$a_{0j} = b_{0j}, \qquad 1 \le j \le m.$$

Then $\dim \mathcal{S}_2 = N_2$, where

$$N_2 = N_{(k,m,k)} + N_{(l,m,l)} - m.$$

So $\mathcal{S}_2$ can be identified with $\mathbb{R}^{N_2}$. Let

$$\mathcal{C}_2 = \mathcal{S}_2 \cap (\mathbf{LR}_{k,m,k} \times \mathbf{LR}_{l,m,l}).$$

Then $\mathcal{C}_2$ is a rational polyhedral cone in $\mathcal{S}_2$, and the intersection

$$\Omega_2 = \mathcal{C}_2 \cap \mathbb{Z}^{N_2}$$

is a lattice cone in $\mathcal{S}_2$.

If $D = (\lambda_1, ..., \lambda_k)$, $E = (\nu_1, ..., \nu_l)$, $F = (\mu_1, ..., \mu_k)$ and $G = (\alpha_1, ..., \alpha_l)$ are Young diagrams, let $\mathcal{C}_2(D, E; F, G)$ be the set of points (A, B) in $\mathcal{C}_2$ such that

$$\begin{cases} A = [a_{ij}], \ B = [b_{ij}] \\ \sum_{q=i}^{k} a_{iq} = \lambda_i, \ \sum_{p=0}^{i} a_{pi} = \mu_i \quad \text{for } 1 \le i \le k, \\ \sum_{q=i}^{l} b_{iq} = \nu_i, \ \sum_{p=0}^{i} b_{pi} = \alpha_i \quad \text{for } 1 \le i \le l. \end{cases}$$

Note that $\mathcal{C}_2(D, E; F, G)$ is a polytope contained in $\mathcal{C}_2$, and $m_2(\rho_n^{D,E}, \rho_n^F \otimes (\rho_n^G)^*)$ is the number of integral points in $\mathcal{C}_2(D, E; F, G)$. Since

$$\Omega_2 = \bigcup_{D,E,F,G} \mathcal{C}_2(D, E; F, G) \cap \mathbb{Z}^{N_2},$$

Ω_2 can be used to label a basis for $\mathfrak{A}_2$.

3.2.4. *A basis for* $\mathfrak{A}_2$. In [HL1], we have constructed a basis $\mathcal{B}_2 = \{\Delta^{(2)}_{(A,B)} : (A, B) \in \Omega_2\}$ for $\mathfrak{A}_2$ with the following properties: Write a typical element of M_{nk} and M_{nl} as $X = (x_{ij})$, $\hat{Y} = (\hat{y}_{ij})$. For $1 \le i \le n$ and $1 \le j \le l$, let $y_{ij} = \hat{y}_{n+i-1,j}$. We also let for $1 \le a \le k$ and $1 \le b \le l$, $r_{ab}^2 = \sum_{i=1}^{n} x_{ia} y_{i,n-b+1}$. Then the set of variables

$$S_2 = \{x_{ab}, y_{cd}, r_{ef}^2 : \ 1 \le a, b, e \le k, \ 1 \le c, d, f \le l\}$$

is algebraically independent. Let $\mathbb{C}[S_2]$ be the subalgebra of $\mathcal{P}(V_{n,k,l})$ generated by S_2. Then $\mathbb{C}[S_2]$ is itself a polynomial algebra and $\mathfrak{A}_2 \subseteq \mathbb{C}[S_2]$. Now we let τ_2 be the following monomial ordering on $\mathbb{C}[S_2]$: it is the graded lexicographic order [CLO] such that

$$\begin{gathered} r_{11}^2 > r_{12}^2 > \cdots > r_{1l}^2 > r_{22}^2 > r_{23}^2 > \cdots > r_{kl}^2 \\ > x_{11} > x_{12} > \cdots > x_{1k} > x_{21} > \cdots > x_{kk} \\ > y_{11} > y_{12} > \cdots > y_{1l} > y_{21} > \cdots > y_{ll}. \end{gathered}$$

Then for each $(A,B) \in \Omega_2$ where $A = [a_{ij}]$ and $B = [b_{ij}]$,

$$\mathrm{in}_{\tau_2}(\Delta^{(2)}_{(A,B)}) = \left(\prod_{j=1}^{m} (r^2_{jj})^{a_{0j}}\right) \left(\prod_{1\le i\le h\le k} x_{ih}^{a_{ih}}\right) \left(\prod_{1\le i\le h\le l} y_{ih}^{b_{ih}}\right).$$

3.3. $\mathfrak{A}_3$: **stable branching algebras for** $(\mathrm{GL}_n, \mathrm{O}_n)$**.** This algebra describes the restriction of an irreducible representation of GL_n to O_n.

3.3.1. *The definition of* $\mathfrak{A}_3$. Let n and k be positive integers, and let $\mathrm{O}_n \times \mathrm{GL}_k$ act on M_{nk} by

$$(g,h).X = gXh^{-1}, \quad g \in \mathrm{O}_n,\ h \in \mathrm{GL}_k,\ X \in \mathrm{M}_{nk}.$$

We extend this action to the algebra $\mathcal{P}(\mathrm{M}_{nk})$ of polynomial functions on M_{nk} in the usual way. Let U_k and U_{SO_n} denote the standard maximal unipotent subgroups of GL_k and SO_n respectively, and consider the algebra $\mathcal{P}(\mathrm{M}_{nk})^{U_{\mathrm{SO}_n}\times U_k}$ of $U_{\mathrm{SO}_n} \times U_k$ invariants in $\mathcal{P}(\mathrm{M}_{nk})$.

We shall assume that $2k < n$, which is the stable range condition. In this case, the algebra structure of $\mathcal{P}(\mathrm{M}_{nk})^{U_{\mathrm{SO}_n}\times U_k}$ describes how an irreducible representation ρ_n^F of GL_n indexed by a Young diagram F with at most k rows decomposes under the action of O_n ([HTW1], [HL2]). In view of this property, we call $\mathcal{P}(\mathrm{M}_{nk})^{U_{\mathrm{SO}_n}\times U_k}$ a *stable branching algebra* for $(\mathrm{GL}_n, \mathrm{O}_n)$. To analyze the structure of this algebra, it is more convenient to replace it by the algebra $\mathfrak{A}_3$ defined as follows.

Let SM_k be the space of all $k \times k$ complex symmetric matrices, and let $\mathrm{GL}_n \times \mathrm{GL}_k$ act on $\mathrm{M}_{nk} \oplus \mathrm{SM}_k$ by

$$(g,h)(T,S) = \left((g^{-1})^t T h^{-1}, (h^{-1})^t S h^{-1}\right), \quad g \in \mathrm{GL}_n, h \in \mathrm{GL}_k.$$

This induces an action of $\mathrm{GL}_n \times \mathrm{GL}_k$ on the polynomial algebra $\mathcal{P}\left(\mathrm{M}_{nk} \oplus \mathrm{SM}_k\right)$. We let

$$\mathfrak{A}_3 = \mathfrak{A}_{(\mathrm{GL}_n,\mathrm{O}_n,k)} = \mathcal{P}\left(\mathrm{M}_{nk} \oplus \mathrm{SM}_k\right)^{U_n\times U_k}$$

be the subalgebra of $U_n \times U_k$-invariants in $\mathcal{P}\left(\mathrm{M}_{nk} \oplus \mathrm{SM}_k\right)$. Then the algebras $\mathfrak{A}_3$ and $\mathcal{P}(\mathrm{M}_{nk})^{U_{\mathrm{SO}_n}\times U_k}$ are isomorphic (see [HL2]).

3.3.2. *Branching multiplicity.* The algebra $\mathfrak{A}_3$ is a module for $A_n \times A_k$, so it decomposes into a sum of joint $A_n \times A_k$ eigenspaces. For each Young diagram D and F with at most k rows, the dimension of the $\psi_n^D \times \psi_k^F$ eigenspace in $\mathfrak{A}_3$ coincides with the multiplicity of σ_n^D in ρ_n^F, which is given by

$$m_3(\sigma_n^D, \rho_n^F) = \sum_{r(E)\le k} c^F_{2E,D}$$

(see [HL2]). Here σ_n^D is the irreducible representation of O_n indexed by the Young diagram D ([Ho], [Wy]).

3.3.3. *The lattice cone* Ω_3. Let $N_3 = k(k+3)/2$ and $\mathcal{C}_3 = \mathbf{LR}_k$. Then $\mathcal{C}_3$ is a rational polyhedral cone in $\mathbb{R}^{N_3}$. We also let

$$L_3 = \{A = [a_{ij}] \in \mathbb{Z}^{N_3} : a_{0j} \text{ is even for } 1 \leq j \leq k \},$$

and

$$\Omega_3 = \mathcal{C}_3 \cap L_3.$$

Then L_3 is a finite index sublattice of $\mathbb{Z}^{N_3}$, so Ω_3 is a lattice cone in $\mathbb{R}^{N_3}$.

If $D = (\lambda_1, ..., \lambda_k)$ and $F = (\mu_1, ..., \mu_k)$ are Young diagrams, let $\mathcal{C}_3(D, F)$ be the set of points $A = [a_{ij}]$ in $\mathcal{C}_3$ such that $\mu_j = \sum_{p=0}^{j} a_{pj}$ and $\lambda_j = \sum_{q=j}^{n} a_{jq}$ for $1 \leq j \leq k$. Then $\mathcal{C}_3(D, F)$ is a polytope contained in $\mathcal{C}_3$, and $m_3(\sigma_n^D, \rho_n^F)$ is the number of points in the intersection $\mathcal{C}_3(D, F) \cap L_3$. Moreover,

$$\Omega_3 = \bigcup_{r(D), r(F) \leq k} L_3 \cap \mathcal{C}_3(D, F).$$

So Ω_3 can be used to label a basis for $\mathfrak{A}_3$.

3.3.4. *A basis for* $\mathfrak{A}_3$. In [HL2], we have constructed a basis $\mathcal{B}_3 = \{\Delta_A^{(3)} : A \in \Omega_3\}$ for $\mathfrak{A}_3$ with the following property: Write the typical elements of M_{nk} and SM_k as $X = (x_{ij})$ and $R = (r_{ij}^2)$ respectively. Let τ_3 be the following monomial ordering on $\mathcal{P}(\mathrm{M}_{nk} \oplus \mathrm{SM}_k)$: it is the graded lexicographic order ([CLO]) such that

$$r_{11}^2 > r_{12}^2 > \cdots > r_{1k}^2 > r_{22}^2 > r_{23}^2 > \cdots > r_{kk}^2 > x_{11} > x_{12} > \cdots > x_{1k} > x_{21} > \cdots > x_{nk}.$$

Then for each $A = [a_{ij}] \in \Omega_3$,

$$\mathrm{in}_{\tau_3}(\Delta_A^{(3)}) = \left(\prod_{j=1}^{k} (r_{jj}^2)^{a_{0j}/2} \right) \left(\prod_{1 \leq i \leq h \leq k} x_{ih}^{a_{ih}} \right).$$

3.4. $\mathfrak{A}_4$: **stable branching algebras for** $(\mathrm{O}_{n+m}, \mathrm{O}_n \times \mathrm{O}_m)$**.** This algebra describes the restriction of an irreducible representation of O_{n+m} to $\mathrm{O}_n \times \mathrm{O}_m$.

3.4.1. *The definition of* $\mathfrak{A}_4$. Let n, m and k be positive integers and let $(.,.)$ be the O_{n+m} invariant symmetric bilinear form on $\mathbb{C}^{n+m}$. For $1 \leq i, j \leq k$ and $T \in \mathrm{M}_{n+m,k}$, let $r_{ij}^2(T) = (T_i, T_j)$ where T_i and T_j are the i-th and the j-th columns of T respectively. Let

$$\mathcal{N} = \left\{ T \in \mathrm{M}_{n+m,k} : \ r_{ij}^2(T) = 0 \ \text{ for } 1 \leq i, j \leq k \right\},$$

and let $\mathcal{R}(\mathcal{N})$ be the algebra of regular functions on $\mathcal{N}$. The algebra $\mathcal{R}(\mathcal{N})$ is an $\mathrm{O}_{n+m} \times \mathrm{GL}_k$ module. We consider the subalgebra $\mathcal{R}(\mathcal{N})^{U_{\mathrm{SO}_n} \times U_{\mathrm{SO}_m} \times U_k}$ of $U_{\mathrm{SO}_n} \times U_{\mathrm{SO}_m} \times U_k$ invariants in $\mathcal{R}(\mathcal{N})$.

We shall assume that $2k < \min(n, m)$, which is the stable range condition. In this case, the algebra structure of $\mathcal{R}(\mathcal{N})^{U_{\mathrm{SO}_n} \times U_{\mathrm{SO}_m} \times U_k}$ describes how an irreducible representation σ_{n+m}^D of O_{n+m} indexed by a Young diagram D with at most k rows decomposes under the action of $\mathrm{O}_n \times \mathrm{O}_m$ ([HL2]). In view of this property, we shall call the algebra $\mathcal{R}(\mathcal{N})^{U_{\mathrm{SO}_n} \times U_{\mathrm{SO}_m} \times U_k}$ a *stable branching algebra for* $(\mathrm{O}_{n+m}, \mathrm{O}_n \times \mathrm{O}_m)$. To analyze the algebra structure of $\mathcal{R}(\mathcal{N})^{U_{\mathrm{SO}_n}\times U_{\mathrm{SO}_m}\times U_k}$, it is more convenient to replace it by the algebra $\mathfrak{A}_4$ defined as follows.

Let $\mathrm{GL}_n \times \mathrm{GL}_m \times \mathrm{GL}_k$ act on $\mathrm{M}_{nk} \oplus \mathrm{M}_{mk} \oplus \mathrm{SM}_k$ by

$$(g_1, g_2, h)(X, Y, R) = \big((g_1^{-1})^t X h^{-1}, (g_2^{-1})^t X h^{-1}, (h^{-1})^t R h^{-1}\big)$$

where $(g_1, g_2, h) \in \mathrm{GL}_n \times \mathrm{GL}_m \times \mathrm{GL}_k$, $X \in \mathrm{M}_{nk}$, $Y \in \mathrm{M}_{mk}$ and $R \in \mathrm{SM}_k$. This action induces an action of $\mathrm{GL}_n \times \mathrm{GL}_m \times \mathrm{GL}_k$ on the algebra $\mathcal{P}\left(\mathrm{M}_{nk} \oplus \mathrm{M}_{mk} \oplus \mathrm{SM}_k\right)$ of polynomial functions on $\mathrm{M}_{nk} \oplus \mathrm{M}_{mk} \oplus \mathrm{SM}_k$). We let

$$\mathfrak{A}_4 = \mathfrak{A}_{(\mathrm{O}_{n+m}, \mathrm{O}_n\times \mathrm{O}_m, k)} = \mathcal{P}\left(\mathrm{M}_{nk} \oplus \mathrm{M}_{mk} \oplus \mathrm{SM}_k\right)^{U_n\times U_m\times U_k}$$

be the subalgebra of $U_n \times U_m \times U_k$ invariants in $\mathcal{P}\left(\mathrm{M}_{nk} \oplus \mathrm{M}_{mk} \oplus \mathrm{SM}_k\right)$. Then the algebras $\mathfrak{A}_4$ and $\mathcal{R}(\mathcal{N})^{U_{\mathrm{SO}_n}\times U_{\mathrm{SO}_m}\times U_k}$ are isomorphic (see [HL2]).

3.4.2. *Branching multiplicity.* The algebra $\mathfrak{A}_4$ is a module for $A_n \times A_m \times A_k$, so it decomposes into a sum of joint $A_n \times A_m \times A_k$ eigenspaces. For Young diagrams D, E and F with at most k rows, the dimension of the $\psi_n^D \times \psi_m^E \times \psi_k^F$ eigenspace in $\mathfrak{A}_4$ coincides with the multiplicity of the representation $\sigma_n^D \otimes \sigma_m^E$ of $\mathrm{O}_n \times \mathrm{O}_m$ in the representation σ_{n+m}^F of O_{n+m}, which is given by ([HTW2], [HL2])

$$m_4(\sigma_n^D \otimes \sigma_m^E, \sigma_{n+m}^F) = \sum_{r(G), r(H) \leq k} c_{2H,D}^G c_{G,E}^F.$$

3.4.3. *The lattice cone* Ω_4. Let $\mathcal{S}_4$ be the subspace of $\mathbb{R}^{k(k+3)/2} \oplus \mathbb{R}^{k(k+3)/2}$ containing (A, B) such that $A = [a_{ij}]$, $B = [b_{ij}]$ and

$$\sum_{i=0}^{j} a_{ij} = b_{0j} \quad \text{ for } 1 \leq j \leq k.$$

Then $\dim \mathcal{S}_4 = N_4$ where

$$N_4 = k(k+3) - k = k(k+2).$$

So $\mathcal{S}_4$ can be identified with $\mathbb{R}^{N_4}$. Let

$$\mathcal{C}_4 = \mathcal{S}_4 \cap (\mathbf{LR}_k \times \mathbf{LR}_k),$$

which is a rational polyhedral cone in $\mathcal{S}_4$. Let

$$L_4 = \{(A, B) \in \mathbb{Z}^{N_4} : \ A = [a_{ij}], \ a_{0j} \text{ is even for } 1 \leq j \leq k\}.$$

Then L_4 is a finite index sublattice of $\mathbb{Z}^{N_4}$, so Ω_4 is a lattice cone in $\mathbb{R}^{N_4}$.

Now if $D = (\lambda_1, ..., \lambda_k)$, $E = (\nu_1, ..., \nu_k)$ and $F = (\mu_1, ..., \mu_k)$ are Young diagrams, let $\mathcal{C}_4(D, E, F)$ be the set of points (A, B) in $\mathcal{C}_4$ such that $A = [a_{ij}]$, $B = [b_{ij}]$, $\lambda_j = \sum_{q=j}^k a_{jq}$, $\nu_j = \sum_{q=j}^k b_{jq}$ and $\mu_j = \sum_{p=0}^j b_{pj}$, for $1 \leq j \leq k$. Then $\mathcal{C}_4(D, E, F)$ is a polytope contained in $\mathcal{C}_4$, and $m_4(\sigma_n^D \otimes \sigma_m^E, \sigma_{n+m}^F)$ is the number of points in the intersection $\mathcal{C}_4(D, E, F) \cap L_4$. Note that

$$\Omega_4 = \bigcup_{r(D), r(E), r(F) \leq k} \mathcal{C}_4(D, E, F) \cap L_4.$$

So Ω_4 can be used to label a basis for $\mathfrak{A}_4$.

3.4.4. *A basis for* $\mathfrak{A}_4$. In [HL2], we have constructed a basis $\mathcal{B}_4 = \{\Delta^{(4)}_{(A,B)} : (A,B) \in \Omega_4\}$ for $\mathfrak{A}_4$ with the following property: Write the typical elements in M_{nk}, M_{mk} and SM_k as $X = (x_{ij})$, $Y = (y_{ih})$ and $R = (r^2_{ab})$ respectively. Let τ_4 be the following monomial ordering on $\mathcal{P}(\mathrm{M}_{nk} \oplus \mathrm{M}_{mk} \oplus \mathrm{SM}_k)$: it is the graded lexicographic order ([CLO]) such that

$$\begin{array}{l} r^2_{11} > r^2_{12} > \cdots > r^2_{1k} > r^2_{22} > r^2_{23} > \cdots > r^2_{kk} \\ > x_{11} > x_{12} > \cdots > x_{1k} > x_{21} > \cdots > x_{nk} \\ > y_{11} > y_{12} > \cdots > y_{1l} > y_{21} > \cdots > y_{mk}. \end{array}$$

Then for each $(A,B) \in \Omega_4$ where $A = [a_{ij}]$ and $B = [b_{ij}]$,

$$\mathrm{in}_{\tau_4}(\Delta^{(5)}_{(A,B)}) = \left(\prod_{j=1}^{k} (r^2_{jj})^{a_{0j}/2}\right)\left(\prod_{1\le i\le h\le k} x^{a_{ih}}_{ih}\right)\left(\prod_{1\le i\le h\le k} y^{b_{ih}}_{ih}\right).$$

3.5. $\mathfrak{A}_5$: **stable branching algebra for** $(\mathrm{Sp}_{2n}, \mathrm{GL}_n)$**.** This algebra describes the restriction of an irreducible representation of Sp_{2n} to GL_n.

3.5.1. *The definition of* $\mathfrak{A}_5$. Let n and k be positive integers. Consider the following subgroup of Sp_{2n}:

$$\left\{ \begin{pmatrix} g & 0 \\ 0 & (g^{-1})^t \end{pmatrix} : g \in \mathrm{GL}_n \right\}.$$

It is isomorphic to GL_n. So we shall denote this subgroup also by GL_n, and its standard unipotent subgroup by U_n.

We shall write a typical element of $\mathrm{M}_{2n,k}$ as $T = \begin{pmatrix} X \\ \hat{Y} \end{pmatrix}$ where

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nk} \end{pmatrix} \quad \text{and} \quad \widehat{Y} = \begin{pmatrix} \hat{y}_{11} & \hat{y}_{12} & \cdots & \hat{y}_{1k} \\ \hat{y}_{21} & \hat{y}_{22} & \cdots & \hat{y}_{2k} \\ \vdots & \vdots & & \vdots \\ \hat{y}_{n1} & \hat{y}_{n2} & \cdots & \hat{y}_{nk} \end{pmatrix}.$$

For $1 \le i < j \le k$, let $\xi_{ij}(T) = \sum_{a=1}^{n} (x_{ai}\hat{y}_{aj} - \hat{y}_{ai}x_{aj})$. These polynomials generate the algebra of Sp_{2n} invariants in $\mathcal{P}(\mathrm{M}_{2n,k})$. Let

$$\mathcal{V} = \{T \in \mathrm{M}_{2n,k} : \ \xi_{ij}(T) = 0, \quad \text{for } 1 \le i,j \le k\},$$

and let $\mathcal{R}(\mathcal{V})$ be the algebra of regular functions on $\mathcal{V}$. Then $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ act on $\mathcal{R}(\mathcal{V})$ by

$$[(g,h).f](T) = f(gTh^t), \qquad g \in \mathrm{Sp}_{2n}, h \in \mathrm{GL}_k, f \in \mathcal{R}(\mathcal{V}), T \in \mathrm{M}_{2n,k},$$

and we consider the subalgebra $\mathcal{R}(\mathcal{V})^{U_n \times U_k}$ of $U_n \times U_k$ invariants in $\mathcal{R}(\mathcal{V})$.

We shall assume that $2k \le n$, which is the stable range condition. In this case, the algebra structure of $\mathcal{R}(\mathcal{V})^{U_n \times U_k}$ describes how an irreducible representation τ^F_{2n} of Sp_{2n} indexed by a Young diagram F with at most k rows decomposes under the action of GL_n ([HL2]). In view of this property, we shall call $\mathcal{R}(\mathcal{V})^{U_n\times U_k}$ a *stable branching algebra for* $(\mathrm{Sp}_{2n},\mathrm{GL}_n)$, and we will denote it by

$$\mathfrak{A}_5=\mathfrak{A}_{(\mathrm{Sp}_{2n},\mathrm{GL}_n,k)}.$$

3.5.2. *Branching multiplicity.* The algebra $\mathfrak{A}_5$ is a module for $A_n\times A_k$. For Young diagrams D, E and F with at most k rows, the dimension of the $\psi_n^{D,E}\times\psi_k^F$ eigenspace of $\mathcal{R}(\mathcal{V})^{U_n\times U_k}$ coincides with the multiplicity of the representation $\rho_n^{D,E}$ of GL_n in the representation τ_{2n}^F of Sp_{2n} indexed by the Young diagram F, which is given by

$$m_5(\rho_n^{D,E},\tau_{2n}^F)=\sum_{r(G),r(H)\le k}c_{2H,D}^{G}c_{G,E}^{F}$$

(see [HTW2], [HL2]).

3.5.3. *The lattice cone* Ω_5. We note that the multiplicity $m_5(\rho_n^{D,E},\tau_{2n}^F)$ and the multiplicity $m_4(\sigma_n^D\otimes\sigma_m^E,\sigma_{n+m}^F)$ given in Section 3.4.2 are identical. This suggests that we can use the same lattice cone as the algebra $\mathfrak{A}_4$. Specifically we let $\mathcal{S}_5=\mathcal{S}_4$, $\mathcal{C}_5=\mathcal{C}_4$, $L_5=L_4$, and $\Omega_5=\Omega_4$.

3.5.4. *A basis for* $\mathfrak{A}_5$. In [HL2], we have constructed a basis $\mathcal{B}_5=\{\Delta^{(5)}_{(A,B)}:\ (A,B)\in\Omega_5\}$ for $\mathfrak{A}_5$ with the following property: For $1\le a\le b\le k$, let $w_{ab}=\sum_{i=1}^n(x_{ia}\hat{y}_{ib}+x_{ib}\hat{y}_{ia})$. Then the set of polynomials

$$S_5=\{x_{ij},\hat{y}_{n-i+1,j}:1\le i,j\le k\}\cup\{w_{ab}:1\le a\le b\le k\}$$

of $\mathcal{P}(\mathrm{M}_{2n,k})$ is algebraically independent. Let $\mathbb{C}[S_5]$ be the subalgebra of $\mathcal{P}(\mathrm{M}_{2n,k})$ generated by S_5. Then $\mathbb{C}[S_5]$ is itself a polynomial algebra and $\mathfrak{A}_5$ can be identified with a subalgebra of $\mathbb{C}[S_5]$ ([HL2]). Let τ_5 be the following monomial ordering on $\mathbb{C}[S_5]$: it is the graded lexicographic order ([CLO]) such that

$$\begin{aligned}&w_{11}>w_{12}>\cdots>w_{1k}>w_{22}>w_{23}>\cdots>w_{kk}\\&\quad>x_{11}>x_{12}>\cdots>x_{1k}>x_{21}>\cdots>x_{kk}\\&\quad>y_{11}>y_{12}>\cdots>y_{1k}>y_{21}>\cdots>y_{kk},\end{aligned}$$

where $y_{ij}=\hat{y}_{n-i+1,j}$ for $1\le i\le n$ and $1\le j\le k$. Then for each $(A,B)\in\Omega_5$ where $A=[a_{ij}]$ and $B=[b_{ij}]$,

$$\mathrm{in}_{\tau_5}(\Delta^{(5)}_{(A,B)})=\left(\prod_{j=1}^{k}(w_{jj})^{a_{0j}/2}\right)\left(\prod_{1\le i\le h\le k}x_{ih}^{a_{ih}}\right)\left(\prod_{1\le i\le h\le k}y_{ih}^{b_{ih}}\right).$$

3.6. $\mathfrak{A}_6$: **stable branching algebra for** $(\mathrm{GL}_{2n},\mathrm{Sp}_{2n})$**.** This algebra describes the restriction of an irreducible representation of GL_{2n} to Sp_{2n}.

3.6.1. *The definition of* $\mathfrak{A}_6$. Let n and k be positive integers and let $\mathcal{P}(\mathrm{M}_{2n,k})$ be the algebra of polynomial functions on $\mathrm{M}_{2n,k}$. Let $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ act on $\mathcal{P}(\mathrm{M}_{2n,k})$ by

$$[(g,h)f](T) = f(gTh^t), \quad g \in \mathrm{Sp}_{2n},\ h \in \mathrm{GL}_k,\ f \in \mathcal{P}(\mathrm{M}_{2n,k}),\ T \in \mathrm{M}_{2n,k}.$$

Let $U_{\mathrm{Sp}_{2n}}$ denote the standard maximal unipotent subgroup of Sp_{2n}, and consider the subalgebra $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_k}$ of $U_{\mathrm{Sp}_{2n}} \times U_k$ invariants in $\mathcal{P}(\mathrm{M}_{2n,k})$.

We shall assume that $k \leq n$, which is the stable range condition. In this case, the algebra structure of $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_k}$ describes how an irreducible representation ρ_{2n}^F of GL_{2n} indexed by a Young diagram F, with at most k rows, decomposes under the action of Sp_{2n} ([HTW2], [HL3]). In view of this property, we call the algebra $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_k}$ a *stable branching algebra for* for $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$. To analyze the algebra structure of $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_k}$, it is more convenient to replace $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_k}$ by the algebra $\mathfrak{A}_6$ defined as follows.

Let AM_k denote the space of all $k \times k$ complex skew-symmetric matrices, and let $\mathrm{GL}_{2n} \times \mathrm{GL}_k$ act on $\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k$ by

$$(g,h)(T,S) = \left((g^{-1})^t T h^{-1}, (h^{-1})^t S h^{-1}\right), \quad g \in \mathrm{GL}_{2n},\ h \in \mathrm{GL}_k,\ T \in \mathrm{M}_{2n,k},\ S \in \mathrm{AM}_k.$$

This action induces an action of $\mathrm{GL}_{2n} \times \mathrm{GL}_k$ on the algebra $\mathcal{P}\left(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k\right)$ of polynomial functions on $\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k$. We let

$$\mathfrak{A}_6 = \mathfrak{A}_{(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n}, k)} = \mathcal{P}\left(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k\right)^{U_{2n} \times U_k}$$

be the subalgebra of $U_{2n} \times U_k$ invariants in $\mathcal{P}\left(\mathrm{M}_{2n,k} \oplus \mathrm{AM}_k\right)$. Then the algebras $\mathfrak{A}_6$ and $\mathcal{P}(\mathrm{M}_{2n,k})^{U_{\mathrm{Sp}_{2n}} \times U_k}$ are isomorphic (see [HL3]).

3.6.2. *Branching multiplicity.* The algebra $\mathfrak{A}_6$ is a module for $A_{2n} \times A_k$, so it decomposes into a sum of joint $A_{2n} \times A_k$ eigenspaces. For Young diagrams D and F with at most k rows, the dimension of the $\psi_{2n}^D \times \psi_k^F$ eigenspace in $\mathfrak{A}_6$ coincides with the multiplicity of the representation τ_{2n}^D of Sp_{2n} in the representation ρ_{2n}^F of GL_{2n}, which is given by (see [HTW2], [HL3])

$$m_6(\tau_{2n}^D, \rho_{2n}^F) = \sum_{E \in \mathcal{E}_k} c_{E,D}^F$$

where

$$\mathcal{E}_k = \{E = (\mu_1, \mu_1, \mu_2, \mu_2, ..., \mu_m, \mu_m) :\ \mu_1 \geq \cdots \geq \mu_m \geq 0,\ 2m \leq k\}. \tag{3.1}$$

3.6.3. *The lattice cone* Ω_6. For a real number t, let $[t]$ denote the greatest integer less than or equal to t. Let $\mathcal{S}_6$ be the subspace of $\mathbb{R}^{N_{(k,2[k/2],k)}}$ consisting of $A = [a_{ij}]$ such that $a_{0,2j-1} = a_{0,2j}$ for $1 \leq j \leq [k/2]$. Then $\dim \mathcal{S}_6 = N_6$ where

$$N_6 = N_{(k,2[k/2],k)} - [k/2].$$

Let

$$\mathcal{C}_6 = \mathcal{S}_6 \cap \mathbf{LR}_{k,2[k/2],k}.$$

Then $\mathcal{C}_6$ is a rational polyhedral cone in $\mathcal{S}_6$, so the intersection

$$\Omega_6 = \mathcal{C}_6 \cap \mathbb{Z}^{N_6}$$

is a lattice cone in $\mathbb{R}^{N_6}$.

If $D = (\lambda_1, ..., \lambda_k)$ and $F = (\mu_1, ..., \mu_k)$ are Young diagrams, let $\mathcal{C}_6(D, F)$ be the set of points $A = [a_{ij}]$ in $\mathcal{C}_6$ such that $\mu_j = \sum_{p=0}^{j} a_{pj}$ and $\lambda_j = \sum_{q=j}^{k} a_{jq}$ for $1 \leq j \leq k$. Then $\mathcal{C}_6(D, F)$ is a polytope contained in $\mathcal{C}_6$, and $m_6(\tau_{2n}^D, \rho_{2n}^F)$ is the number of integral points in $\mathcal{C}_6(D, F)$. Since

$$\Omega_6 = \bigcup_{D,F} \mathcal{C}_6(D, F) \cap \mathbb{Z}^{N_6},$$

Ω_6 can be used to label a basis for $\mathfrak{A}_6$.

3.6.4. *A basis for* $\mathfrak{A}_6$. In [HL3], we have constructed a basis $\mathcal{B}_6 = \{\Delta_A^{(6)} : A \in \Omega_6\}$ for $\mathfrak{A}_6$ with the following property: Write typical elements of $\mathrm{M}_{2n,k}$ and AM_k as $X = (x_{ij})$ and $\xi = (\xi_{ij})$ respectively. Let τ_6 be the following monomial ordering: it is the graded lexicographic order ([CLO]) such that

$$\xi_{12} > \cdots > \xi_{1k} > \xi_{23} > \cdots > \xi_{2n} > \xi_{34} > \cdots > \xi_{k-1,k}$$
$$> x_{11} > x_{12} > \cdots > x_{1k} > x_{21} > \cdots > x_{nk}.$$

Then for each $A = [a_{ij}] \in \Omega_6$,

$$\mathrm{in}_{\tau_6}(\Delta_A^{(6)}) = \left(\prod_{j=1}^{[k/2]} \xi_{2j-1,2j}^{a_{0,2j}} \right) \left(\prod_{1 \leq i \leq h \leq k} x_{ih}^{a_{ih}} \right).$$

3.7. $\mathfrak{A}_7$: **stable branching algebras for** $(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$**.** This algebra describes the restriction of an irreducible representation of $\mathrm{Sp}_{2(n+m)}$ to $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$.

3.7.1. *The definition of* $\mathfrak{A}_7$. Let n, m and k be positive integers and let $\langle ., . \rangle$ be the $\mathrm{Sp}_{2(n+m)}$-invariant skew-symmetric bilinear form on $\mathbb{C}^{2(n+m)}$. For $1 \leq i < j \leq k$ and $T \in \mathrm{M}_{2(n+m),k}$, let $\xi_{ij}(T) = \langle T_i, T_j \rangle$ where T_i and T_j are the i-th and the j-th columns of T respectively. Let

$$\mathcal{V} = \left\{ T \in \mathrm{M}_{2(n+m),k} : \ \xi_{ij}(T) = 0 \ \text{ for } 1 \leq i < j \leq k \right\},$$

and $\mathcal{R}(\mathcal{V})$ the algebra of regular functions on $\mathcal{V}$. Let $\mathrm{Sp}_{2(n+m)} \times \mathrm{GL}_k$ act on $\mathcal{R}(\mathcal{V})$ by

$$[(g, h)f](T) = f(gTh^t), \quad g \in \mathrm{Sp}_{2(n+m)}, h \in \mathrm{GL}_k, f \in \mathcal{R}(\mathcal{V}), T \in \mathrm{M}_{2(n+m),k}.$$

We consider the subalgebra $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_k}$ of $U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_k$ invariants in $\mathcal{R}(\mathcal{V})$.

We shall assume that $k \leq \min(n, m)$, which is the stable range condition. In this case, the algebra structure of $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_k}$ describes how an irreducible representation $\tau_{2(n+m)}^D$ of $\mathrm{Sp}_{2(n+m)}$ indexed by a Young diagram D with at most k rows decomposes under the action of $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$ ([HL3]). In view of this property, we call the algebra $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_k}$ a *stable branching algebra for* $(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$. To analyze the algebra structure of $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_k}$, it is more convenient to replace it by the algebra $\mathfrak{A}_7$ defined as follows.

Let $\mathrm{GL}_{2n} \times \mathrm{GL}_{2m} \times \mathrm{GL}_k$ act on $\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k$ by

$$(g_1, g_2, h)(X, Y, Z) = \left((g_1^{-1})^t X h^{-1}, (g_2^{-1})^t X h^{-1}, (h^{-1})^t Z h^{-1}\right)$$

where $(g_1, g_2, h) \in \mathrm{GL}_{2n} \times \mathrm{GL}_{2m} \times \mathrm{GL}_k$, $X \in \mathrm{M}_{nk}$, $Y \in \mathrm{M}_{mk}$ and $Z \in \mathrm{AM}_k$. This action induces an action of $\mathrm{GL}_{2n} \times \mathrm{GL}_{2m} \times \mathrm{GL}_k$ on the algebra $\mathcal{P}\left(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k\right)$ of polynomial functions on $\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k$. We let

$$\mathfrak{A}_7 = \mathfrak{A}_{(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}, k)} = \mathcal{P}\left(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k\right)^{U_{2n} \times U_{2m} \times U_k}$$

be the subalgebra of $U_{2n} \times U_{2m} \times U_k$ invariants in $\mathcal{P}\left(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k\right)$. Then the algebras $\mathfrak{A}_7$ and $\mathcal{R}(\mathcal{V})^{U_{\mathrm{Sp}_{2n}} \times U_{\mathrm{Sp}_{2m}} \times U_k}$ are isomorphic (see [HL3]).

3.7.2. *Branching multiplicity.* The algebra $\mathfrak{A}_7$ is a module for $A_{2n} \times A_{2m} \times A_k$, so it decomposes into a sum of joint $A_{2n} \times A_{2m} \times A_k$ eigenspaces. For Young diagrams D, E and F with at most k rows, the dimension of the $\psi_{2n}^D \times \psi_{2m}^E \times \psi_k^F$ eigenspace in $\mathfrak{A}_7$ coincides with the multiplicity of the representation $\tau_{2n}^D \otimes \tau_{2m}^E$ in the representation $\tau_{2(n+m)}^F$, which is given by

$$m_7(\tau_{2n}^D \otimes \tau_{2m}^E, \tau_{2(n+m)}^F) = \sum_{G \in \mathcal{E}_k, r(H) \leq k} c_{G,D}^H c_{H,E}^F$$

where the set $\mathcal{E}_k$ of Young diagrams is defined in equation (3.1) ([HTW2],[HL3]).

3.7.3. *The lattice cone* Ω_7. Let $\mathcal{S}_7$ be the subspace of $\mathbb{R}^{N_{(k,2[k/2],k)}} \oplus \mathbb{R}^{N_{(k,k,k)}}$ consisting of (A, B) such that $A = [a_{ij}]$, $B = [b_{ij}]$, $a_{0,2j-1} = a_{0,2j}$ for $1 \leq j \leq [k/2]$ and $\sum_{i=0}^{j} a_{ij} = b_{0j}$ for $1 \leq j \leq k$. Then $\dim \mathcal{S}_7 = N_7$ where

$$N_7 = N_{(k,2[k/2],k)} + N_{(k,k,k)} - [k/2] - k.$$

So $\mathcal{S}_7$ can be identified with $\mathbb{R}^{N_7}$. Let

$$\mathcal{C}_7 = \mathcal{S}_7 \cap \left(\mathbf{LR}_{k,2[k/2],k} \times \mathbf{LR}_k\right).$$

Then $\mathcal{C}_7$ is a rational polyhedral cone in $\mathcal{S}_7$, so the intersection

$$\Omega_7 = \mathcal{C}_7 \cap \mathbb{Z}^{N_7}$$

is a lattice cone in $\mathcal{S}_7$.

Now if $D = (\lambda_1, ..., \lambda_k)$, $E = (\nu_1, ..., \nu_k)$ and $F = (\mu_1, ..., \mu_k)$ are Young diagrams, let $\mathcal{C}_7(D, E, F)$ be the set of points (A, B) in $\mathcal{C}_8$ such that $A = [a_{ij}]$, $B = [b_{ij}]$, $\lambda_j = \sum_{q=j}^{k} a_{jq}$, $\nu_j = \sum_{q=j}^{k} b_{jq}$ and $\mu_j = \sum_{p=0}^{j} b_{pj}$, for $1 \leq j \leq k$. Then $\mathcal{C}_7(D, E, F)$ is a polytope contained in $\mathcal{C}_7$ and $m_7(\tau_{2n}^D \otimes \tau_{2m}^E, \tau_{2(n+m)}^F)$ is the number of integral points in $\mathcal{C}_7(D, E, F)$. Since

$$\Omega_7 = \bigcup_{D,E,F} \mathcal{C}_7(D, E, F) \cap \mathbb{Z}^{N_7},$$

Ω_7 can be used to label a basis for $\mathfrak{A}_7$.

3.7.4. *A basis for* $\mathfrak{A}_7$. In [HL3], we have constructed a basis $\mathcal{B}_7 = \{\Delta^{(7)}_{(A,B)} : (A,B) \in \Omega_7\}$ for $\mathfrak{A}_7$ with the following property: Write the typical elements in M_{nk}, M_{mk} and AM_k as $X = (x_{ij})$, $Y = (y_{ih})$ and $Z = (\nu_{ab})$ respectively. Let τ_7 be the following monomial ordering on $\mathcal{P}\left(\mathrm{M}_{2n,k} \oplus \mathrm{M}_{2m,k} \oplus \mathrm{AM}_k\right)$: it is the graded lexicographic order ([CLO]) such that

$$\begin{aligned} &\nu_{12} > \nu_{13} > \cdots > \nu_{1k} > \nu_{23} > \nu_{24} > \cdots > \nu_{k-1,k} \\ &\quad > x_{11} > x_{12} > \cdots > x_{1k} > x_{21} > \cdots > x_{nk} \\ &\quad > y_{11} > y_{12} > \cdots > y_{1l} > y_{21} > \cdots > y_{nl}. \end{aligned}$$

Then for each $(A,B) \in \Omega_7$ where $A = [a_{ij}]$ and $B = [b_{ij}]$,

$$\mathrm{in}_{\tau_7}(\Delta^{(7)}_{(A,B)}) = \left(\prod_{j=1}^{[k/2]} \nu_{2j-1,2j}^{a_{0,2j}}\right)\left(\prod_{1\le i\le h\le k} x_{ih}^{a_{ih}}\right)\left(\prod_{1\le i\le h\le k} y_{ih}^{b_{ih}}\right).$$

3.8. $\mathfrak{A}_8$: **stable branching algebras for** $(\mathrm{O}_{2n}, \mathrm{GL}_n)$**.** This algebra $\mathfrak{A}_8$ describes the restriction of an irreducible representation of O_{2n} to GL_n.

3.8.1. *The definition of* $\mathfrak{A}_8$. Let J be the $n \times n$ matrix with 1's on its anti-diagonal and 0 elsewhere. We consider the following subgroup of O_{2n}:

$$\left\{\begin{pmatrix} a & 0 \\ 0 & J(a^{-1})^t J \end{pmatrix} : \ a \in \mathrm{GL}_n\right\}.$$

It is isomorphic to GL_n. So we shall denote this subgroup also by GL_n, and its standard unipotent subgroup by U_n.

We shall write a typical element in $\mathrm{M}_{2n,k}$ as $\begin{pmatrix} Z \\ W \end{pmatrix}$ where

$$Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1k} \\ z_{21} & z_{22} & \dots & z_{2k} \\ \vdots & \vdots & & \vdots \\ z_{n1} & z_{n2} & \dots & z_{nk} \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1k} \\ w_{21} & w_{22} & \dots & w_{2k} \\ \vdots & \vdots & & \vdots \\ w_{n1} & w_{n2} & \dots & w_{nk} \end{pmatrix}.$$

For $1 \le i,j \le k$, let $r^2_{ij}(T) = \sum_{a=1}^n (z_{a,i}w_{n-a+1,j} + w_{n-a+1,i}z_{a,j})$. These polynomials generate the algebra of O_{2n} invariants in $\mathcal{P}(\mathrm{M}_{2n,k})$. Let

$$\mathcal{N} = \left\{T \in \mathrm{M}_{2n,k} : \ r^2_{ij}(T) = 0 \ \text{ for } 1 \le i,j \le k\right\},$$

and let $\mathcal{R}(\mathcal{N})$ be the algebra of regular functions on $\mathcal{N}$. Then O_{2n} and GL_k act on $\mathcal{R}(\mathcal{N})$ by

$$[(g,h)f](T) = f(g^tTh), \quad g \in \mathrm{O}_{2n}, \ h \in \mathrm{GL}_k, \ f \in \mathcal{R}(\mathcal{N}), \ T \in \mathcal{N},$$

and we consider the subalgebra $\mathcal{R}(\mathcal{N})^{U_n \times U_k}$ of $U_n \times U_k$ invariants in $\mathcal{R}(\mathcal{N})$.

We shall assume that $2k \le n$, which is the stable range condition. In this case, the algebra structure of $\mathcal{R}(\mathcal{N})^{U_n \times U_k}$ describes how an irreducible representation σ^D_{2n} of O_{2n} indexed by a Young diagram D with at most k rows decomposes under the action of GL_n ([HL3]). In view of this property, we call the algebra $\mathcal{R}(\mathcal{N})^{U_n\times U_k}$ a *stable branching algebra for* $(\mathrm{O}_{2n}, \mathrm{GL}_n)$, and we denote it by

$$\mathfrak{A}_8 = \mathfrak{A}_{(\mathrm{O}_{2n},\mathrm{GL}_n,k)}.$$

3.8.2. *Branching multiplicity.* The algebra $\mathfrak{A}_8$ is a module for $A_n \times A_k$. For Young diagrams D, E and F with at most k rows, the dimension of the $\psi_n^{D,E} \times \psi_k^F$ eigenspace of $\mathcal{R}(\mathcal{N})^{U_n\times U_k}$ coincides with the multiplicity of the representation $\rho_n^{D,E}$ of GL_n in the representation σ_{2n}^F of O_{2n}, which is given by

$$m_8(\rho_n^{D,E}, \sigma_{2n}^F) = \sum_{G\in\mathcal{E}_k, r(H)\le k} c^H_{G,D} c^F_{H,E}.$$

3.8.3. *The lattice cone* Ω_8. We note that the multiplicity $m_8(\rho_n^{D,E}, \sigma_{2n}^F)$ and the multiplicity $m_7(\tau_{2n}^D \otimes \tau_{2m}^E, \tau^F_{2(n+m)})$ given in Section 3.7.2 are identical. This suggests that we can use the same lattice cone as the algebra $\mathfrak{A}_7$. Specifically we let $\mathcal{S}_8 = \mathcal{S}_7$, $\mathcal{C}_8 = \mathcal{C}_7$, so that $\Omega_8 = \mathcal{C}_8 \cap \mathbb{Z}^{N_8} = \Omega_7$.

3.8.4. *A basis for* $\mathfrak{A}_8$. In [HL3], we have constructed a basis $\mathcal{B}_8 = \{\Delta^{(8)}_{(A,B)} : (A,B) \in \Omega_8\}$ for $\mathfrak{A}_8$ with the following property: For $1 \le a \le b \le k$, let $t_{ab} = \sum_{i=1}^n (z_{ia}w_{n-i+1,b} - z_{ib}w_{n+1-i,a})$. Then the set

$$S_8 = \{z_{ij}, w_{n-i+1,j} : 1 \le i,j \le k\} \cup \{t_{ab} : 1 \le a \le b \le k\}$$

of $\mathcal{P}(\mathrm{M}_{2n,k})$ is algebraically independent. Let $\mathbb{C}[S_8]$ be the subalgebra of $\mathcal{P}(\mathrm{M}_{2n,k})$ generated by S_8. Then $\mathbb{C}[S_8]$ is itself a polynomial algebra and $\mathfrak{A}_8$ can be identified with a subalgebra of $\mathbb{C}[S_8]$ ([HL3]). We now let τ_8 be the following monomial ordering on $\mathbb{C}[S_8]$: it is the graded lexicographic order ([CLO]) such that

$$\begin{aligned} t_{12} > t_{13} > \cdots > t_{1k} > t_{23} > t_{24} > \cdots > t_{k-1,k} \\ > z_{11} > z_{12} > \cdots > z_{1k} > z_{21} > \cdots > z_{nk} \\ > w_{11} > w_{12} > \cdots > w_{1k} > w_{21} > \cdots > w_{nk}. \end{aligned}$$

Then for each $(A,B) \in \Omega_8$ where $A = [a_{ij}]$ and $B = [b_{ij}]$,

$$\mathrm{in}_{\tau_8}(\Delta^{(8)}_{(A,B)}) = \left(\prod_{j=1}^k (t_{2j-1,2j})^{a_{0j}/2}\right)\left(\prod_{1\le i\le h\le k} x_{ih}^{a_{ih}}\right)\left(\prod_{1\le i\le h\le k} y_{ih}^{b_{ih}}\right).$$

3.9. $\mathfrak{A}_9$: **stable O_n tensor product algebras.** This algebra describes the decomposition of the tensor product of two irreducible representations of O_n in the stable range.

3.9.1. *The definition of* $\mathfrak{A}_9$. Let $2k < n$ and let $\mathrm{O}_n \times \mathrm{GL}_k$ act on the algebra $\mathcal{P}(\mathrm{M}_{nk})$ of polynomial functions on M_{nk} by

$$[(g,h).f](X) = f(gXh^t), \quad g \in \mathrm{O}_n, h \in \mathrm{GL}_k, f \in \mathcal{P}(\mathrm{M}_{nk}), X \in \mathrm{M}_{nk}.$$

Let $(.,.)$ be the O_n-invariant symmetric bilinear form on $\mathbb{C}^n$. For $1 \leq i,j \leq k$ and $T \in \mathrm{M}_{n,k}$, let $r_{ij}^2(T) = (T_i, T_j)$, where T_i and T_j are the i-th and j-th column of T respectively. Let I_{nk} be the ideal of $\mathcal{P}(\mathrm{M}_{nk})$ generated by $\{r_{ij}^2 : \ 1 \leq i,j \leq k\}$. Then the quotient algebra $\mathcal{P}(\mathrm{M}_{nk})/I_{nk}$ is again an $\mathrm{O}_n \times \mathrm{GL}_k$ module. Similarly we form the quotient algebra $\mathcal{P}(\mathrm{M}_{n\ell})/I_{n\ell}$ where $2\ell < n$. Consider the tensor product $(\mathcal{P}(\mathrm{M}_{nk})/I_{nk}) \otimes (\mathcal{P}(\mathrm{M}_{n\ell})/I_{n\ell})$ and let

$$\mathfrak{A}_9 = \mathfrak{A}_{(\mathrm{O}_n \times \mathrm{O}_n, \Delta(\mathrm{O}_n),(k,\ell))} = \left\{ (\mathcal{P}(\mathrm{M}_{nk})/I_{nk})^{U_k} \otimes (\mathcal{P}(\mathrm{M}_{n\ell})/I_{n\ell})^{U_\ell} \right\}^{U_{\mathrm{SO}_n}}$$

be its subalgebra of $U_{\mathrm{SO}_n} \times U_k \times U_\ell$ invariants.

We shall assume that $n > 2(k+\ell)$, which is the stable range condition. In this case, the algebra structure of $\mathfrak{A}_9$ describes how a tensor product of the form $\sigma_n^D \otimes \sigma_n^E$ where D and E are Young diagrams with at most k rows and ℓ rows respectively decomposes into a sum of irreducible representations of O_n ([HTW1],[HTW2]). In view of this property, we call $\mathfrak{A}_9$ a *stable* O_n *tensor product algebra*.

3.9.2. *Branching multiplicity.* Let A_{SO_n} be the diagonal torus of SO_n. For a Young diagram F with $r(F) \leq n$, let ϕ_n^F be the restriction of ψ_n^F to A_{SO_n}. Then the algebra $\mathfrak{A}_9$ is a module for $A_{\mathrm{SO}_n} \times A_k \times A_\ell$, so it decomposes into a sum of joint $A_{\mathrm{SO}_n} \times A_k \times A_\ell$ eigenspaces. For Young diagrams F, D and E such that $r(F) \leq n$, $r(D) \leq k$ and $r(E) \leq \ell$, the dimension of the $\phi_n^F \times \psi_k^D \times \psi_\ell^E$ eigenspace in $\mathfrak{A}_9$ coincides with the multiplicity of the representation σ_n^F of O_n in the tensor product $\sigma_n^D \otimes \sigma_n^E$, which is given by ([HTW2])

$$m_9(\sigma_n^F, \sigma_n^D \otimes \sigma_n^E) = \sum_{\substack{r(G) \leq k, r(H) \leq \ell \\ r(L) \leq \min(k,\ell)}} c_{G,H}^F c_{L,G}^D c_{H,L}^E.$$

3.9.3. *The lattice cone* Ω_9. Let $s = \min(k,\ell)$, and let $\mathcal{S}_9$ be the subspace of $\mathbb{R}^{N_{(n,k,\ell)}} \oplus \mathbb{R}^{N_{(k,s,k)}} \oplus \mathbb{R}^{N_{(\ell,\ell,s)}}$ consisting of (A,B,C) such that $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}]$ and

$$a_{0i} = \sum_{p=i}^{k} b_{ip}, \ b_{0j} = \sum_{q=j}^{k} c_{jq}, \ c_{0t} = \sum_{r=i}^{n} a_{tr}$$

for $1 \leq i \leq k$, $1 \leq j \leq s$ and $1 \leq t \leq \ell$. Then $\dim \mathcal{S}_9 = N_9$ where

$$N_9 = N_{(n,k,\ell)} + N_{(k,s,k)} + N_{(\ell,\ell,s)} - k - s - \ell.$$

So $\mathcal{S}_9$ can be identified with $\mathbb{R}^{N_9}$. Let

$$\mathcal{S}_9 = \mathcal{S}_9 \cap (\mathbf{LR}_{n,k,\ell} \times \mathbf{LR}_{k,s,k} \times \mathbf{LR}_{\ell,\ell,s}).$$

Then $\mathcal{C}_9$ is a rational polyhedral cone in $\mathcal{S}_9$, so the intersection

$$\Omega_9 = \mathcal{C}_9 \cap \mathbb{Z}^{N_9}$$

is a lattice cone in $\mathbb{R}^{N_9}$.

If $F = (\lambda_1, ..., \lambda_n)$, $D = (\mu_1, ..., \mu_k)$ and $E = (\nu_1, ..., \nu_\ell)$ are Young diagrams, let $\mathcal{C}_9(F; D, E)$ be the set of points (A, B, C) in $\mathcal{C}_9$ such that $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}]$,

$$\begin{cases} \sum_{p=0}^{i} a_{pi} = \lambda_i, & 1 \le i \le n, \\ \sum_{q=0}^{j} b_{qj} = \mu_j, & 1 \le j \le k, \\ \sum_{r=0}^{m} c_{rm} = \nu_m, & 1 \le m \le \ell. \end{cases}$$

Then $\mathcal{C}_9(F; D, E)$ is a polytope contained in $\mathcal{C}_9$, and $m_9(\sigma_n^F, \sigma_n^D \otimes \sigma_n^E)$ is the number of integral points in the intersection $\mathcal{C}_9(F; D, E) \cap \mathbb{Z}^{N_9}$. Since

$$\Omega_9 = \bigcup_{D,E,F} \mathcal{C}_9(F; D, E) \cap \mathbb{Z}^{N_9},$$

Ω_9 can be used to label a basis for $\mathfrak{A}_9$. However, there is still no construction of such a basis.

3.10. $\mathfrak{A}_{10}$: **stable** Sp_{2n} **tensor product algebra.** This algebra describes the decomposition of the tensor product of two irreducible representations of Sp_{2n} in the stable range.

3.10.1. *The definition of* $\mathfrak{A}_{10}$. Let $k \le n$ and let $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ act on the algebra $\mathcal{P}(\mathrm{M}_{2n,k})$ of polynomial functions on $\mathrm{M}_{2n,k}$ by

$$[(g, h).f](X) = f(gXh^t), \quad g \in \mathrm{O}_n, h \in \mathrm{GL}_k, f \in \mathcal{P}(\mathrm{M}_{nk}), X \in \mathrm{M}_{nk}.$$

Let $\langle ., . \rangle$ be the Sp_{2n} invariant symplectic form on $\mathbb{C}^{2n}$. For $1 \le i < j \le k$ and $T \in \mathrm{M}_{2n,k}$, let $\xi_{ij}(T) = \langle T_i, T_j \rangle$, where T_i and T_j are the i-th and j-th column of T respectively. Let $I_{2n,k}$ be the ideal of $\mathcal{P}(\mathrm{M}_{2n,k})$ generated by $\{\xi_{ij} : 1 \le i < j \le k\}$. Then the quotient algebra $\mathcal{P}(\mathrm{M}_{2n,k})/I_{2n,k}$ is again a $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ module. Similarly we form the quotient algebra $\mathcal{P}(\mathrm{M}_{2n,\ell})/I_{2n,\ell}$ where $\ell \le n$. Consider the tensor product $(\mathcal{P}(\mathrm{M}_{2n,k})/I_{2n,k}) \otimes (\mathcal{P}(\mathrm{M}_{2n,\ell})/I_{n\ell})$ and let

$$\mathfrak{A}_{10} = \mathfrak{A}_{(\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}, \Delta(\mathrm{Sp}_{2n}), (k,\ell))} = \left\{ (\mathcal{P}(\mathrm{M}_{2n,k})/I_{2n,k})^{U_k} \otimes (\mathcal{P}(\mathrm{M}_{2n,\ell})/I_{2n,\ell})^{U_\ell} \right\}^{U_{\mathrm{Sp}_{2n}}}$$

be its subalgebra of $U_{\mathrm{Sp}_{2n}} \times U_k \times U_\ell$ invariants.

We shall assume that $n \ge k + \ell$, which is the stable range condition. In this case, the algebra structure of $\mathfrak{A}_{10}$ describes how a tensor product of the form $\tau_{2n}^D \otimes \tau_{2n}^E$ where D and E are Young diagrams with at most k rows and ℓ rows respectively decomposes into a sum of irreducible representations of Sp_{2n}. In view of this property, we call $\mathfrak{A}_{10}$ a *stable* Sp_{2n} *tensor product algebra.*

3.10.2. *Branching multiplicity.* Let $A_{\mathrm{Sp}_{2n}}$ be the diagonal torus of Sp_{2n}. For a Young diagram F with $r(F) \le n$, let χ_n^F be the restriction of ψ_n^F to $A_{\mathrm{Sp}_{2n}}$. Then the algebra $\mathfrak{A}_{10}$ is a module for $A_{\mathrm{Sp}_{2n}} \times A_k \times A_\ell$, so it decomposes into a sum of joint $A_{\mathrm{Sp}_{2n}} \times A_k \times A_\ell$ eigenspaces. For Young diagrams D, E and F such that $r(F) \le n$, $r(D) \le k$ and $r(E) \le \ell$, the dimension of the $\chi_n^F \times \psi_k^D \times \psi_\ell^E$ eigenspace in $\mathfrak{A}_{10}$ coincides with the multiplicity of the representation τ_{2n}^F of Sp_{2n} in the tensor product $\tau_{2n}^D \otimes \tau_{2n}^E$, which is given by ([HTW2])

$$m_{10}(\tau_{2n}^F, \tau_{2n}^D \otimes \tau_{2n}^E) = \sum_{\substack{r(G)\leq k, r(H)\leq \ell \\ r(L)\leq \min(k,\ell)}} c_{G,H}^F c_{G,L}^D c_{H,L}^E$$

3.10.3. *The lattice cone* Ω_{10}. We note that the multiplicity $m_{10}(\tau_{2n}^F, \tau_{2n}^D \otimes \tau_{2n}^E)$ and the multiplicity $m_9(\sigma_n^F, \sigma_n^D \otimes \sigma_n^E)$ given in Section 3.9.2 are identical. This suggests that we can use the same lattice cone as the algebra $\mathfrak{A}_9$. Specifically, we let $\mathcal{S}_{10} = \mathcal{S}_9$, $\mathcal{C}_{10} = \mathcal{C}_9$, so that $\Omega_{10} = \mathcal{C}_{10} \cap \mathbb{Z}^{N_{10}} = \Omega_9$. So Ω_{10} may be used to label a basis for $\mathfrak{A}_{10}$. However, there is still no construction for such a basis.

4. Main theorem

In this section, we prove that the stable branching algebra $\mathfrak{A}_i$ $(0 \leq i \leq 8)$ defined in Section 2 and Sections 3.1-3.8 are flat deformation of semigroup algebras.

MAIN THEOREM. *Let* $\mathfrak{A}_0, ..., \mathfrak{A}_8$ *be the stable branching algebras defined in Section 2 and in Sections 3.1-3.8. Then for each* $0 \leq i \leq 8$*, there exists a lattice cone* Ω_i *and a flat one-parameter family of complex algebras with general fibre* $\mathfrak{A}_i$ *and special fibre* $\mathbb{C}[\Omega_i]$.

Proof. The proof is similar to the case of the GL_n tensor product algebra $\mathfrak{A}_0$. Let $0 \leq i \leq 8$. Recall from Section 2 and 3 that $\mathfrak{A}_i$ has the following properties:

(a) $\mathfrak{A}_i$ can be identified with a subalgebra of a polynomial algebra $\mathcal{P}(V_i)$.

(b) $\mathfrak{A}_i$ has a basis $\mathcal{B}_i = \{\Delta_A^{(i)} : A \in \Omega_i\}$ where Ω_i is a lattice cone in $\mathbb{R}^{N_i}$.

(c) There is a monomial ordering τ_i on $\mathcal{P}(V_i)$ such that the leading monomials $\mathrm{in}_{\tau_i}(\Delta_A^{(i)})$ of the basis elements are all distinct, and the set

$$\mathrm{in}_{\tau_i}(\mathcal{B}_i) = \{\mathrm{in}_{\tau_i}(\Delta_A^{(i)}) : A \in \Omega_i\}$$

is an affine semigroup isomorphic to Ω_i.

It follows from similar arguments as in Proposition 2.4.3 that $\mathcal{B}_i$ is a SAGBI basis for $\mathfrak{A}_i$. Consequently the initial algebra $\mathrm{in}_{\tau_i}(\mathfrak{A}_i)$ is generated by $\mathrm{in}_{\tau_i}(\mathcal{B}_i)$, and since $\mathrm{in}_{\tau_i}(\mathcal{B}_i) \cong \Omega_i$,

$$\mathrm{in}_{\tau_i}(\mathfrak{A}_i) \cong \mathbb{C}[\Omega_i].$$

Since Ω_i is a finitely generated semigroup, the algebra $\mathbb{C}[\Omega_i]$ is also finitely generated. It follows that the initial algebra $\mathrm{in}_{\tau_i}(\mathfrak{A}_i)$ is finitely generated. The theorem now follows from Proposition 2.5.2. □

Remarks: If Ω_i is generated by $A_1, ..., A_m$, then $\{\Delta_{A_1}^{(i)}, ..., \Delta_{A_m}^{(i)}\}$ is a finite SAGBI basis for $\mathfrak{A}_i$.

For the remaining two stable branching algebras, we have the following conjecture:

Conjecture: *For i =9,10, there exists a affine semigroup* Ω_i *and a flat one-parameter family of complex algebras with general fibre* $\mathfrak{A}_i$ *and special fibre* $\mathbb{C}[\Omega_i]$.

References

[AB] V. Alexeev and M. Brion, *Toric degenerations of spherical varieties*, Selecta Math. (N.S.) 10 (2004), 453 – 478.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Univ. Press, 1998, 453 pp.

[BZ] A. Berenstein and A. Zelevinsky, *Triple multiplicities for* $\mathfrak{sl}(r+1)$ *and the spectrum of the exterior algebra of the adjoint representation*, J. of Alg. Comb. 1 (1992), 7 – 22.

[Ca] P. Caldero, *Toric degenerations of Schubert varieties*, Transf. Groups 7 (2002), No. 1, 51 – 60.

[CHV] A. Conca, J. Herzog and G. Valla, *SAGBI basis with applications to blow-up algebras*, J. Reine Angew. Math. 474 (1996), 113 – 138.

[CLO] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Second edition, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1997.

[EH] D. Eisenbud and J. Harris, *The Geometry of Schemes*, Graduate Texts in Mathematics 197, Springer-Verlag, New York, 2000.

[Fu1] W. Fulton, *Intrduction to Toric Varieties*, Ann. Math. Studies, Princeton University Press, Princeton, New Jersey, 1993.

[Fu2] W. Fulton, *Young Tableaux*, Cambridge University Press, Cambridge, UK, 1997.

[GL1] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transf. Groups 1 (1996), No. 3, 215 – 248.

[GL2] N. Gonciulea and V. Lakshmibai, *Schubert varieties, toric varieties and ladder determinant varieties*, Ann. Inst. Fourier (Grenoble) 47 (1997), 1013 – 1064.

[GW] R. Goodman and N. Wallach, *Representations and Invariants of the Classical Groups*, Cambridge Univ. Press, 1998.

[Hd] W. Hodge, *Some enumerative results in the theory of forms*, Proc. Cam. Phi. Soc. 39 (1943), 22 – 30.

[Ho] R. Howe, *Perspectives on invariant theory*, The Schur Lectures, I. Piatetski-Shapiro and S. Gelbart (eds.), Israel Mathematical Conference Proceedings, 1995, 1 – 182.

[HL1] R. Howe and S. T. Lee, *Bases for some reciprocity algebras I*, Trans. Amer. Math. Soc. 359 (2007), 4359-4387.

[HL2] R. Howe and S. T. Lee, *Bases for some reciprocity algebras II*, Adv. in Math. 206 (2006), 145-210.

[HL3] R. Howe and S. T. Lee, *Bases for some reciprocity algebras III*, Compositio Math. 142 (2006), 1594-1614.

[HTW1] R. Howe, E-C. Tan and J. Willenbring, *Reciprocity algebras and branching for classical symmetric pairs*, in "Groups and Analysis - the Legacy of Hermann Weyl," London Mathematical Society Lecture Note Series No. 354, Cambridge University Press, 2008.

[HTW2] R. Howe, E-C. Tan and J. Willenbring, *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. 357 (2005) 1601 – 1626.

[HTW3] R. Howe, E-C. Tan and J. Willenbring, *A basis for* GL_n *tensor product algebra*, Adv. in Math. 196 (2005), 531 – 564.

[Ka] K. Kaveh, *SAGBI bases and degeneration of spherical varieties to toric varieties*, Michigan Math. J., 53 (2005), 109 – 121.

[Ki] S. Kim, *Standard monomial theory for flag algebras of* $GL(n)$ *and* $Sp(2n)$, J. Algebra 320 (2008), 534-568.

[KM] M. Kogan and E. Miller, *Toric degeneration of Schubert varieties and GelfandTsetlin polytopes*, Adv. in Math. 193 (2005), 1-17.

[KT1] A. Knutson and T. Tao, *The honeycomb model of* $GL_n(\mathbb{C})$ *tensor products I: Proof of the saturation conjecture*, J. Amer. Math. Soc. 12 (1999), 1055 – 1090.

[KT2] A. Knutson, T. Tao and C. Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determin facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. 17 (2003), No. 1, 19 – 48.

[Ma] I. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1995.

[MS] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics 227, Springer-Verlag, New York, 2005.

[PV] I. Pak and E. Vallejo, *Combinatorics and geometry of Littlewood-Richardson cones*, Europ. J. Combinatorics **26** (2005), 995-1008

[RS] L. Rubbiano and M. Sweedler, *Subalgebra bases*, Proceedings Salvador 1988 Eds. W. Bruns, A. Simis, Lecture Notes Math 1430 (1990), 61 – 87.

[St1] B. Sturmfels, *Algorithms in Invariant Theory*, Texts and Monographs in Symbolic Computation, Springer, Berlin, 1993.

[St2] B. Sturmfels, *Grobner Beses and Convex Polytopes*, Univ. Lecture Series, Vol. 8, Amer. Math. Soc., Providence, RI, 1996.

[Wy] H. Weyl, *The Classical Groups*, Princeton University Press, Princeton, N.J., 1946.

[Zh] D. Zhelobenko, *Compact Lie Groups and their Representations*, Transl. of Math. Mono. 40, AMS, Providence, R.I., 1973.

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