Invariant Polynomials on Tensors Under the Action of a Product of Orthogonal Groups

by

Lauren Kelly Williams

A Dissertation Submitted in

Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

in

Mathematics

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Abstract

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Under the Supervision of Professor Jeb F. Willenbring

Let K be the product $O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$ of orthogonal groups. Let $V = \otimes_{i=1}^{r} \mathbb{C}^{n_i}$, the r-fold tensor product of defining representations of each orthogonal factor. We compute a stable formula for the dimension of the K-invariant algebra of degree d homogeneous polynomial functions on V. To accomplish this, we compute a formula for the number of matchings which commute with a fixed permutation. Finally, we provide formulas for the invariants and describe a bijection between a basis for the space of invariants and the isomorphism classes of certain r-regular graphs on d vertices, as well as a method of associating each invariant to other combinatorial settings such as phylogenetic trees.

To my mother, Jean. You are deeply missed.

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List of Symbols

$\mathbb{C}$	The complex numbers
$\mathbb{N}$	The natural numbers $\{1, 2, 3, 4, \ldots\}$
$\mathbb{R}$	The real numbers
$\mathbb{Z}$	The integers
$\mathbf{n}$	r-tuple of natural numbers $(n_1, n_2, \ldots, n_r)$
$\dim V$	Complex dimension of a $\mathbb{C}$-vector space V
GL_n	General linear group of $n \times n$ invertible matrices over $\mathbb{C}$
O_n	Complex orthogonal group of $n \times n$ matrices
F_n^λ	Regular representation of GL_n indexed by λ
W_n^λ	Irreducible representation of S_n indexed by λ
$\mathcal{O}[X]$	Regular functions on a variety X
$\mathcal{P}(V)$	Complex polynomial functions on a complex vector space V
$\mathcal{P}^d(V)$	Degree d homogeneous functions in $\mathcal{P}(V)$
S_n	The symmetric group on n letters
$U(n)$	Unitary group of $n \times n$ matrices
V^*	Dual of a $\mathbb{C}$-vector space V
$V \otimes W$	Tensor product of vector spaces V and W, tensored over $\mathbb{C}$
$\otimes^r V$	r-fold tensor product of vector space V
$\lvert X \rvert$	Number of elements in a set X

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Chapter 1

Introduction

This thesis is motivated by the following situation. Let $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ be a fixed r-tuple of natural numbers, with $r \in \mathbb{N}$, and let $K(\mathbf{n})$ denote the product group $K(\mathbf{n}) = K(n_1) \times K(n_2) \times \cdots \times K(n_r)$ where $K(n_i)$ is a compact subgroup of $U(n_i)$. Let $V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_r}$ be the representation of $K(\mathbf{n})$ under the standard action on each tensor factor. Obtaining a description of the $K(\mathbf{n})$-orbits in $V(\mathbf{n})$ is an interesting but challenging problem in representation theory.

A possible approach to finding such a description is considered in [26] and [37]. Let $\mathcal{P}_{\mathbb{R}}(V)$ denote the algebra of complex valued polynomial functions on a finite dimensional complex vector space V when viewed as a real vector space. A compact subgroup, K, of the unitary operators on V acts on $\mathcal{P}_{\mathbb{R}}(V)$ in the standard way: for $k \in K$, $v \in V$, and $f \in \mathcal{P}_{\mathbb{R}}(V)$, we have $k \cdot f(v) = f(k^{-1}v)$. We then have

Theorem 1. *[26] If $v, w \in V$ then $f(v) = f(w)$, for all $f \in \mathcal{P}_{\mathbb{R}}(V)^K$ if and only if v and w are in the same K-orbit.*

An understanding of the K-orbits in V can now be obtained via the invariant theory of K. The K-invariant subspace of $\mathcal{P}_{\mathbb{R}}(V)$, which we denote by $\mathcal{P}_{\mathbb{R}}(V)^K$, is known to be finitely generated [16]. However, a description of these generators is incomplete except for particular cases, and therefore motivates the subject of classical invariant theory.

A polynomial $f \in \mathcal{P}_{\mathbb{R}}(V)$ is said to be homogeneous of degree d if $f(cv) = c^d v$ for all scalars c and $v \in V$. Any polynomial can be written as a sum of its homogeneous parts, and so we have the standard gradation $\mathcal{P}_{\mathbb{R}}(V) = \bigoplus_{d=0}^{\infty} \mathcal{P}_{\mathbb{R}}^d(V)$, where $\mathcal{P}_{\mathbb{R}}^d(V)$ is the subspace of degree d homogeneous polynomials on V. Moreover, the K-invariant subalgebra inherits this gradation. That is, $\mathcal{P}^d_{\mathbb{R}}(V)^K = \mathcal{P}^d_{\mathbb{R}}(V) \cap \mathcal{P}_{\mathbb{R}}(V)^K$.

We will complexify this picture. $\mathcal{P}(V)$ will be the complex valued polynomial functions on the complexification of the real vector space V. For example, if $V = \mathbb{R}^n$, $\mathcal{P}(V) \cong \mathbb{C}[z_1, \ldots, z_n]$, and if $V = \mathbb{C}^n$ viewed as a real vector space, then $\mathcal{P}(V) \cong \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. The real vector space $\mathbb{R}^n$ is the defining representation of the group[1] $O(n) = O_n(\mathbb{R})$. The complex group $O_n = O_n(\mathbb{C})$ acts on the complexification $\mathbb{C}^n$. The unitary group, $U(n)$, acts on the real vector space $\mathbb{C}^n$ (the defining representation). The complexification of $U(n)$ is $\mathrm{GL}_n(\mathbb{C})$, which acts on the complexification of $\mathbb{C}^n$.

The group $K(\mathbf{n})$ acts on $V(\mathbf{n})$ by

$$(g_1, g_2, \ldots, g_r) \cdot (v_1, v_2, \ldots, v_r) = (g_1 v_1) \otimes (g_2 v_2) \otimes \cdots \otimes (g_r v_r)$$

for $g_i \in O_{n_i}$, $v_i \in \mathbb{C}^{n_i}$. Part of the purpose of this thesis is to describe the space $\mathcal{P}^d(V(\mathbf{n}))^{K(\mathbf{n})}$ when $K(\mathbf{n}) = \prod_{i=1}^r O_{n_i}$, a product of orthogonal groups, where $\mathcal{P}(V(\mathbf{n}))$ denotes the complex valued polynomials on $V(\mathbf{n})$ viewed as a complex space.

In general, describing the invariants of this action remains an open and difficult problem. However, if we require $n_i \geq d$ for $1 \leq i \leq r$, we are able to determine a formula for the dimension of the space of $K(\mathbf{n})$-invariant degree d homogeneous polynomials on $V(\mathbf{n})$. We will refer to these inequalities as the *stable range*. We derive a formula for this dimension in Chapter 3. Explicit formulas for the invariants in this stable range are then provided in Chapter 4, along with a graph theoretic interpretation. We also examine correspondences between the invariants and various combinatorial settings. Specifically, we consider r-tuples of matchings (i.e., order two permutations without fixed points) counted up to simultaneous conjugation.

[1]The orthogonal group, $O(n)$, is the group of real $n \times n$ matrices M such that $M^T M = I$, where M^T denotes the transpose of M.

Our main result, derived in Chapter 3, is

Theorem 2. *Let d be a positive integer and let $K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$ and $V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_r}$, where $n_1, n_2, \ldots, n_r \geq d$. Then the dimension of the space of $K(\mathbf{n})$-invariant degree d homogeneous polynomial functions on $V(\mathbf{n})$ is zero when d is odd. When d is even, we have*

$$\dim \mathcal{P}^d(V(\mathbf{n}))^{K(\mathbf{n})} = \sum_{\lambda \vdash d} \frac{N(\lambda)^r}{z_\lambda}$$

where $N(\lambda)$ is the number of matchings that commute with a permutation with shape λ, and z_λ is the order of the centralizer of a permutation with shape λ.

We determine the number of matchings which commute with a given permutation in Section 3.4. In particular, we have

Theorem 3. *Given a permutation σ with shape $\lambda = (1^{b_1} 2^{b_2} \cdots t^{b_t})$, the number of matchings that commute with σ is given by*

$$N(\lambda) = N((1^{b_1})) \cdot N((2^{b_2})) \cdot \cdots \cdot N((t^{b_t}))$$

where

$$N((a^b)) = \begin{cases} 0 & \text{if } a \text{ is odd and } b \text{ is odd} \\ \frac{b! a^{b/2}}{2^{b/2} (\frac{b}{2})!} & \text{if } a \text{ is odd and } b \text{ is even} \\ \sum_{i=1, i \text{ odd}}^{b} \frac{b! a^{(b-i)/2}}{i! (\frac{b-i}{2})! 2^{(b-i)/2}} & \text{if } a \text{ is even and } b \text{ is odd} \\ \sum_{i=0, i \text{ even}}^{b} \frac{b! a^{(b-i)/2}}{i! (\frac{b-i}{2})! 2^{(b-i)/2}} & \text{if } a \text{ is even and } b \text{ is even} \end{cases}$$

In Chapter 4, we show that given an r-tuple of matchings, $(\tau_1, \tau_2, \ldots, \tau_r)$, each on $d = 2m$ letters[2], we can describe an invariant in $\mathcal{P}^d(V(\mathbf{n}))^{K(\mathbf{n})}$. The symmetric group S_d acts on such a tuple by simultaneous conjugation; that is, for $g \in S_d$, we

[2]Note that a matching is a permutation in S_d, where d is even.

define

$$g \cdot (\tau_1, \tau_2, \ldots, \tau_r) = (g\tau_1 g^{-1}, g\tau_2 g^{-1}, \ldots, g\tau_r g^{-1}).$$

We provide an explicit basis for $\mathcal{P}^d(V(\mathbf{n}))^{K(\mathbf{n})}$. The orbits of the above action are in bijective correspondence with this basis, as well as isomorphism classes of certain graphs.

In [14], an analogous problem is considered when $K(\mathbf{n}) = \prod_{i=1}^{r} U(n_i)$, a product of unitary groups[3]. A formula for the dimension of these invariant functions is provided within a stable range. That is, when the rank of each group in the product is sufficiently large. This result is recalled in Chapter 3. Formulas for the $K(\mathbf{n})$-invariants in $\mathcal{P}^d(V(\mathbf{n}))$, as well as a new graphical interpretation of these invariants, is provided in [15] and recalled in Chapter 4. This situation has a physical interpretation, as it is related to the quantum mechanical state space of a multi-particle system in which each particle has finitely many outcomes upon observation. Moreover, these invariant functions separate the entangled and unentangled states, and are therefore viewed as measurements of quantum entanglement [26].

The First Fundamental Theorem of Invariant Theory

Our problem, when $K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$, is in sharp contrast to the situation covered by the First Fundamental Theorem of Invariant Theory (FFT), which we now recall following the notation of [17]. Let V be a vector space, and let $G \subset \mathrm{GL}(V)$ be a group of linear transformations on V. Then the FFT describes the algebra of G-invariants $\mathcal{P}(V^\ell \oplus (V^*)^m)^G$, where V^* denotes the dual of V and G acts on V^* by the contragrediant of its action on V.

We now review the FFT for $G = O_n$. The orthogonal group O_n acts on $\bigoplus^m \mathbb{C}^n$ (which we may view as the space of $n \times m$ matrices) by left multiplication. O_n also

[3]The unitary group, $U(n)$, is the group of complex $n \times n$ matrices M such that $M^*M = I$, where M^* denotes the conjugate transpose of M.

preserves the inner product

$$(u, w) = \sum_{i=1}^{n} u_i w_i$$

where $u = (u_1, \ldots, u_n)$ and $w = (w_1, \ldots, w_n)$ are n-tuples of complex numbers. Let $v = (v_1, \ldots, v_m)$ be an element of $\bigoplus^m \mathbb{C}^n$, where each v_i is itself an n-tuple $v_i = (v_{i_1}, v_{i_2}, \ldots, v_{i_n})$.

Theorem 4. *(First fundamental theorem of invariant theory for O_n) The invariant algebra $\mathcal{P}(\bigoplus^m \mathbb{C}^n)^{O_n}$ is generated by second order polynomials of the form*

$$(v_i, v_j) = \sum_{k=1}^{n} v_{i_k} v_{j_k}$$

for $1 \leq i \leq j \leq n$.

Example 1. *Consider the case $n = 2$. Following Theorem 4, the degree 2 homogeneous polynomial function $f(x, y)$ invariant under O_2 is*

$$f(x, y) = x^2 + y^2$$

for $(x, y) \in \mathbb{C}^2$. To verify that this polynomial is invariant, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 orthogonal matrix. By definition, we have

$$A^T A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $a^2+c^2=b^2+d^2=1$, *and* $ab+cd=0$. *Observe that*

$$\begin{aligned} f(A(x,y)) &= f(ax+by,cx+dy) = (ax+by)^2+(cx+dy)^2 \\ &= ((a^2+c^2)x^2+2(ab+cd)xy+(b^2+d^2)y^2 = x^2+y^2 = f(x,y) \end{aligned}$$

and so f *is indeed invariant under* O_2.

The situation considered in this thesis is distinct from the First Fundamental Theorem. In particular, we replace the direct sums of the FFT by tensor products, and look for invariants of the product group $O_{n_1}\times\cdots\times O_{n_r}$ in place of O_n.

Schur Weyl Duality

Classical Schur-Weyl duality (see Section 3.1) describes a relationship between irreducible finite dimensional representations of the general linear group and the symmetric group. The natural actions of each of these groups on the space $\bigotimes^k\mathbb{C}^n$ centralize each other, resulting in the multiplicity free decomposition

$$\bigotimes^k \mathbb{C}^n \cong \bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\le n}} F^\lambda\otimes W^\lambda$$

where F^λ and W^λ are the irreducible representations of GL_n and S_k, respectively, associated to the partition λ.

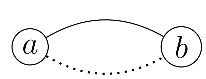
There is an analogous version of Schur Weyl duality when GL_n is replaced by the complex orthogonal group. This theory was developed by Richard Brauer. Both of these pictures give rise to an understanding of invariant tensors, but not, directly, an understanding of the invariant polynomial functions on tensors, which lie deeper.

Classical Kostant-Rallis Theory

When $r = 2$ and $K(\mathbf{n})$ is a product of orthogonal or unitary groups, the situation is well known, and is an instance of Kostant-Rallis theory (see also [12, Ch. 12]). We now recall the situation for the orthogonal group in detail. That is, we consider $\mathcal{P}^d(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})^{O_{n_1} \times O_{n_2}}$. Here, we may think of $V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ as the space of $n_1 \times n_2$ matrices with entries in $\mathbb{C}$. Denote these matrices by $X = (x_{ij})$. We now recall formulas for these invariants, as related to this thesis. As we show in Chapter 3, there are no invariants when d is odd, so we review the situation when $d = 2, 4$, and 6.

An invariant polynomial in the $r = 2$ case will be shown to correspond to an 2-regular graph on d vertices in Section 4.2.2. We represent the two colors of these graphs with solid and dotted edge. For any such graph, we can label a set of exactly half of these vertices $\{a_1, a_2, \ldots, a_{d/2}\}$ so that no solid or dotted edge connects any two elements of the set. As d increases, we may have multiple choices for this set. Let $\{b_1, b_2, \ldots, b_{d/2}\}$ denote the remaining vertices. We create a general term of the associated invariant polynomial by taking products of the indeterminants $x_{a_i b_j}$ whenever there is an edge (of any color) between vertices a_i and b_j. Taking sums of these terms as each a_i ranges from 1 to n_1 and each b_i ranges from 1 to n_2 yields the invariant. Note that other choices for labeling the vertices may be possible, but will result in equivalent polynomials.

We begin with the simplest case, $d = 2$. By Theorem 2, the space $\mathcal{P}^2(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})^{O_{n_1} \times O_{n_2}}$ is generated by only one polynomial. Additionally, we have only one 2-regular graph on two vertices a and b, up to isomorphism:

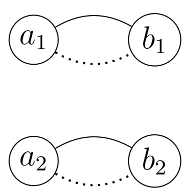


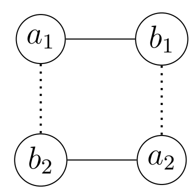
The associated invariant is written

$$f(x_{11}, \ldots, x_{n_1 n_2}) = \sum_{a=1}^{n_1} \sum_{b=1}^{n_2} x_{ab}^2$$

with the above notation. We may recognize this polynomial as $\mathrm{Tr}(X^T X)$, the trace of the matrix $X^T X$.

If we next consider $d = 4$, we find two 2-regular graphs on 4 vertices, up to isomorphism. In the presentations below, we have already made a choice of labels for the vertices:





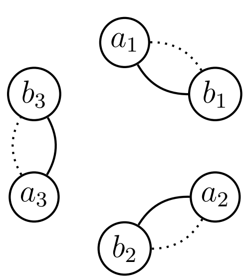
The space $\mathcal{P}^4(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})^{O_{n_1} \times O_{n_2}}$ has two basis elements. Using the method described above, we can write these invariants, respectively, as

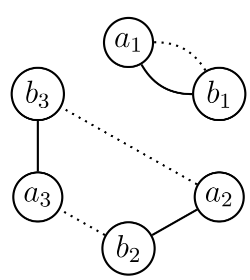
$$f_1(x_{11}, \ldots, x_{n_1 n_2}) = \sum_{a_1, a_2 = 1}^{n_1} \sum_{b_1, b_2 = 1}^{n_2} x_{a_1 b_1}^2 x_{a_2 b_2}^2$$

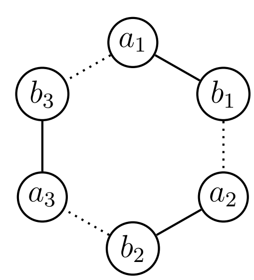
$$f_2(x_{11}, \ldots, x_{n_1 n_2}) = \sum_{a_1, a_2 = 1}^{n_1} \sum_{b_1, b_2 = 1}^{n_2} x_{a_1 b_1} x_{a_1 b_2} x_{a_2 b_1} x_{a_2 b_2}$$

In this case, we recognize f_1 to be $[\mathrm{Tr}(X^T X)]^2$, and f_2 to be $\mathrm{Tr}[(X^T X)^2]$.

Finally, we consider $d = 6$. Here we have a total of 3 graphs, up to isomorphism. Again, we have already chosen the sets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ in each case:







The space $\mathcal{P}^6(\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2})^{O_{n_1}\times O_{n_2}}$ has three generators, each associated to a graph shown above:

$$f_1(x_{11},\ldots,x_{n_1n_2}) = \sum_{a_1,a_2,a_3=1}^{n_1}\sum_{b_1,b_2,b_3=1}^{n_2} x_{a_1b_1}^2 x_{a_2b_2}^2 m_{a_3b_3}^2$$

$$f_2(x_{11},\ldots,x_{n_1n_2}) = \sum_{a_1,a_2,a_3=1}^{n_1}\sum_{b_1,b_2,b_3=1}^{n_2} x_{a_1b_1}^2 x_{a_2b_2}x_{a_2b_3}x_{a_3b_2}x_{a_3b_3}$$

$$f_3(x_{11},\ldots,x_{n_1n_2}) = \sum_{a_1,a_2,a_3=1}^{n_1}\sum_{b_1,b_2,b_3=1}^{n_2} x_{a_1b_1}x_{a_1b_3}x_{a_2b_1}x_{a_2b_2}x_{a_3b_2}x_{a_3b_3}$$

Again, we note that f_1, f_2, and f_3 are $[\mathrm{Tr}(X^TX)]^3$, $\mathrm{Tr}(X^TX)\cdot\mathrm{Tr}[(X^TX)^2]$, and $\mathrm{Tr}[(X^TX)^3]$, respectively.

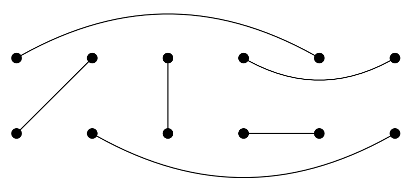
Combinatorics of matchings

The idea of a matching will arise throughout this thesis. Matchings have surprisingly deep connections to a variety of areas of research. Recall that Schur-Weyl duality relates the irreducible finite dimensional representations of the general linear and symmetric groups. In 1937, Richard Brauer defined an algebra which plays the role of the group algebra of S_n in a similar statement on the representation theory of the orthogonal group [3] [38] [20]. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ denote the *conjugate partition*, where λ'_j is the number of boxes in the jth column of the Young diagram of λ. We have

$$\otimes^k\mathbb{C}^n \cong \bigoplus_{i=0}^{\lfloor k/2\rfloor}\bigoplus_{\substack{\lambda\vdash(k-2i)\\ \lambda'_1+\lambda'_2\le n}} U^\lambda\otimes V^\lambda$$

where U^λ and V^λ denote, respectively, the irreducible representations of the orthogonal group and the Brauer algebra. Let $\mathcal{B}_k(n)$ be the centralizer algebra of the diagonal action of O_n on $\otimes^k\mathbb{C}^n$. The dimension of the algebra $\mathcal{B}_k(n)$ is $(2k -$

1)$(2k-3)\cdots 3\cdots 1$, the number of matchings on $2k$ elements. These basis elements of the algebra are frequently depicted as graphs; the diagram below represents a basis element of the algebra $\mathcal{B}_6(n)$:



A full rooted binary tree, together with a labeling of its leaves, is called a *phylogenetic tree*. Diaconis and Holmes, in [6], have described a bijection between matchings on $2n$ elements and phylogenetic trees with $n+1$ leaves, which we recount in Section 4.2.3. Note that there is more than one possible rule to create such a correspondence. The notion of a phylogenetic tree is motivated by concepts in biology, where the trees are used to illustrate inferred evolutionary relationships.

An orbit of $\mathcal{P}^d(V(\mathbf{n}))^{K(\mathbf{n})}$ will consist of polynomials which correspond to the conjugacy classes of ordered r-tuples of matchings on d letters, under simultaneous conjugation by the symmetric group. We'll show in Chapter 4 that each invariant therefore corresponds to what we introduce as a phylogenetic forest.

1.1 Organization of this thesis

After preliminary notation and definitions are reviewed in Chapter 2, we prove the stable dimension formula in Chapter 3, and provide explicit invariants along with a graph theoretic interpretation in Chapter 4. Finally, in Appendix A, we provide some data related to the nonstable cases. In Appendix B, we provide several Sage [36] programs used to verify our results and calculate the values in Appendix A.

Chapter 2

Preliminaries

2.1 Partitions and Permutations

We now review some definitions and notation used throughout this thesis. We say λ is a *partition* of a positive integer n, denoted $\lambda \vdash n$, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers such that $\sum_{i=1}^{\ell} \lambda_i = n$. In this case, we say λ has *size* n and *length* ℓ. If λ has b_1 ones, b_2 twos, b_3 threes, etc, we say that λ has *shape* $(1^{b_1}, 2^{b_2} 3^{b_3}, \ldots)$. For example, $\lambda = (3, 2, 2, 1)$ is a partition of 8 with shape $(1^1 2^2 3^1)$. Finally, we say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is an *even* partition if every part, λ_i, is an even integer.

Let S_n denote the symmetric group whose elements are permutations of $\{1, 2, \ldots, n\}$. Recall that there are $n!$ such permutations, each of which may be denoted in a variety of ways. A particularly useful convention for our purposes is *disjoint cycle notation*. We write a permutation σ as a product of cycles of the form $(x_1\ x_2\ x_3\ \cdots\ x_l)$ if $\sigma(x_1) = x_2$, $\sigma(x_2) = x_3$, ... , $\sigma(x_{l-1}) = x_l$, and $\sigma(x_l) = x_1$. Every element in S_n can be written in this way, with the contents of each cycle forming disjoint sets. Since disjoint cycles commute, it is possible to represent a permutation in multiple ways; additionally, the notation for an individual cycle is not unique, and depends on the starting point x_1 chosen. For example,

$$(1\ 2\ 7)(3)(4\ 5)(6) = (1\ 2\ 7)(4\ 5)(3)(6) = (7\ 1\ 2)(3)(4\ 5)(6)$$

all denote the same element of S_7 written in cycle notation. It is our convention to write the cycles in decreasing order of length with the leading entry of each cycle chosen to be the smallest number within that cycle, as in the center example above.

Cycle notation allows us to quickly identify some importation characteristics of each permutation. A permutation σ is said to have a *fixed point* x if $\sigma(x) = x$. A fixed point appears as a cycle containing the single entry x. The permutation in the above example has two fixed points, 3 and 6. A permutation σ that exchanges two elements x and y (that is, $\sigma(x) = y$ and $\sigma(y) = x$) will contain a cycle of length two, $(x\ y)$, called a *transposition*.

We can form a partition of n by listing the lengths of each cycle of an element of S_n. This partition is called the *cycle type* of the permutation. Note that two permutations may have the same cycle type. For instance, the permutations $\sigma_1 = (1\ 3\ 6)(4\ 7)(2\ 5)$ and $\sigma_2 = (1\ 5\ 6)(2\ 7)(3\ 4)$ both have cycle type $\lambda = (3, 2, 2)$. We may also refer to the cycle type as the shape of the permutation. Thus the permutations σ_1 and σ_2 may be said to have cycle type $(2^2 3^1)$. Two permutations have the same cycle type if and only if they are conjugate in S_n, and so the conjugacy classes of S_n can be indexed by the partitions of n.

Given a permutation $\sigma \in S_n$ with cycle type $\lambda = (1^{b_1} 2^{b_2} 3^{b_3}, \ldots)$, the number of permutations commuting with σ (that is, the order of the centralizer Z_σ) is denoted by z_λ. The cardinality of the conjugacy class is [30]

$$|Z_\sigma| = \frac{n!}{z_\lambda} = \frac{n!}{1^{b_1} 2^{b_2} 3^{b_3} \cdots b_1! b_2! b_3! \cdots} \tag{2.1}$$

A permutation σ is called an *involution* if $\sigma^{-1} = \sigma$. Note that a permutation is an involution if and only if its cycles have length at most two; that is, it is composed only of fixed points and transpositions. An involution without fixed points is therefore composed only of transpositions, and is called a *matching*.

Aside from simply writing a partition λ as a sequence, we can denote it in a variety of other ways. Ferrers and Young diagrams are two visual means of portraying integer partitions that are commonly used. In a *Ferrers diagram*, dots are arranged into rows according to the size of each part of the partition. In a *Young diagram*, these dots are replaced by boxes. As partitions are conventionally written as weakly decreasing sequences, the rows of these diagrams will weakly decrease from top to bottom. The beginning of each row is left justified, resulting in columns that weakly decrease from left to right, as in Figure 2.1. The size and shape of such a diagram is the same as the size and shape of the corresponding partition. Each Young diagram corresponds to an irreducible representation (over $\mathbb{C}$) of the symmetric group. [30] [24] [7]

(a) Ferrers Diagram

(b) Young Diagram

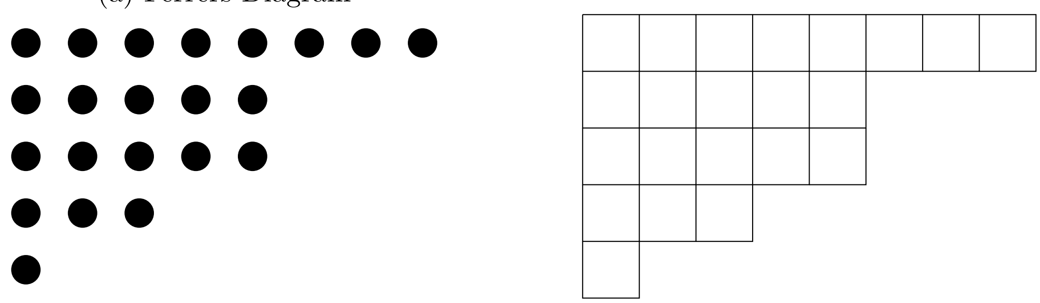


Figure 2.1: Diagrams representing the partition $(8, 5, 5, 3, 1)$

A *Young tableau* is formed by filling the boxes of a Young diagram with symbols. The filling is called the *content* of the tableau. A *standard Young tableau* (sometimes referred to simply as a *tableau*) of size n is filled with the integers $\{1, 2, \ldots, n\}$ so that the numbers are strictly increasing across the rows from left to right, and along each column from top to bottom. A *semistandard Young tableau* (also called *column strict*) contains numbers that weakly increase across rows, but strictly increase down columns. The *weight* of a tableau is a sequence recording the number of times each entry appears in the tableau. Hence, a standard tableau with n boxes has weight $(1, 1, \ldots, 1)$ and content $(1, 2, \ldots, n)$. The *Kostka numbers*, denoted $K_{\lambda,\mu}$, is defined to be the number of semistandard tableaux with shape λ and weight μ.

Example 2. *For* $\lambda = (4, 3, 2)$ *and* $\mu = (3, 3, 2, 1)$*, we have* $K_{\lambda,\mu} = 4$*. The four* *semistandard tableaux with shape* λ *and content* $(1,1,1,2,2,2,3,3,4)$ *are*

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2&3\\\cline{1-3} 3&4\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2&4\\\cline{1-3} 3&3\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&2&2\\\cline{1-3} 3&4\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|c|}\hline 1&1&1&4\\\hline 2&2&2\\\cline{1-3} 3&3\\\cline{1-2}\end{array}$$

2.1.1 Schur Polynomials

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, we can define a symmetric polynomial s_λ in n variables called the *Schur polynomial*. Various formulas for the Schur polynomials exist. One of the earliest definitions is attributed to Cauchy [18], and is described by a ratio of determinants. For indeterminants $x_1, \ldots, x_n$, define

$$A = \begin{pmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \cdots & x_n^{\lambda_2+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \cdots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{pmatrix}$$

and

$$B = \begin{pmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

The Schur polynomial s_λ can be written as

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det(A)}{\det(B)}$$

Note that the numerator is an alternating polynomial and the denominator is the Vandermonde determinant, which guarantees that s_λ will be a symmetric polynomial.

An equivalent definition can be written in terms of semistandard Young tableaux.

Let T be a semistandard tableau with weight $(t_i, \ldots, t_n)$. Then

$$s_\lambda(x_1, \ldots, x_n) = \sum_T x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$$

where the sum is over all semistandard tableaux with shape λ. An immediate corollary of this description is that $s_\lambda(1, 1, \ldots, 1)$ gives us the number of semistandard tableaux with shape λ.

2.1.2 The Robinson-Schensted-Knuth Correspondence

A particularly useful fact, originally described by Gilbert Robinson [28], is the correspondence between permutations in S_n and ordered pairs (P, Q) of standard tableaux of size n with the same shape [35]. The algorithm to determine the pair of tableaux associated to a particular permutation was later improved by Craige Schensted [32]; their combined effort is referred to as the *Robinson-Schensted correspondence.* The study of this correspondence has given rise to some remarkable combinatorics. For instance, the lengths of the longest increasing and decreasing subsequences of a permutation can be identified by the lengths of the first row and column, respectively, of the associated tableaux [32] [35]. The process was later generalized by Donald Knuth[1] to a correspondence between matrices with nonnegative integer entries and pairs of semistandard tableaux with the same shape.

Much of the work throughout this thesis focuses on matchings, the permutations formed by a product of transpositions. The Robinson-Schensted correspondence gives rise to an interesting way of encoding a matching. It can be shown that the number of fixed points of an involution is equal to the number of columns of odd length in the associated tableaux. Additionally, inverting the permutation corresponds to interchanging P and Q. Therefore, involutions are in bijective correspondence to standard tableaux. Given an involution, the number of odd columns is

[1]A Milwaukee native.

equal to the number of fixed points. As a consequence, each matching corresponds to a standard tableau whose columns have even length.

2.2 The Orthogonal Group

Let $O(n, F)$ denote the set of all $n \times n$ invertible matrices g with entries in a field F such that $g^T g$ is the identity matrix. Then $O(n, F)$ is a group, called the *orthogonal group*, whose elements are the *orthogonal matrices*. The orthogonal matrices have determinant ± 1. The subgroup of orthogonal matrices with determinant 1 is called the *special orthogonal group*, denoted $SO(n, F)$.

Throughout this thesis, we let $O_n = O(n, \mathbb{C})$ denote the complex orthogonal matrices. The real orthogonal matrices will be denoted by $O_n(\mathbb{R})$. Note that neither O_n nor $O_n(\mathbb{R})$ form connected groups. They instead are the union of two connected components, one of which is the special orthogonal group, and the other being the set of orthogonal matrices with determinant -1.

The group $O_n(\mathbb{R})$ forms a compact Lie group, with dimension $\frac{n(n-1)}{2}$. Similarly, the group O_n forms a linear algebraic group over $\mathbb{C}$ with Krull dimension $\frac{n(n-1)}{2}$. However, in the Euclidean topology, the complex group O_n is not compact for $n > 1$.

The definition of the complex orthogonal group is similar to the definition of the *unitary group*, $U(n)$, which consists of $n \times n$ matrices g with entries in $\mathbb{C}$ such that $g^* g$ is the identity matrix, where g^* is the conjugate transpose of g. Note that there is a distinction between the groups O_n and $U(n)$. For instance, the matrices of the form

$$\begin{bmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{bmatrix}$$

for $z \in \mathbb{C}$ are complex orthogonal matrices, but are not necessarily unitary. The unitary group $U(n)$ forms a Lie group of dimension n^2, and is both compact and connected.

2.3 Representations

2.3.1 Definitions

Recall that the general linear group, $\mathrm{GL}_n(\mathbb{C})$, is the group of $n \times n$ invertible matrices with complex entries. For $g \in \mathrm{GL}_n(\mathbb{C})$, let a_{ij} denote the entry in the ith row and jth column of g. The determinant of g is a polynomial in these matrix entries. The regular functions on $\mathrm{GL}_n(\mathbb{C})$ are elements of the commutative algebra

$$\mathcal{O}[\mathrm{GL}_n(\mathbb{C})] = \mathbb{C}[a_{11}, a_{12}, \ldots, a_{21}, a_{22}, \ldots, a_{nn}, \overset{-1}{\det}].$$

Let G be a group, and let V be a complex vector space. We denote by $\mathrm{GL}(V)$ the group of automorphisms of V. A representation of a G is a pair (π, V) such that $\pi : G \to \mathrm{GL}(V)$ is a group homomorphism. We will often refer to this representation simply by V. The *dimension* of the representation is the dimension of the space V.

Given an n-dimensional complex vector space V, we can define the group isomorphism $\psi : \mathrm{GL}(V) \to \mathrm{GL}_n(\mathbb{C})$ by choosing a basis for V. We then define the algebra of regular functions on $\mathrm{GL}(V)$, denoted $\mathcal{O}[\mathrm{GL}(V)]$, to be

$$\mathcal{O}[\mathrm{GL}(V)] = \{f \circ \psi \ : \ f \in \mathcal{O}[\mathrm{GL}_n(\mathbb{C})]\}.$$

The group $\mathrm{GL}(V)$ has the structure of an affine variety. The group operations are compatible with the variety structure. A Zariski closed subgroup of $\mathrm{GL}(V)$ will be called a *linear algebraic group*. Set

$$\mathcal{O}[G] = \{f|_G \ : \ f \in \mathcal{O}[\mathrm{GL}(V)]\},$$

the algebra of regular functions on G.

Let G be a linear algebraic group. Denote by V^* the complex dual of the vector space V, defined as the space consisting of all $\mathbb{C}$-linear functionals on V. For the representation (π, V) as above, we define the *matrix coefficients* of π to be the maps

$$g \mapsto v^*(\pi(g)v)$$

for $g \in G$, $v \in V$, $v^* \in V^*$. If V is finite dimensional and the matrix coefficients of π are regular functions, then we say that the representation is *regular*.

Suppose V has a subspace W that is preserved under the action of G; that is, for $g \in G$ and $w \in W$, we have $g.w \in W$. In this case, we say that W is a *subrepresentation* of V. If V has no nontrivial proper subrepresentations (that is, the only subrepresentations are $\{0\}$ and V itself), it is called *irreducible*.

A representation V of G is called *decomposable* if it can be written as a direct sum of two nontrivial subrepresentations. We say V is *completely reducible* if it can be written as a direct sum of irreducible components. If each irreducible representation appears in the decomposition only once, we say the representation is *multiplicity free*. If G is a linear algebraic group, all of whose regular representations are completely reducible, then we say G is *reductive*.

Suppose a representation V can be written

$$V = \bigoplus_{i=1}^{k} m_i W^{(i)}$$

where the $W^{(i)}$ are inequivalent[2] irreducible representations and m_i is the multiplicity of $W^{(i)}$ in the sum. Let d_i denote the dimension of $W^{(i)}$. Then

$$\dim V = \sum_{i=1}^{k} m_i d_i$$

That is, we can determine the dimension of the space if we are able to find the

[2]Two representations are equivalent if there is a G-equivariant map between them.

dimensions and multiplicities of its components.

2.3.2 Representations of the Symmetric Group

The conjugacy classes of the symmetric group S_n can be described by partitions of n [30] [7] [19]. As a result of the representation theory of finite groups, the irreducible representations of S_n over $\mathbb{C}$ are indexed by these partitions. Equivalently, these representations correspond to Young diagrams with n boxes, since there are the same number of each. Our indexing follows [12].

A basis for an irreducible S_n representation indexed by a partition λ, which we denote by π_λ, can be described by standard Young tableaux. Thus, the number of standard tableaux with a given shape tells us the dimension of the representation. There is a convenient formula for finding this dimension. Given a Young diagram D_λ with shape λ, we define the *hook length* of a box x in the Young diagram, denoted $\mathrm{hook}(x)$, to be the total number of boxes to the right of and below x, plus one for x itself. The *hook length formula* gives us the dimension of the representation π_λ:

$$\dim \pi_\lambda = \frac{n!}{\Pi_{x \in D_\lambda} \mathrm{hook}(x)}.$$

For any n, the symmetric group S_n has a one-dimensional representation, called the *trivial representation.* It assigns every permutation $\sigma \in S_n$ to the identity map, $\pi(\sigma)x = x$ for $x \in \mathbb{C}$. The trivial representation of S_n corresponds to the Young diagram with one row and n columns.

If $n \geq 2$, the group S_n has another one-dimensional representation, called the *sign representation* or *alternating representation.* Every permutation $\sigma \in S_n$ can be written as a product of transpositions. We define the *sign* of σ, denoted $\mathrm{sgn}(\sigma)$, to be $+1$ if σ can be written as a product of an even number of transpositions, and to be -1 if σ is a product of an odd number of transpositions. The sign of a permutation is well defined, and given two permutations $\sigma_1, \sigma_2 \in S_n$, we have

$\operatorname{sgn}(\sigma_1\sigma_2) = \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2)$. Hence, we can define the sign representation to be $\pi(\sigma)x = \operatorname{sgn}(\sigma)x$ for $x \in \mathbb{C}$. The sign representation of S_n corresponds to the Young diagram with one column and n rows.

For $n \geq 2$, the trivial and sign representations are the only one-dimensional irreducible representations of S_n. We now define another important representation of S_n of dimension $n-1$ that occurs when $n > 2$. Choose a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$, and define an action of S_n on $\mathbb{C}^n$ by

$$\sigma(a_1e_1 + \cdots a_ne_n) = a_1e_{\sigma(1)} + \cdots a_ne_{(\sigma(n))}$$

for $\sigma \in S_n$. This action defines an n-dimensional representation of S_n called the *permutation representation*, which is not irreducible. It has a one-dimensional subspace spanned by the sum of basis vectors $e_1 + \cdots + e_n$. The orthogonal complement of this subspace spanned by vectors $(v_1, \ldots, v_n) \in \mathbb{C}^n$ such that $v_1 + \cdots + v_n = 0$ is an $n-1$ dimensional irreducible representation, called the *standard representation* of S_n. The standard representation corresponds to a Young diagram with two rows, the first containing $n-1$ boxes, and the second containing a single box.

Example 3. *Consider the standard representation of S_5, associated to the Young diagram of shape $(4,1)$. There are four standard tableaux with this shape:*

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1}\end{array}$$

and so the dimension of this representation is 4. The hook lengths of the boxes along the first row, from left to right, of the Young diagram are 5, 3, 2, and 1, respectively. The hook length of the single box in the second row is 1. We find the same result using the hook length formula:

$$\dim \pi_{(4,1)} = \frac{5!}{5 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 4$$

In addition to providing information about the irreducible representations of S_n, Young diagrams can be used to describe the decomposition of restricted representations from S_n to S_{n-1}. Each irreducible representation π_λ of S_n also gives us a representation of S_{n-1}, but this representation will not necessarily be irreducible. Upon restriction to S_{n-1}, π_λ is multiplicity free. The irreducible factors of the restricted representation can be obtained by removing a single box from the diagram of shape λ. By Frobenius reciprocity [12], the induced representation of π_λ can be found by taking a direct sum of irreducible representations of S_{n+1} obtained by adding an additional box to the diagram with shape λ. Each of these irreducibles appears with multiplicity one. For example,

$$\mathrm{Res}_{S_4}^{S_5}\left(\ydiagram{3,2}\right) = \ydiagram{2,2} \oplus \ydiagram{3,1}$$

and

$$\mathrm{Ind}_{S_4}^{S_5}\left(\ydiagram{3,1}\right) = \ydiagram{4,1} \oplus \ydiagram{3,2} \oplus \ydiagram{3,1,1}$$

2.3.3 Highest Weight Theory

Let $\mathfrak{g}$ be a complex reductive Lie algebra with Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. An example is $\mathfrak{g} = \mathfrak{gl}_\mathfrak{n}$, the set of $n \times n$ matrices. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and a Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. In the case of $\mathfrak{g} = \mathfrak{gl}_\mathfrak{n}$, we take $\mathfrak{h}$ to be the diagonal matrices and $\mathfrak{b}$ to be the upper triangular matrices. A linear functional $\lambda : \mathfrak{h} \to \mathbb{C}$ is called a *weight* of $\mathfrak{g}$.

A representation of $\mathfrak{g}$ is a linear map $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ so that

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

for all $X, Y \in \mathfrak{g}$, where $\mathfrak{gl}(V)$ is the Lie algebra of endomorphisms of V under the usual commutator bracket. If V is a representation of $\mathfrak{g}$, we define the *weight space* of V with weight λ to be

$$V^\lambda = \{v \in V \mid X.v = \lambda(X).v \text{ for all } X \in \mathfrak{h}\}$$

Note that V^λ is a subspace of V which generalizes the notion of an eigenspace. A weight of the representation V is a weight λ of $\mathfrak{g}$ such that $V^\lambda \neq \{0\}$.

The *adjoint representation* of a Lie algebra $\mathfrak{g}$, denoted *ad*, is the linear representation defined by

$$ad(X)(Y) = [X, Y]$$

for $X, Y \in \mathfrak{g}$. The zero weight space of $\mathfrak{g}$ is the Cartan. The nonzero weights of the adjoint representation of $\mathfrak{g}$ are called *roots*, and the corresponding weight space is called the *root space*, denoted $\mathfrak{g}_\alpha$. We denote the set of roots by Φ. We may now choose a set of *positive roots*, denoted by Φ^+, such that for each $\alpha \in \Phi$, exactly one of α or $-\alpha$ are in Φ^+, and so that if $\alpha, \beta \in \Phi$ and $\alpha + \beta$ is a root, then $\alpha + \beta \in \Phi^+$. We choose Φ^+ so that $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. If a positive root α cannot be written as a sum of other positive roots, we say α is a *simple root*. We denote the set of simple roots by Π.

Once we have fixed Φ^+, we may define a partial ordering on the weights of a representation V of $\mathfrak{g}$. Let λ, μ be two such weights. We say $\mu \preccurlyeq \lambda$ if $\lambda - \mu$ is a nonnegative linear combination of simple roots. Under this ordering, if λ is larger than or equal to all other weights of V, we call λ a *highest weight* of V. If V is irreducible, there will be a unique highest weight.

The finite dimensional irreducible representations of $\mathfrak{g}$ are uniquely determined by their highest weights [12]. This yields a convenient method for indexing the irreducible representations of $\mathfrak{g}$, and hence, irreducible representations of G with $G = \mathfrak{g}$.

For this thesis, we will only need to understand $G = \mathrm{GL}_n$. If λ is a partition of

k with at most n parts, each of which is a positive integer, there is an associated irreducible polynomial representation of GL_n with highest weight λ, which we denote by V^λ [12] [8]. The dimension of this representation is equal to the number of semistandard Young tableaux with shape λ.

Of particular interest are the irreducible polynomial representations of GL_n associated to the partitions whose Young diagrams have a single row or a single column. If $\lambda = (k)$ (that is, λ is the partition whose Young diagram consists of a single row of k boxes), then V^λ is the kth symmetric power, $\mathrm{Sym}^k(\mathbb{C}^n)$. If $\lambda = (1, 1, \ldots, 1)$ is the partition whose Young diagram consists of a single column, then the representation of GL_n with λ as its highest weight is the kth exterior power, $\wedge^k(\mathbb{C}^n)$, when $k \leq n$.

2.4 Gelfand Pairs and Symmetric Pairs

2.4.1 Gelfand Pairs

Suppose G is a finite group, and let H be a subgroup of G. Let 1_H denote the *trivial representation* of H, the one-dimensional representation such that $1_H(h) = 1$ for all $h \in H$. If the induced representation 1_H^G is multiplicity free, then we say (G, H) is a *Gelfand pair* [10]. Equivalently, (G, H) is a Gelfand pair if $HgH = Hg^{-1}H$ for all $g \in G$ [24, Ch. VII, 1.2].

The notion of a Gelfand pair (G, H) extends to a variety of settings. Here, we focus on the case where G is a finite group, though similar ideas exist in other contexts.

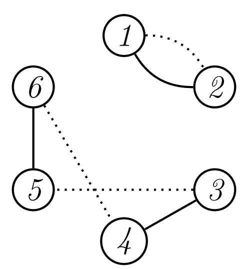
The Gelfand pairs $(S_n, \_\_)$, where S_n denotes the symmetric group on n letters, were classified in [31]. An example of particular interest to this thesis, define

$$\tau_0 = (1\ 2)(3\ 4)(5\ 6)\cdots(2n-1\ 2n)$$

and let $H_n = \{\rho \in S_{2n} \mid \rho\tau = \tau\rho\}$, the centralizer of τ_0. We now show (S_{2n}, H_n) is a Gelfand pair, following [24].

For each permutation $\sigma \in S_{2n}$, we create an undirected edge colored graph $\mathcal{G}(\sigma)$ with $2n$ vertices, numbered 1 through $2n$, as follows. Begin by drawing a solid edge ε_i between vertices $2i-1$ and $2i$, resulting in n solid edges. Next, draw a dotted edge, denoted $\sigma\varepsilon_i$, from $\sigma(2i-1)$ to $\sigma(2i)$. The resulting graph $\mathcal{G}(\sigma)$ will have connected components. The number of edges in each connected component will be called the *length* of the component. Each of these components will have even length. Let $2p_i$ denote the length of the ith connected component. We can then form a partition $P_\sigma = (p_1 \geq p_2 \geq \cdots)$ associated to σ, called the *coset type* of the permutation.

Example 4. *Let* $\sigma = (1\ 5)(2\ 3\ 6)(4)$. *We construct the graph* $\mathcal{G}(\sigma)$ *and find the partition* P_σ. *We have four solid edges,* $\varepsilon_1 = \{1,2\}$, $\varepsilon_2 = \{3,4\}$, *and* $\varepsilon_3 = \{5,6\}$. *We also have four dotted edges,* $\sigma\varepsilon_1 = \{\sigma(1), \sigma(2)\} = \{5,3\}$, $\sigma\varepsilon_2 = \{\sigma(3), \sigma(4)\} = \{6,4\}$, *and* $\sigma\varepsilon_3 = \{\sigma(5), \sigma(6)\} = \{1,2\}$. *The resulting graph is*



Thus we have two connected components:

$$1 \leftrightarrow 2 \Leftrightarrow 1 \qquad \textit{and} \qquad 3 \leftrightarrow 4 \Leftrightarrow 6 \leftrightarrow 5 \Leftrightarrow 3$$

where $\leftrightarrow$ *denotes a solid edge and* $\Leftrightarrow$ *denotes a dotted edge. The lengths of these components are 4 and 2, and so the coset type of* σ *is* $P_\sigma = (2,1)$.

It is possible for two permutations σ_1, σ_2 to result in graphs $\mathcal{G}(\sigma_1), \mathcal{G}(\sigma_2)$ which are isomorphic as edge colored graphs; this occurs if and only if the permutations have the same coset type. In other words, σ_1, σ_2 have the same coset type if and only if there is some permutation $\tau \in S_{2n}$ that preserves edge colored graphs and maps $\mathcal{G}(\sigma_1)$ onto $\mathcal{G}(\sigma_2)$. Suppose $\mathcal{G}(\sigma_1)$ and $\mathcal{G}(\sigma_2)$ are isomorphic, with $\tau(\mathcal{G}(\sigma_1)) = \mathcal{G}(\sigma_2)$. Since this permutation τ preserves the edges ε_i, it must be an element of H_n. Additionally, the edges $\sigma_2\varepsilon_i$ are a permutation of the edges $\tau\sigma_1\varepsilon_i$ of $\tau(\mathcal{G}(\sigma_1))$. Then the edges ε_i are permutations of the $\sigma_2^{-1}\tau\sigma_1\varepsilon_i$. Thus, $\sigma_2^{-1}\tau\sigma_1 \in H_n$. We have now shown that

Proposition 5. *[24] Two permutations $\sigma_1, \sigma_2 \in S_{2n}$ have the same coset type if and only if $\sigma_2 \in H_n\sigma_1 H_n$.*

Next, note that we have $\sigma\mathcal{G}(\sigma^{-1}) = \mathcal{G}(\sigma)$. Thus the graphs of $\mathcal{G}(\sigma^{-1})$ and $\mathcal{G}(\sigma)$ are isomorphic, and so

Proposition 6. *[24] σ and σ^{-1} have the same coset type.*

Combined with the definition of Gelfand pairs for finite groups, these two propositions yield

Theorem 7. *[24, Thm 2.2] (S_{2n}, H_n) is a Gelfand pair.*

2.4.2 Symmetric Pairs

Let G be a connected reductive group over $\mathbb{C}$, with reductive subgroup K. If there exists a regular involution ϕ on G such that K is the group of ϕ-invariant elements, G^ϕ. In this case, we say (G, K) is a *symmetric pair*. The pair (GL_n, O_n) is an example of a symmetric pair, where $\phi : \mathrm{GL}_n \to \mathrm{GL}_n$ is the inverse transpose: $\phi(g) = (g^{-1})^T$.

The Cartan-Helgason theorem gives us a useful result regarding symmetric pairs [11] [17]. The theorem states in part that the G decomposition of regular functions on G/K is multiplicity free whenever (G, K) is a symmetric pair. In particular, we have

Theorem 8. *For the irreducible representation F^λ of $GL_n(\mathbb{C})$ with highest weight λ, we have*

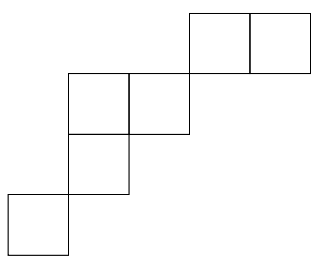
$$\dim(F^\lambda)^{O_n} = \begin{cases} 1 & \textit{if } \lambda \textit{ is an even partition} \\ 0 & \textit{otherwise} \end{cases}$$

A direct proof of this theorem uses the complexified Iwasawa decomposition for $\mathrm{GL}_n(\mathbb{R})$ [12], a generalization of Gram-Schmidt orthogonalization.

2.5 Decomposition of Tensor Products of Irreducible Representations

2.5.1 Littlewood-Richardson coefficients

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ be integer partitions such that $r \geq s$ and $\nu_i \geq \lambda_i$ for $1 \leq i \leq r$. We form a *skew diagram* of shape ν/λ by deleting the first λ_i boxes (starting from the left) of the ith row of the Young diagram of ν. For example, if $\nu = (5, 3, 2, 1)$ and $\lambda = (3, 1, 1)$, we obtain the skew diagram



As with Young diagrams, we can fill the boxes of a skew diagram. We obtain a *semistandard skew tableau* by filling the boxes of a skew diagram with integers that are weakly increasing across its rows and strictly increasing down its columns. The integers used to fill these diagrams form a sequence $(u_1, u_2, \ldots, u_t)$ where u_i is the number of times the number i appears in the diagram. This sequence is called the *weight* of the tableau, and will typically form a partition. The diagrams in Figure 2.2 show examples of semistandard skew tableaux.

(a) $\nu = (5,4,2,1)$, $\lambda = (4,1)$, $\mu = (3,2,1,1)$

(b) $\nu = (5,3,2,1)$, $\lambda = (2,2,1)$, $\mu = (4,1,1)$

Figure 2.2: Semistandard skew tableaux with shape ν/λ and weight μ

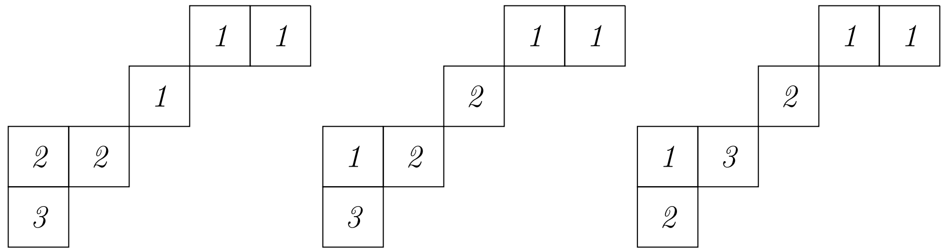
Given such a semistandard skew tableau, we may form a sequence T by concatenating the entries of the rows in reverse order (from right to left), beginning with the topmost row and working down. The sequences formed in this manner for the tableaux shown in Figure 2.2 would then be $(2,3,1,1,2,1,4)$ and $(1,1,1,2,1,3)$. We call this sequence a *lattice word* if each number i appears at least as often as the number $i+1$ at any point in the sequence when read from left to right. The sequence T for the diagram in Figure 2.2 (a) does not form a lattice word, while the sequence formed in this way for the diagram in Figure 2.2 (b) does. A tableau with this property is called a *Littlewood-Richardson tableau.*

Let F^λ and F^μ denote the irreducible representations of general linear groups indexed by partitions λ and μ, respectively (note, we do not require these partitions to be of the same size). We can decompose the tensor product of these representations, $F^\lambda \otimes F^\mu$, according to the well known Littlewood-Richardson rule [23] [24] [28]:

$$F^\lambda \otimes F^\mu = \bigoplus_\nu c^\nu_{\lambda,\mu} F^\nu$$

where the multiplicities $c^\nu_{\lambda,\mu}$, called *Littlewood-Richardson coefficients*, are the number of Littlewood-Richardson tableaux with shape ν/λ and weight μ.

Example 5. *To determine the multiplicity of* $F^{(5,3,2,1)}$ *in the decomposition of the tensor product* $F^{(3,2)} \otimes F^{(3,2,1)}$*, we enumerate the Littlewood-Richardson tableaux with shape* $(5,3,2,1)/(3,2)$ *and weight* $(3,2,1)$*. In this case, we have three such tableaux:*



and so $c^{(5,3,2,1)}_{(3,2),(3,2,1)} = 3$.

The Littlewood-Richardson coefficients may also be described by Schur functions (see Section 2.1.1). They appear as the multiplicity of the Schur functions (as defined in [24]) s_ν in the product of s_λ and s_μ:

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$$

For smaller partitions λ, μ, and ν, it may be possible to determine $c^\nu_{\lambda,\mu}$ by brute force, simply listing all possible skew tableaux and counting those that satisfy the required properties. Fortunately, several algorithms may be employed to find these numbers in more complicated cases. There also exist a number of necessary conditions that must be satisfied for $c^\nu_{\lambda,\mu} > 0$. Several methods for finding the value of $c^\nu_{\lambda,\mu}$ for specific partitions λ, μ, and ν are detailed in [7].

2.5.2 Kronecker coefficients

We now consider a similar decomposition when dealing with tensor products of irreducible representations of symmetric groups[3]. For a partition π, let W^π denote the irreducible representation of S_n indexed by $\pi \vdash n$, as in Section 2.3.2. Given $W^\lambda \otimes W^\mu$, we denote the multiplicity of the irreducible representation W^ν in the product by $g^\nu_{\lambda,\mu}$. That is,

$$W^\lambda \otimes W^\mu = \bigoplus_\nu g^\nu_{\lambda,\mu} W^\nu$$

[3]Here, S_n acts on tensors by the diagonal action.

where $g^{\nu}_{\lambda,\mu}W^{\nu}$ denotes the direct sum of $g^{\nu}_{\lambda,\mu}$ copies of W^{ν}.

The multiplicities $g^{\nu}_{\lambda,\mu}$ are called the *Kronecker coefficients* [9]. A "simple" combinatorial rule for determining these coefficients for general choices of (λ, μ, ν) is unknown, although algorithms for finding them do exist. Formulas for certain cases have been studied, such as when μ, ν are rectangular [25], or μ, ν are two row shapes or hook shapes [29].

As with the Littlewood-Richardson coefficients, the Kronecker coefficients can be defined via Schur functions. While the Littlewood-Richardson coefficients describe the multiplicities in a product of Schur functions, the Kronecker coefficients count the multiplicities in the Kronecker product of Schur functions:

$$s_\lambda * s_\mu = \sum_{\nu} g^{\nu}_{\lambda,\mu} s_\nu$$

We may also consider a tensor product of more than two representations. Suppose we have an r-tuple of partitions $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ of n. We have the decomposition

$$W^{\lambda^{(1)}} \otimes W^{\lambda^{(2)}} \otimes \cdots \otimes W^{\lambda^{(r)}} = \bigoplus_{\mu \vdash m} g_{\lambda^{(1)}\lambda^{(2)}\ldots,\lambda^{(r)}\mu}\, W^{\mu}$$

We also refer to the numbers $g_{\lambda^{(1)}\lambda^{(2)}\ldots\lambda^{(r)}\mu}$ as Kronecker coefficients.

Let $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ be an r-tuple of positive integers. In [14], it is shown that when $K(\mathbf{n}) = U(n_1) \times U(n_2) \times \cdots \times U(n_r)$ is a product of unitary groups and $V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_r}$, we have

$$\dim \mathcal{P}^{2m}_{\mathbb{R}}(V(\mathbf{n}))^{K(\mathbf{n})} = \sum_{(\mu^{(1)},\mu^{(2)},\ldots,\mu^{(r)})} g^2_{\mu^{(1)}\mu^{(2)}\ldots\mu^{(r)}(m)}$$

where the sum is taken over all r-tuples of partitions of m such that the length of $\mu^{(i)}$ is at most n_i. That is, the dimensions of the space of invariant polynomials on $V(\mathbf{n})$ under the action of $K(\mathbf{n})$ is a sum of squares of Kronecker coefficients.

The goal of this thesis is to consider the analogous problem when $K(\mathbf{n})$ is a product of orthogonal groups, $K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$. We will show in Section 3.3 that the Kronecker coefficients arise in this setting as well. In particular, we obtain the sum

$$\dim \mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})} = \sum_{(\mu^{(1)},\mu^{(2)},\ldots,\mu^{(r)})} g_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}}$$

where each $\mu^{(i)}$ is an even partition of $2m$.

Chapter 3

The Dimension of the Invariant Space

In this chapter, we seek a formula for the dimension of the space of degree d homogeneous polynomials on the complex vector space $V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_r}$ under the action of the product group $K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$, for positive integers d and r where each $n_i \geq d$. In Section 3.3, we prove the following result:

Theorem. *Let d be a positive, even integer and let $K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$ and $V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_r}$, where $n_1, n_2, \ldots, n_r \geq d$. Then dimension of the space of $K(\mathbf{n})$-invariant degree d homogeneous polynomial functions on $V(\mathbf{n})$ is*

$$\dim \mathcal{P}^d(V(\mathbf{n}))^{K(\mathbf{n})} = \sum_{\lambda \vdash d} \frac{N(\lambda)^r}{z_\lambda}$$

where $N(\lambda)$ is the number of matchings that commute with a permutation of shape λ.

As it turns out, the number of matchings that commute with a permutation depends on the shape of the permutation. In particular, if the permutation has an odd number of cycles of odd length, there are no matchings which will commute with it. Otherwise, we have the following result, which we show in Section 3.4:

Theorem. *Given a permutation σ with shape $\lambda = (1^{b_1} 2^{b_2} \cdots t^{b_t})$, the number of matchings that commute with σ is given by*

$$N(\lambda) = N((1^{b_1})) \cdot N((2^{b_2})) \cdot \cdots \cdot N((t^{b_t}))$$

where

$$N((a^b)) = \begin{cases} 0 & \textit{if } a \textit{ odd and } b \textit{ odd} \\ \frac{b! a^{b/2}}{2^{b/2}(\frac{b}{2})!} & \textit{if } a \textit{ odd and } b \textit{ even} \\ \sum_{i=1, i \textit{ odd}}^{b} \frac{b! a^{(b-i)/2}}{i!(\frac{b-i}{2})! 2^{(b-i)/2}} & \textit{if } a \textit{ even and } b \textit{ odd} \\ \sum_{i=0, i \textit{ even}}^{b} \frac{b! a^{(b-i)/2}}{i!(\frac{b-i}{2})! 2^{(b-i)/2}} & \textit{if } a \textit{ even and } b \textit{ even} \end{cases}$$

Finally, in Section 3.5, we display some related data generated by these results.

3.1 General Setup

For the remainder of this section, let K denote a compact Lie group, which acts $\mathbb{C}$-linearly on a complex vector space V. We can then define an action of K on the algebra $\mathcal{P}(V)$ of complex valued polynomial functions on V by $k \cdot f(v) = f(k^{-1}v)$ for $k \in K$, $v \in V$, and $f \in \mathcal{P}(V)$. As a graded representation, we have $\mathcal{P}_{\mathbb{R}}(V) \cong \mathcal{P}(V \oplus \overline{V})$ where $\overline{V}$ is the complex vector space with the opposite complex structure (see [26]). We denote the dual representation of V by V^*, and observe that V^* is equivalent, as a representation of K, to $\overline{V}$.

Let $\mathcal{P}^d(V)$ denote the degree d homogeneous polynomials on V; that is, $\mathcal{P}^d(V)$ is the subalgebra of $\mathcal{P}(V)$ such that $f(cv) = c^d f(v)$ for $c \in \mathbb{C}$, $v \in V$, and $f \in \mathcal{P}^d(V)$. We have the standard gradation

$$\mathcal{P}(V) = \bigoplus_{d=0}^{\infty} \mathcal{P}^d(V)$$

Moreover, if $\mathcal{P}(V)^K$ denotes the polynomials in $\mathcal{P}(V)$ invariant under the action of K, we have

$$\mathcal{P}(V)^K = \bigoplus_{d=0}^{\infty} \mathcal{P}^d(V)^K$$

Let G denote the complexification of K, and note that a complex representation of K extends to G. Define $\widehat{G}$ to be the equivalence classes of irreducible rational representations of G. We have

Proposition 9. *[14] Let G be a complex reductive group, and let V be a finite dimensional complex rational representation. Assume that an irreducible representation of G occurs in $\mathcal{P}(V)$ at most in one degree. Then $\mathcal{P}^{2m+1}(V \oplus V^*)^G = 0$ and*

$$\dim \mathcal{P}^{2m}(V \oplus V^*)^G = \sum_{\rho \in \widehat{G}} mult(m, \rho)^2$$

where $mult(m, \rho)$ denotes the multiplicity of the representation ρ in $\mathcal{P}^m(V)$.

Peter-Weyl Decomposition

We now review a useful theorem of Hermann Weyl and Fritz Peter [27]. We focus on an algebraic version of only part of this theorem, following [12].

Let G be a compact group, and let $\mathcal{O}[G]$ denote the regular functions on G. The product group $G \times G$ acts on $\mathcal{O}[G]$ by left and right translation. Let $\widehat{G}$ denote the equivalence classes of irreducible representations of G. Fix a representation (π^λ, F^λ) for each $\lambda \in \widehat{G}$. Finally, for $g \in G$, $\lambda \in \widehat{G}$, $v \in F^\lambda$, and $v^* \in F^{\lambda^*}$ where λ^* is the contragradient representation to λ, define

$$\phi_\lambda(v^* \otimes v)(g) = \langle v^*, \pi^\lambda(g)v \rangle$$

and extend ϕ_λ by linearity to a map from $F^{\lambda^*} \otimes F^\lambda$ to $\mathcal{O}[G]$. We have

Theorem 10. *[12, Thm 4.2.7] Under the action of $G \times G$, we have the decomposition*

$$\mathcal{O}[G] = \bigoplus_{\lambda \in \widehat{G}} \phi_\lambda(F^{\lambda^*} \otimes F^\lambda)$$

Schur-Weyl Duality

Recall that the symmetric group S_m is the group of permutations σ on m letters. Let $V = \bigotimes^m \mathbb{C}^n$, and note that the group S_m acts on V by permuting tensor factors.

That is, given $\sigma \in S_m$ and $v_1, v_2, \ldots, v_m \in \mathbb{C}^n$, we have

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

By linearity, this action extends to all of $\otimes^m \mathbb{C}^n$. The general linear group GL_n also acts on the space V, by the diagonal action

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = (g \cdot v_1) \otimes (g \cdot v_2) \otimes \cdots \otimes (g \cdot v_m)$$

for $g \in \mathrm{GL}_n$. These two actions can be easily seen to commute with each other.

Let $\mathrm{End}(V)$ denote the algebra of endomorphisms of $V = \bigotimes^m \mathbb{C}^n$; that is, the linear maps $f : V \to V$. We have

Theorem 11. *[17] The algebras spanned by the images of GL_n and S_m acting on V as described above are mutual commutants in $End(V)$.*

Consequently, we have

Theorem 12. *[17] Let F_n^λ denote the irreducible rational representation of GL_n with highest weight indexed by λ. Let W_m^λ denote the irreducible complex representation of S_m indexed by λ. Under the joint action of $GL_n \times S_m$ on $\bigotimes^m \mathbb{C}^n$, we have the multiplicity free decomposition*

$$\bigotimes^m \mathbb{C}^n \cong \bigoplus_\lambda F_n^\lambda \otimes W_m^\lambda$$

where the sum is over all partitions λ of m with at most n parts. Note that all irreducible representations of S_m appear in the decomposition when $n \geq m$.

So far, we have only considered the groups S_m and GL_n acting on the vector space $V = \bigotimes^m \mathbb{C}^n$. Note that if V is a representation of a group G, then the tensor $V \otimes V$ is also a representation of G, under the diagonal action. Hence, we may consider the multiplicity of each irreducible representation of G in the decomposition of a tensor product of representations of G under the diagonal action.

Let $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)}$, and λ be partitions of m, and let $W_m^{\mu^{(i)}}$ denote the irreducible S_m-representation indexed by $\mu^{(i)}$. Then the tensor product $W_m^{\mu^{(1)}} \otimes W_m^{\mu^{(2)}} \otimes \cdots \otimes W_m^{\mu^{(r)}}$ is also a representation of S_m. We define $g_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}\lambda}$ to be the multiplicity of the irreducible representation W_m^{λ} in the decomposition of this tensor product; that is,

$$W_m^{\mu^{(1)}} \otimes W_m^{\mu^{(2)}} \otimes \cdots \otimes W_m^{\mu^{(r)}} \cong \bigoplus_{\lambda \vdash m} g_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}\lambda} W_m^{\lambda}$$

Again, let $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)}$ denote partitions, this time with the length of each partition $\mu^{(i)}$ at most n_i for some positive integer n_i. Let $n = n_1 n_2 \cdots n_r$, and let λ be a partitions with length at most n. Let $F_{(n_i)}^{\mu^{(i)}}$ denote the irreducible GL_{n_i}-representation with highest weight indexed by $\mu^{(i)}$, and F_n^{λ} denote the irreducible GL_n-representation with highest weight indexed by λ. Finally, denote

$$G(\mathbf{n}) = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_r}$$

This group $G(\mathbf{n})$ acts on the space $V = \bigotimes_{i=1}^{r} \mathbb{C}^{n_i}$ by

$$(g_1, g_2, \ldots, g_r) \cdot v_1 \otimes v_2 \otimes \cdots \otimes v_r = (g_1 v_1) \otimes (g_2 v_1) \otimes \cdots \otimes (g_r v_r) \tag{3.1}$$

Under this action, we define $k_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}\lambda}$ to be the multiplicity of $F_{n_1}^{\mu^{(1)}} \otimes F_{n_2}^{\mu^{(2)}} \otimes \cdots \otimes F_{n_r}^{\mu^{(r)}}$ in the decomposition of F_n^{λ}; that is,

$$F_n^{\lambda} \cong \bigoplus_{\mu^{(1)},\mu^{(2)},\ldots,\mu^{(r)}} k_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}\lambda} F_{n_1}^{\mu^{(1)}} \otimes F_{n_2}^{\mu^{(2)}} \otimes \cdots \otimes F_{n_r}^{\mu^{(r)}}$$

Theorem 13. *[14] Let $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)}$ and λ be partitions of m with the length* *of $\mu^{(i)}$ at most n_i for all i and the length of λ to be at most $n_1 n_2 \cdots n_r$. Then*

$$g_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}\lambda} = k_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}\lambda}$$

In the case where $\lambda = (m)$ for a positive integer m, we have

$$F_n^\lambda = F_n^{(m)} \cong \mathcal{P}^m((\mathbb{C}^n)^*)$$

and so the decomposition of $\mathcal{P}^m(V(\mathbf{n})^*)$ under the action of $G(\mathbf{n})$ is obtained by computing $k_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}\lambda}$. Note that the partition $\lambda = (m)$ corresponds to the trivial representation, W_m^λ, of S_m.

Counting Orbits

A central result related to counting orbits, known to Augustin Cauchy in 1845 and later attributed to Ferdinand Frobenius by Burnside [4]. This result is popularly known as *Burnside's Lemma*, which we recall below.

Theorem 14. *[4] Let G be a finite group which acts on a set X. Denote by X^g the set of elements in X fixed by the element g of G. The number of orbits of this action is*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |X^g|$$

3.2 The Unitary Case

A goal of this thesis is to determine a formula for the dimension of the subspace of degree d homogeneous polynomials on $V(\mathbf{n}) = \bigotimes_{i=1}^r \mathbb{C}^{n_i}$ invariant under the action of the product $K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$, for an r-tuple $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ with $n_i \in \mathbb{Z}_+$. A similar problem was considered in [14], where a formula is given for these dimensions when the group $K(\mathbf{n})$ is replaced by a product of unitary groups,

$K(\mathbf{n}) = U(n_1)\times U(n_2)\times\cdots\times U(n_r)$. We now review these results using the notation of Section 3.1 and [14], before proceeding to the proof of our main theorem.

Let $K(\mathbf{n}) = U(n_1)\times U(n_2)\times\cdots\times U(n_r)$. The complexification of $K(\mathbf{n})$ is $G(\mathbf{n})$. Recall that $\widehat{G(\mathbf{n})}$ is the set of equivalence classes of irreducible representations of $G(\mathbf{n})$. Let $mult(m,\rho)$ denote the multiplicity of ρ in $\mathcal{P}^m(V(\mathbf{n}))$. By Proposition 9, we have

$$\dim \mathcal{P}^{2m}_{\mathbb{R}}(V(\mathbf{n}))^{K(\mathbf{n})} = \sum_{\rho\in\widehat{G(\mathbf{n})}} mult(m,\rho)^2$$

The problem of finding these multiplicities is now equivalent to decomposing $\mathcal{P}(V(\mathbf{n}))$ and $\mathcal{P}(V(\mathbf{n})^*)$ under the action of $G(\mathbf{n})$. Each irreducible representation of $G(\mathbf{n})$ occurs in $\mathcal{P}(V(\mathbf{n}))$ if and only if its dual occurs in $\mathcal{P}(V(\mathbf{n})^*)$. Moreover, these representations will occur with the same multiplicity. Hence, we only need to consider the decomposition of $\mathcal{P}^m(V(\mathbf{n})^*)$.

Let $\lambda = (m)$. We have

$$\mathcal{P}^m(V(\mathbf{n})^*) \cong \bigoplus_{\substack{\mu^{(i)}\vdash m\\ \ell(\mu^{(i)})\le n_i}} g_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}(m)} F^{\mu^{(1)}}_{(n_1)}\otimes F^{\mu^{(2)}}_{(n_2)}\otimes\cdots\otimes F^{\mu^{(r)}}_{(n_r)}$$

with the action defined in 3.1. Thus $mult(m,\rho) = g_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}(m)}$. Additionally, since (m) corresponds to the trivial representation of S_m, we have $g_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}(m)} = g_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}}$; hence

$$\dim \mathcal{P}^{2m}_{\mathbb{R}}(V(\mathbf{n}))^{K(\mathbf{n})} = \sum_{(\mu^{(1)},\mu^{(2)},\ldots,\mu^{(r)})} g^2_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(r)}(m)}$$

where each $\mu^{(i)}\vdash m$ and the length of $\mu^{(i)}$ is at most n_i. In general, the left hand side of the above equation may not be easily computed. However, if we add the requirement that $m \le \min(n_1, n_2, \ldots, n_r)$, the values stabilize to a non-negative integer depending on m and r. In particular, we have

Theorem 15. *[14] For $V(\mathbf{n}) = \bigotimes_{i=1}^{r} \mathbb{C}^{n_i}$ and $K(\mathbf{n}) = U(n_1) \times \cdots \times U(n_2)$*

$$\dim \mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})} = \sum_{\lambda \vdash m} z_\lambda^{r-2}$$

where $n_i \geq m$ for $1 \leq i \leq r$.

3.3 The Orthogonal Case

We now return to the central theme of this thesis, and let $K(\mathbf{n})$ denote a product of orthogonal groups. For $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ define

$$K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$$

where O_{n_i} is the complex orthogonal group of rank n_i.

For each partition λ of d with at most n parts, we again let F_n^λ denote the irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight indexed by λ. If $\lambda^{(i)} \vdash d$ for $1 \leq i \leq r$, then

$$F_{n_1}^{\lambda^{(1)}} \otimes F_{n_2}^{\lambda^{(2)}} \otimes \cdots \otimes F_{n_r}^{\lambda^{(r)}}$$

is an irreducible representation of $G(\mathbf{n})$ which embeds in $\mathcal{P}^d(V(\mathbf{n}))$ with multiplicity denoted by $g_{\lambda^{(1)}\lambda^{(2)}\ldots\lambda^{(r)}}$, where again $G(\mathbf{n})$ denotes the product of general linear groups $\mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{C})$. That is,

$$\mathcal{P}^d(V(\mathbf{n})) \cong \bigoplus_{\substack{\lambda^{(i)} \vdash d \\ \ell(\lambda^{(i)}) \leq n_i}} g_{\lambda^{(1)}\lambda^{(2)}\ldots\lambda^{(r)}} F_{n_1}^{\lambda^{(1)}} \otimes F_{n_2}^{\lambda^{(2)}} \otimes \cdots \otimes F_{n_r}^{\lambda^{(r)}}$$

We can now write the $K(\mathbf{n})$-invariants as

$$\begin{aligned}\left[\mathcal{P}^d(V(\mathbf{n}))\right]^{K(\mathbf{n})} &\cong \bigoplus_{\substack{\lambda^{(i)}\vdash d\\ \ell(\lambda^{(i)})\leq n_i}} g_{\lambda^{(1)}\lambda^{(2)}\cdots\lambda^{(r)}}\left(F_{n_1}^{\lambda^{(1)}}\otimes F_{n_2}^{\lambda^{(2)}}\otimes\cdots\otimes F_{n_r}^{\lambda^{(r)}}\right)^{K(\mathbf{n})}\\ &\cong \bigoplus_{\substack{\lambda^{(i)}\vdash d\\ \ell(\lambda^{(i)})\leq n_i}} g_{\lambda^{(1)}\lambda^{(2)}\cdots\lambda^{(r)}}\left(F_{n_1}^{\lambda^{(1)}}\right)^{O_{n_1}}\otimes\left(F_{n_2}^{\lambda^{(2)}}\right)^{O_{n_2}}\otimes\cdots\otimes\left(F_{n_r}^{\lambda^{(r)}}\right)^{O_{n_r}}\end{aligned}$$

The Cartan-Helgason theorem (Theorem 8) tells us that if F_n^λ is an irreducible representation of $\mathrm{GL}_n(\mathbb{C})$, then $\dim\left(F_n^\lambda\right)^{O_n}$ is at most one. In particular, we have

$$\dim\left(F_n^\lambda\right)^{O_n} = \begin{cases}1 & \text{if } \lambda \text{ is an even partition}\\ 0 & \text{otherwise}\end{cases}$$

and so

$$\dim\left[\mathcal{P}^d(V(\mathbf{n}))\right]^{K(\mathbf{n})} = \sum_{\substack{\lambda^{(1)},\ldots,\lambda^{(r)}\\ \lambda^{(i)}\vdash d\\ \lambda^{(i)}\text{ even}}} g_{\lambda^{(1)}\lambda^{(2)}\cdots\lambda^{(r)}} \tag{3.2}$$

An immediate consequence of Equation 3.2 is that $\dim\left[\mathcal{P}^d(V(\mathbf{n}))\right]^{K(\mathbf{n})} = 0$ whenever d is odd (this fact is also evident in Proposition 9). Hence, for the remainder of this chapter, we write $d = 2m$ for a positive integer m. In addition, we observe that these dimensions stabilize when the sum on the right is over all even partitions of $d = 2m$. This occurs when each of the n_i's are sufficiently large; that is, we require $n_i \geq 2m$ for all $1 \leq i \leq r$. We will assume this stability condition is met for $\mathbf{n} = (n_1, \ldots, n_r)$ in what is to follow.

The summands appearing in Equation 3.2 are in fact the same multiplicities that appeared in Section 3.2. An immediate difference is that the summands here are not squared, as the exponent in the unitary case was a result of doubling the space. However, the same challenge is faced as in the previous section: a closed formula for these multiplicities is unknown, and so a sum of these terms is not particularly useful.

Schur-Weyl duality allows us to find another interpretation of Equation 3.2. As noted in Section 3.1, while GL_n acts on the space $\otimes^m \mathbb{C}^n$ by simultaneous matrix multiplication, the symmetric group acts on the same space by permuting tensor factors. That is, given $x \in \mathrm{GL}_n$ and a permutation $\sigma \in S_m$, we have

$$x(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = xv_1 \otimes xv_2 \otimes \cdots \otimes xv_m$$

and

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

for $v_1, v_2, \ldots, v_m \in \mathbb{C}^n$. By Theorem 11, we obtain the multiplicity free decomposition

$$\otimes^m \mathbb{C}^n \cong F_n^\lambda \otimes W_m^\lambda$$

where W_m^λ is an irreducible representation of S_m indexed by the partition λ.

Let W^λ denote the irreducible representation of the symmetric group S_{2m} indexed by the partition λ. We have

$$\dim(W^{\lambda^{(1)}} \otimes W^{\lambda^{(2)}} \otimes \cdots \otimes W^{\lambda^{(r)}})^{S_{2m}} = g_{\lambda^{(1)}\lambda^{(2)}\cdots\lambda^{(r)}} \tag{3.3}$$

where each $\lambda^{(i)} \vdash 2m$. The numbers $g_{\lambda^{(1)}\lambda^{(2)}\cdots\lambda^{(r)}}$ are the Kronecker coefficients, detailed in Section 2.5.

Fix $\tau_0 = (1\ \ 2)(3\ \ 4)\cdots(2m-1\ \ 2m)$ in S_{2m}. Denote by H_m the centralizer of τ_0 in S_{2m}. The group H_m is isomorphic to the hyperoctohedral group of degree m, and is also the wreath product of S_2 and S_m. Furthermore, (S_{2m}, H_m) is a Gelfand pair [24] (see Section 2.4). Of particular interest is the following theorem:

Theorem 16. *[24, Thm 2.5] Let W^λ be the irreducible representation of S_{2m} cor-*

responding to a partition λ of $2m$. We have

$$\dim(W^\lambda)^{H_m} = \begin{cases} 1 & \textit{if } \lambda \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$$

We now have

$$\begin{aligned} \dim[W^{\lambda^{(1)}} \otimes \cdots \otimes W^{\lambda^{(r)}}]^{H_m \times \cdots \times H_m} &= \dim(W^{\lambda^{(1)}})^{H_m} \cdot \dim(W^{\lambda^{(2)}})^{H_m} \cdot \cdots \cdot \dim(W^{\lambda^{(r)}})^{H_m} \\ &= \begin{cases} 1 & \text{if } \lambda^{(i)} \text{ is even for all } i \in \{1, 2, \ldots, r\} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.4}$$

As an immediate consequence of the Peter-Weyl decomposition presented in Theorem 10, we have

$$\mathbb{C}[S_{2m}] \cong \bigoplus_\lambda W^\lambda \otimes W^\lambda$$

We can now describe the group algebra of a product of r copies of S_{2m} as follows:

$$\begin{aligned} \mathbb{C}[S_{2m} \times S_{2m} \times \cdots \times S_{2m}] &\cong \mathbb{C}[S_{2m}] \otimes \mathbb{C}[S_{2m}] \otimes \cdots \otimes \mathbb{C}[S_{2m}] \\ &\cong \left(\bigoplus_\lambda W^\lambda \otimes W^\lambda\right) \otimes \left(\bigoplus_\lambda W^\lambda \otimes W^\lambda\right) \otimes \cdots \otimes \left(\bigoplus_\lambda W^\lambda \otimes W^\lambda\right) \\ &\cong \bigoplus_{\lambda^{(1)}, \ldots, \lambda^{(r)}} \left(W^{\lambda^{(1)}} \otimes W^{\lambda^{(1)}}\right) \otimes \cdots \otimes \left(W^{\lambda^{(r)}} \otimes W^{\lambda^{(r)}}\right) \end{aligned} \tag{3.5}$$

S_{2m} acts on the product $S_{2m}/H_m \times \cdots \times S_{2m}/H_m$ by left multiplication:

$$\sigma(g_1 H_m, \ldots, g_r H_m) = (\sigma g_1 H_m, \ldots, \sigma g_r H_m)$$

for $\sigma, g_1, \ldots, g_r \in S_{2m}$. Additionally, S_{2m} acts on $\mathbb{C}[(S_{2m}/H_m)^r]$ on the left. Thus,

$$\begin{aligned}\mathbb{C}[S_{2m} \times \cdots \times S_{2m}]^{S_{2m} \times H_m \times \cdots \times H_m} &= \mathbb{C}[S_{2m}/H_m \times \cdots \times S_{2m}/H_m]^{S_{2m}} \\ &= \{f : S_{2m}^r \to \mathbb{C} \mid f(g^{-1}xh) = f(x)\}\end{aligned}$$

for $g \in S_{2m}$, $x = (x_1, \ldots, x_r) \in S_{2m}^r$, and $h = (h_1, \ldots, h_r) \in H_m^r$. Now by Equations 3.3, 3.4, and 3.5, we obtain

$$\begin{aligned}&\dim \mathbb{C}[S_{2m} \times S_{2m} \times \cdots \times S_{2m}]^{S_{2m} \times H_m \times \cdots \times H_m} \\ &\quad = \sum_{\substack{\lambda^{(1)}, \ldots, \lambda^{(r)} \\ \lambda^{(i)} \vdash 2m}} \dim \left[\left(W^{\lambda^{(1)}} \otimes \cdots \otimes W^{\lambda^{(r)}} \right)^{S_{2m}} \otimes \left(W^{\lambda^{(1)}} \otimes \cdots \otimes W^{\lambda^{(r)}} \right)^{H_m \times \cdots \times H_m} \right] \\ &\quad = \sum_{\substack{\lambda^{(1)}, \ldots, \lambda^{(r)} \\ \lambda^{(i)} \vdash 2m \\ \lambda^{(i)} \text{ even}}} g_{\lambda^{(1)} \lambda^{(2)} \ldots \lambda^{(r)}}\end{aligned}$$

So far, we have shown that

$$\dim \left[\mathcal{P}^d(V(\mathbf{n}))\right]^{K(\mathbf{n})} = \dim \mathbb{C}[S_{2m} \times S_{2m} \times \cdots \times S_{2m}]^{S_{2m} \times H_m \times \cdots \times H_m}$$

For another interpretation, we define the set

$$\tilde{I}_{2m} = \{\tau \in S_{2m} \ : \ \tau^2 = \mathrm{id}, \tau(i) \neq i \text{ for all } i \leq n\}$$

That is, $\tilde{I}_{2m}$ is the set of matchings on $2m$ letters. As an S_{2m}-set, we have $\tilde{I}_{2m} \cong S_{2m}/H_m$. We will denote the product of r copies of $\tilde{I}_{2m}$ by $\tilde{I}_{2m}^r$.

Recall that given a group G and $H \subset G$, we have $\dim \mathbb{C}[G]^H = \dim \mathbb{C}[G/H]$.

Thus, by the previous results

$$\sum_{\substack{\lambda^{(1)},\ldots,\lambda^{(r)} \\ \lambda^{(i)}\vdash 2m \\ \lambda^{(i)}\text{even}}} g_{\lambda^{(1)}\lambda^{(2)}\cdots\lambda^{(r)}} = \dim \mathbb{C}[S_{2m} \times S_{2m} \times \cdots \times S_{2m}]^{S_{2m}\times H_m \times \cdots \times H_m}$$

$$= \dim \mathbb{C}[S_{2m}/H_m \times \cdots \times S_{2m}/H_m]^{S_{2m}}$$

$$= \dim \mathbb{C}[\tilde{I}_{2m}^r]^{S_{2m}}$$

where S_{2m} acts on r-tuples of matchings by simultaneous conjugation. That is, given $(\tau_1, \tau_2, \ldots, \tau_r) \in \tilde{I}_{2m}^r$, $\sigma \in S_{2m}$, we define the action

$$\sigma \cdot (\tau_1, \tau_2, \ldots, \tau_r) = (\sigma\tau_1\sigma^{-1}, \sigma\tau_2\sigma^{-1}, \ldots, \sigma\tau_r\sigma^{-1})$$

Given a group G and a set X, it is well known that the dimension of the space of invariants $\mathbb{C}[X]^G$ is equal to the number of orbits of the action of G on X. By Theorem 14, the number of such orbits is the average number of $x \in X$ fixed by $g \in G$; that is,

$$\dim \mathbb{C}[X]^G = \frac{1}{|G|}\sum_{g\in G} |X^g|$$

where $|X^g|$ denotes the cardinality of the set of points in X fixed by g. In our setting, where $G = S_{2m}$ and $X = \tilde{I}_{2m}^r$, we have

$$\dim \mathbb{C}[\tilde{I}_{2m}^r]^{S_{2m}} = \frac{1}{|S_{2m}|}\sum_{\sigma\in S_{2m}} |(\tilde{I}_{2m}^r)^\sigma|$$

We easily see that

$$|(\tilde{I}_{2m}^r)^\sigma| = \Big(|(\tilde{I}_{2m})^\sigma|\Big)^r$$

and so, given $\sigma \in S_{2m}$, it remains only to find a formula for the number of matchings $\tau \in \tilde{I}_{2m}$ such that $\sigma\tau\sigma^{-1} = \tau$. Clearly this number is the same for two permutations with the same cycle type, as we can simply relabel the entries of each cycle. Thus, if $\sigma, \rho \in S_{2m}$ have the same cycle type, we have $|(\tilde{I}_{2m})^\sigma| = |(\tilde{I}_{2m})^\rho|$.

Recall that two elements in S_{2m} are conjugate if they have the same cycle type. Denote by $\widehat{S_{2m}}$ the set of conjugacy classes in S_{2m}, indexed by integer partitions μ of $2m$. Hence we can define a class function $N : \widehat{S_{2m}} \to \mathbb{N}$ by setting $N(\lambda)$ equal to the number of matchings $\tau \in \tilde{I}_{2m}$ that commute with a permutation $g \in S_{2m}$ with cycle type μ. Finally, we have shown that for $V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_r}$, $K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$ and $n_i \geq 2m$ for $1 \leq i \leq r$, we have

$$\dim \left[\mathcal{P}^{2m}(V(\mathbf{n}))\right]^{K(\mathbf{n})} = \dim \mathbb{C}[\tilde{I}_{2m}^r]^{S_{2m}} = \frac{1}{|S_{2m}|} \sum_{\sigma \in S_{2m}} |(\tilde{I}_{2m}^r)^\sigma| = \frac{1}{(2m)!} \sum_{\lambda \vdash 2m} N(\lambda)^r$$

The formula for $N(\lambda)$ is presented in Theorem 3, which we prove in the next section.

3.4 The Number of Matchings Which Commute With A Fixed Permutation

We now determine the formula for $N(\lambda)$, the number of matchings which commute with a permutation with cycle type λ. Our strategy will be to focus initially on permutations with shape (m^2). We then consider "brick" permutations with shape (a^b), where $ab = 2m$, and finally we generalize to all permutations.

The formula $N(\lambda)$ has appeared in [13]. In particular, it is shown that if $\chi^\rho(\nu)$ denotes the value of the irreducible character of S_n indexed by ρ at λ, where $\lambda \vdash n$, we have [24, Thm VII.2.4] [13]:

$$N(\lambda) = \sum_{\substack{\rho \,\vdash\, n \\ \rho \text{ is even}}} \chi^\rho(\lambda)$$

3.4.1 Case I: $\lambda = (m^2)$

We begin with:

Lemma 17. *Let $g = (\alpha_1\ \alpha_2 \cdots\ \alpha_m)(\beta_1\ \beta_2\ \cdots \beta_m)$ be a product of two cycles of equal length m. If σ is a matching that commutes with g, then either $\sigma = g^{m/2}$ or σ has form*

$$\sigma = (\alpha_1\ \beta_j)(\alpha_2\ \beta_{j+1})(\alpha_3\ \beta_{j+2}) \cdots (\alpha_m\ \beta_{j-1})$$

where $0 \leq i < m$. Hence, if m is even there are exactly $m + 1$ matchings that commute with g, and m such matchings if m is odd.

Proof. Suppose first that σ permutes the elements within each cycle of g. Let $\sigma(\alpha_k) = \alpha_m$ for some k with $1 \leq k \leq m$; that is, assume the transposition $(\alpha_k\ \alpha_m)$ appears in σ. Then

$$\alpha_{k+1} = g(\alpha_k) = \sigma g \sigma(\alpha_k) = \sigma g(\alpha_m) = \sigma(\alpha_1)$$

and so $(\alpha_1\ \alpha_{k+1})$ is a transposition in σ. Similarly,

$$\alpha_{k+2} = g(\alpha_{k+1}) = \sigma g \sigma(\alpha_{k+1}) = \sigma g(\alpha_1) = \sigma(\alpha_2)$$

and so $(\alpha_2\ \alpha_{k+2})$ is a transposition in σ. Continuing in this way, we find σ contains the transpositions $(\alpha_1\ \alpha_{k+1}), (\alpha_2\ \alpha_{k+2}), (\alpha_3\ \alpha_{k+3}), \ldots, (\alpha_k\ \alpha_m)$. Now

$$\sigma(\alpha_m)\alpha_k = g(\alpha_{k-1}) = \sigma g \sigma(\alpha_{k-1}) = \sigma g(\alpha_{2k-1}) = \sigma(\alpha_{2k})$$

and hence $m = 2k$. Then we have only one possible choice for k, $k = m/2$. Thus

$$\sigma = (\alpha_1\ \alpha_{\frac{m}{2}+1})(\alpha_2\ \alpha_{\frac{m}{2}+2})(\alpha_3\ \alpha_{\frac{m}{2}+3}) \cdots (\alpha_{\frac{m}{2}}\ \alpha_m) = (\alpha_1\ \alpha_2 \cdots\ \alpha_m)^{m/2}$$

Thus if σ permutes the elements within the cycles of g, we must have $\sigma = g^{m/2}$. Note such a σ only exists if m is even.

Now suppose σ interchanges elements between the two cycles of g. Assume

$\sigma(\alpha_1) = \beta_j$ for some j with $1 \leq j \leq m$. Then $(\alpha_1\ \beta_j)$ is a transposition appearing in σ. Since $\sigma g \sigma = g$, we have

$$\alpha_2 = g(\alpha_1) = \sigma g \sigma(\alpha_1) = \sigma g(\beta_j) = \sigma(\beta_{j+1})$$

and so $(\alpha_2\ \beta_{j+1})$ appears in σ. Similarly,

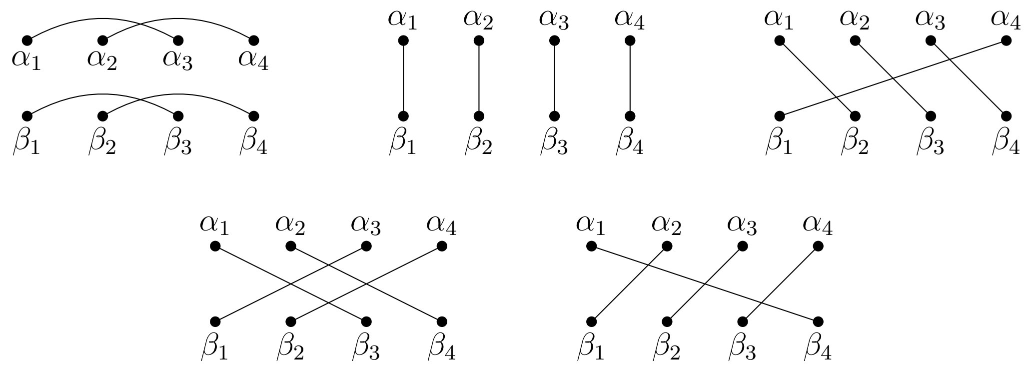
$$\alpha_3 = g(\alpha_2) = \sigma g \sigma(\alpha_2) = \sigma g(\beta_{j+1}) = \sigma(\beta_{j+2})$$

and so $(\alpha_3\ \beta_{j+2})$ appears in σ. Continuing in this way, we have

$$\sigma = (\alpha_1\ \beta_j)(\alpha_2\ \beta_{j+1})(\alpha_3\ \beta_{j+2}) \cdots (\alpha_m\ \beta_{j-1})$$

$\square$

The matchings that will commute with a fixed permutation with two cycles of equal length can easily be interpreted as diagrams. With g as in the above proof, we draw two rows of m nodes. We label the nodes along the top row with $\alpha_1, \alpha_2, \ldots, \alpha_m$, and along the bottom row with $\beta_1, \beta_2, \ldots, \beta_m$. Drawing edges so that each node is connected to exactly one other node defines a matching. When we consider only the diagrams corresponding to matchings which commute with g, an immediate pattern emerges. If m is even, the first such diagram is obtained by drawing an edge from α_1 to $\alpha_{\frac{m}{2}+1}$, an edge from α_2 to $\alpha_{\frac{m}{2}+2}$, and so on. The nodes labeled β_i are matched in an identical way, resulting in a diagram which represents the matching $g^{m/2}$. The remaining m diagrams (for odd or even values of m) are obtained by drawing edges from α_i to β_i for each i, and then cyclically permuting the second row of unlabeled nodes. The five matchings that commute with $g = (\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)(\beta_1\ \beta_2\ \beta_3\ \beta_4)$, for instance, are:



3.4.2 Case II: $\lambda = (a^b)$

Suppose now that our permutation g has b cycles of length a, where $ab = 2m$. These "brick" permutations have rectangular Young diagrams with b rows and a columns.

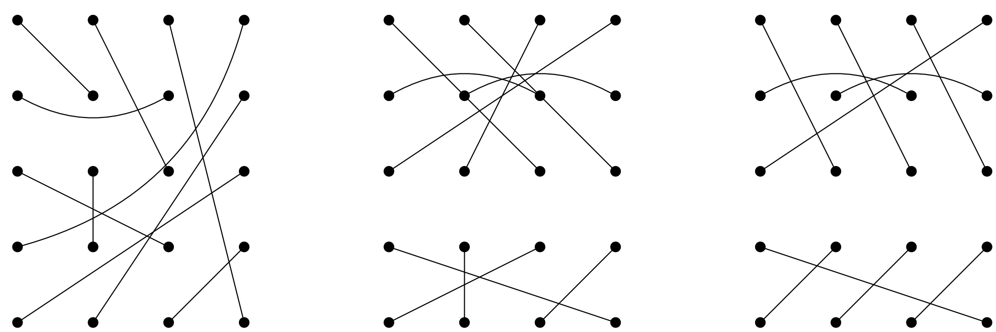
Recall that we can conjugate g by a permutation σ by applying σ to each symbol of each cycle of g. That is, if $g = (\alpha_1\ \alpha_2\ \cdots\ \alpha_a)(\beta_1\ \beta_2\ \cdots\ \beta_a)\cdots$, we have

$$\sigma g \sigma^{-1} = (\sigma(\alpha_1)\ \sigma(\alpha_2)\ \cdots\ \sigma(\alpha_a))(\sigma(\beta_1)\ \sigma(\beta_2)\ \cdots\ \sigma(\beta_a))\cdots$$

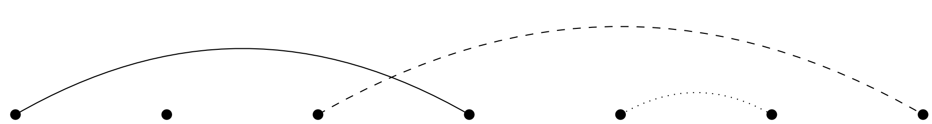
It follows that if σ commutes with g, conjugation by σ sends one cycle of g to another cycle of g. Thus we can take pairs of cycles of g and look for the matchings on $2a$ numbers that commute with the product of these pairs following Lemma 17. If we have an odd number of cycles, the unpaired cycle will commute with a power of itself, that power being $a/2$. The product of these will be a matching which commutes with g itself.

To illustrate, we'll quickly compute a matching which commutes with the permutation $g = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8)(9\ 10\ 11\ 12)$. Suppose we first pair the first and third cycle, and look for matchings which commute with $(1\ 2\ 3\ 4)(9\ 10\ 11\ 12)$. By Lemma 17, we have five to choose from, such as $(1\ 10)(2\ 11)(3\ 12)(4\ 9)$. We still have one cycle of g left, and there is only one matching to commute with it: $(5\ 7)(6\ 8)$. Hence, a matching that commutes with g is $(1\ 10)(2\ 11)(3\ 12)(4\ 9)(5\ 7)(6\ 8)$.

Once again, we turn to diagrams to simplify. After fixing a cycle expression for g, we draw a diagram with b rows of a nodes, and label the nodes of the ith row with the entries of the ith cycle of g, as before. Again, we can define a matching by drawing an edge between pairs of nodes. The following diagrams illustrate the case where g has cycle type $\lambda = (4^5)$. The diagram on the left displays a matching, but not a matching that will commute with the permutation g, as it fails to map one cycle of g to another. The center diagram meets this requirement, but does not satisfy Lemma 17. Finally, the diagram on the right corresponds to a matching that will commute with g. For simplicity, we have suppressed labeling of the nodes.



We have now established a convenient way of establishing which matchings commute with a permutation of shape (a^b), it is simply a matter of counting them. To do this, we will compress the diagrams which correspond to an eligible matching as follows: for each cycle of the permutation, draw a single node. By Proposition 17, there are a matchings that will commute with two distinct cycles. Assign a color to each of these choices, and color the edge between the corresponding nodes accordingly. If matching contains the $a/2$-th power of a cycle, the corresponding node is left isolated. Thus $N((a^b))$ is obtained by counting graphs of the form



where each edge type (solid, dashed, and dotted) represents a different color.

We consider three sub cases, depending on the parity of a and b (recall we cannot allow both a and b to be odd, as we require ab to be even):

a **odd,** b **even**

If a is odd, any matching that commutes with g cannot contain a power of a cycle of g, so we have no isolated nodes. The number of uncolored diagrams in this case is equal to the number of matchings on b letters, and each of the $b/2$ edges of the diagram can be colored in one of a ways. Hence we have

$$N((a^b)) = \frac{b!a^{b/2}}{2^{b/2}\left(\frac{b}{2}\right)!}$$

a **even,** b **even**

If a and b are even, we can allow an even number of isolated nodes. If our diagram contains i isolated nodes, the number of uncolored diagrams is equal to the number of permutations on b with shape $(1^i 2^{b-i})$, and each of the $(b-i)/2$ edges of the diagram can be colored in one of a ways. Hence we have

$$N((a^b)) = \sum_{i=0, i \text{ even}}^{b} \frac{b!a^{(b-i)/2}}{i!(\frac{b-i}{2})!2^{(b-i)/2}}$$

a **even,** b **odd**

Similarly, if b is odd we can allow an odd number of isolated nodes, and so

$$N((a^b)) = \sum_{i=1, i \text{ odd}}^{b} \frac{b!a^{(b-i)/2}}{i!(\frac{b-i}{2})!2^{(b-i)/2}}$$

3.4.3 Case III: The General Case

Lemma 18. *Let g be a permutation and and let σ be a matching such that $\sigma g \sigma = g$. If $(\alpha\ \beta)$ appears in σ, then $\alpha, \beta \in \{1, 2, \ldots, n\}$ appear in cycles of the same length* *in g.*

Proof. Suppose α appears in the cycle $(\alpha\ \alpha_1\ \alpha_2\ \cdots\ \alpha_k)$ and β appears in the cycle $(\beta\ \beta_1\ \beta_2\ \cdots\ \beta_l)$. Assume without loss of generality that $k \leq l$. We have

$$\alpha_1 = g(\alpha) = \sigma g \sigma(\alpha) = \sigma g(\beta) = \sigma(\beta_1)$$

and so $(\alpha_1\ \beta_1)$ is a transposition appearing in σ. Similarly,

$$\alpha_2 = g(\alpha_1) = \sigma g \sigma(\alpha_1) = \sigma g(\beta_1) = \sigma(\beta_2)$$

and so $(\alpha_2\ \beta_2)$ is also a transposition appearing in σ. Continuing in this way, we find that σ must contain the transpositions

$$(\alpha\ \beta), (\alpha_1\ \beta_1), (\alpha_2\ \beta_2), \ldots, (\alpha_k\ \beta_k)$$

Suppose for contradiction that $k < l$. Then there must exist some $\gamma \in \{1, 2, \ldots, n\}$ such that $\gamma \notin \{\alpha, \alpha_1, \ldots, \alpha_k\}$ and $(\gamma\ \beta_l)$ is a transposition appearing in σ. Then

$$\beta = g(\beta_l) = \sigma g \sigma(\beta_l) = \sigma g(\gamma)$$

By hypothesis $\beta = \sigma(\alpha)$, so $\alpha = g(\gamma)$. But $\alpha = g(\alpha_k)$, a contradiction, since $\gamma \neq \alpha_k$. Hence $k = l$. □

Thus, if g has b_i cycles of length i, we need only count the number of matchings which commute with the product of these b_i cycles, which we can calculate using the formulas already determined. To find the total number of matchings which commute with g, we simply take the product. That is,

$$N(\lambda) = N((1^{b_1}))N((2^{b_2})) \cdots N((t^{b_t}))$$

Example 6. *Consider the permutation*

$$g = (1\ 14\ 7\ 11)(5\ 16\ 12)(6\ 20\ 9)(8\ 17\ 13)(10\ 2\ 15\ 4)(18\ 3\ 19)$$

We note that g has two cycles of length four, and by Lemma 17, we have

$$N((4^2)) = 4 + 1 = 5$$

In addition, g has four cycles of length three, and

$$N((3^4)) = \frac{4!3^{4/2}}{2^{4/2}(4/2)!} = 27$$

So the number of matchings that will commute with g is $5 \cdot 27 = 135$.

3.5 Data

Table 3.1 displays the dimension of $\mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})}$ for several values of r and m generated by the formulas of Theorems 2 and 3, using Sage [36]. The values in this table reflect the stable dimensions, which give us an upper bound on the dimension of the invariant space in the general case. For some data related to the non-stable range (that is, where we do not require that $n_i \geq 2m$ for all i), see Appendix A.

For comparison, Table 3.2 displays the dimension of the stable values of $\mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})}$ when $K(\mathbf{n}) = U(n_1) \times U(n_2) \times \cdots \times U(n_r)$ [14].

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$r=1$	1	1	1	1	1	1
$r=2$	1	2	3	5	7	11
$r=3$	1	5	16	86	448	3580
$r=4$	1	14	132	4154	234004	24791668
$r=5$	1	41	1439	343101	208796298	253588562985
$r=6$	1	122	18373	33884745	196350215004	2634094790963313
$r=7$	1	365	254766	3505881766	185471778824111	27380169200102651288
$r=8$	1	1094	3680582	366831842914	175264150734326927	284615877731708760168866

	$m=7$	$m=8$	$m=9$
$r=1$	1	1	1
$r=2$	15	22	30
$r=3$	34981	448628	6854130
$r=4$	3844630928	809199787472	220685007519070
$r=5$	517068048408385	1635862619286705997	7590088213330525666347
$r=6$	69856539937093983210	3315428327367822097464999	261523007306622221875905931924
$r=7$	9439895841362297019735454	6720389536120974873107885015406	9011878475307752483094496672461123697

	$m=10$	$m=11$	$m=12$	$m=13$	$m=14$
$r=1$	1	1	1	1	1
$r=2$	42	56	77	101	135
$r=3$	121173330	2403140605	52655943500	1260724587515	32726520985365

Table 3.1: $\dim \mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})}$ where $K(\mathbf{n}) = \prod_{i=1}^{r} O_{n_i}$, $V(\mathbf{n}) = \prod_{i=1}^{r} \mathbb{C}^{n_i}$, and $n_i \geq 2m$ for $1 \leq i \leq r$

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$r=1$	1	1	1	1	1	1
$r=2$	1	2	3	5	7	11
$r=3$	1	4	11	43	161	901
$r=4$	1	8	49	681	14721	524137
$r=5$	1	16	251	14491	1730861	373486525
$r=6$	1	32	1393	336465	207388305	268749463729
$r=7$	1	64	8051	7997683	24883501301	193492277719861
$r=8$	1	128	47449	191374041	2985987361161	139314094050615817

Table 3.2: $\dim \mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})}$ where $K(\mathbf{n}) = \prod_{i=1}^{r} U(n_i)$, $V(\mathbf{n}) = \prod_{i=1}^{r} \mathbb{C}^{n_i}$, and $n_i \geq 2m$ for $1 \leq i \leq r$

Chapter 4

The Invariants

In this chapter, we provide formulas for the invariants in the space $\mathcal{P}^d(V(\mathbf{n}))^{K(\mathbf{n})}$ in the stable range, where $n_i \geq d$ for $\mathbf{n} = (n_1, n_2, \ldots, n_r)$. In Section 4.1, we recall the results for the case $K(\mathbf{n}) = U(n_1) \times U(n_2) \times \cdots \times U(n_r)$ from [15]. In Section 4.2, we consider the case $K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$.

4.1 The Unitary Setting

Let S_m denote the symmetric group on the set $\{1, \ldots, m\}$. The r-fold cartesian product, denoted

$$S_m^r = \{\mathbf{s} = (\sigma_1, \sigma_2, \ldots, \sigma_r) \mid \sigma_i \in S_m \text{ for all } i\}$$

is acted upon by $S_m \times S_m$ under the action $(\alpha, \beta) \cdot \mathbf{s} = \alpha \mathbf{s} \beta^{-1}$ where $\alpha \mathbf{s} \beta^{-1} = (\alpha\sigma_1\beta^{-1}, \ldots, \alpha\sigma_r\beta^{-1})$. The orbits of this group action are the double cosets, $\Delta \backslash S_m^r / \Delta$ where $\Delta = \{(\sigma, \ldots, \sigma) \mid \sigma \in S_m\}$. As shown in [14] and reviewed in Chapter 3, the number of these double cosets is

$$\widetilde{h}_{m,r} = \sum_{\lambda \vdash m} z_\lambda^{r-2} \tag{4.1}$$

The number of orbits under the S_m-action of simultaneous conjugation

$$\gamma \mathbf{s} \gamma^{-1} = \left(\gamma\sigma_1\gamma^{-1}, \cdots, \gamma\sigma_{r-1}\gamma^{-1}\right),$$

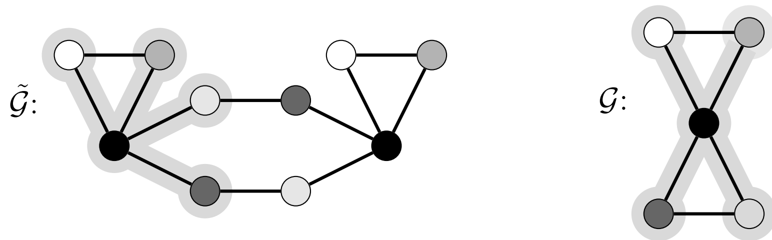


Figure 4.1: Example of a graph covering

on S_m^{r-1} is given by

$$\widetilde{h}_{m,r} = \sum_{\lambda \vdash m} z_\lambda^{r-2}$$

Denote these orbits by $\mathcal{O} = S_m^{r-1}/S_m$. There exists a map $\theta : \Delta\backslash S_m^r/\Delta \to \mathcal{O}$ defined for $\mathbf{s} = (\sigma_1, \ldots, \sigma_r)$ by

$$\theta(\Delta \mathbf{s} \Delta) = \{\gamma(\sigma_1\sigma_r^{-1}, \sigma_2\sigma_r^{-1}, \ldots, \sigma_{r-1}\sigma_r^{-1})\gamma^{-1} \mid \gamma \in S_m\}.$$

It is easy to see that θ is independent of the representative $\mathbf{s}$, and defines a bijective function from $\Delta\backslash S_m^r/\Delta$ to $\mathcal{O}$.

The right hand side of Equation 4.1 may be interpreted by certain graph enumeration problems, which we recall following [22]. Let $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ be a simple connected graph, with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ (that is, $\mathcal{G}$ is an undirected graph with no loops such that there exists a path between any two vertices of $\mathcal{G}$). Let $\beta(\mathcal{G}) = |\mathcal{E}| - |\mathcal{V}| + 1$ denote the first Betti number, which is the number of independent cycles in $\mathcal{G}$. Let $N(v)$ denote the neighborhood[1] of a vertex $v \in \mathcal{V}$. A graph $\tilde{\mathcal{G}}$ is said to be a *covering* of $\mathcal{G}$ with projection $p : \tilde{\mathcal{G}} \to \mathcal{G}$ if there exists a surjection $p : \tilde{\mathcal{V}} \to \mathcal{V}$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any $v \in \mathcal{V}$ and $\tilde{v} \in p^{-1}(v)$. If p is n-to-one, we say $p : \tilde{\mathcal{G}} \to \mathcal{G}$ is an *n-fold covering*. In Figure 4.1, $\tilde{\mathcal{G}}$ is a 2-fold covering of $\mathcal{G}$. We see that the neighborhood of a black vertex of $\tilde{\mathcal{G}}$ maps bijectively onto the neighborhood of the black vertex of $\mathcal{G}$.

[1]The neighborhood $N(v)$ of $v \in \mathcal{V}$ is the set of all vertices in $\mathcal{V}$ adjacent to v.

Two coverings $p_i : \tilde{\mathcal{G}}_i \to \mathcal{G}$, $i = 1, 2$ are said to be *isomorphic* if there exists a graph isomorphism $\Phi : \tilde{\mathcal{G}}_1 \to \tilde{\mathcal{G}}_2$ such that the following diagram commutes:

$$\begin{tikzcd} \tilde{\mathcal{G}}_1 \arrow[rr, "\Phi"] \arrow[dr, "p_1"'] & & \tilde{\mathcal{G}}_2 \arrow[dl, "p_2"] \\ & \mathcal{G} & \end{tikzcd}$$

The author would like to thank Professor Ian Musson for pointing out the following proposition, appearing in [1].

The right hand side of Equation 4.1 is equal to the number of isomorphism classes of m-fold coverings of $\mathcal{G}$ with $\beta(\mathcal{G}) = r - 1$ (see [21]).

In light of this graphical interpretation of Equation 4.1, one anticipates a bijective correspondence between finite graph coverings and measurements of quantum entanglement. Indeed, such a correspondence exists, which we illustrate next.

4.1.1 The correspondence

Recall that if V is a complex vector space, we denote[2] the complex valued polynomial functions on V by $\mathcal{P}(V)$. Suppose that a compact Lie group K acts $\mathbb{C}$-linearly on V. The K-action on V gives rise to an action on $\mathcal{P}(V)$ by $k \cdot f(v) = f(k^{-1}v)$ for $k \in K$, $f \in \mathcal{P}(V)$ and $v \in V$. Both $\mathcal{P}(V)$ and $\mathcal{P}_{\mathbb{R}}(V)$ are complex vector spaces with a natural gradation by degree. As a graded representation, $\mathcal{P}_{\mathbb{R}}(V) \cong \mathcal{P}\left(V \oplus \overline{V}\right)$, where $\overline{V}$ denotes the complex vector space with the opposite complex structure (see [26]). Let V^* refer to the representation on the complex valued linear functionals on V defined by $(k \cdot \lambda)(v) = \lambda(k^{-1}v)$ for $v \in V$, $\lambda \in V^*$, and $k \in K$. As a representation of K, $\overline{V}$ is equivalent to V^*.

In what is to follow, we will complexify the compact group, K, to a complex reductive linear algebraic group. All representations of G will be assumed to be regular. That is, the matrix coefficients are regular functions on the underlying

[2] Here we are viewing V as a *complex space rather than a real space, as we do in defining* $\mathcal{P}_{\mathbb{R}}(V)$.

affine variety G. An irreducible regular representation restricts to an irreducible complex representation of K. Furthermore, since K is Zariski dense in G, regular representations of G (and hence G-invariants) are determined on K. Note that, $G = \mathrm{GL}(n)$ when $K = U(n)$.

We now specialize to $V = V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_r}$, and set up notation for the coordinates in V and V^*. For positive integers k and n, let $\mathrm{Mat}_{n,k}$ denote the vector space of $n \times k$ complex matrices. Let $E^i_j \in \mathrm{Mat}_{n,k}$ denote the matrix with entry in row i and column j equal to 1 and all other entries 0. The group of $n \times n$ invertible matrices with complex number entries will be denoted by $\mathrm{GL}(n)$. This group acts on $\mathrm{Mat}_{n,k}$ by multiplication on the left. We identify $\mathbb{C}^n = \mathrm{Mat}_{n,1}$, which has a distinguished ordered basis consisting of $e_i = E^i_1 \in \mathrm{Mat}_{n,1}$ for $i = 1, \cdots, n$.

In the case of $G = \mathrm{GL}(n)$ we will identify $(\mathbb{C}^n)^*$ with the representation on $\mathrm{Mat}_{1,n}$ defined by the action $g \cdot v = vg^{-1}$ for $v \in \mathrm{Mat}_{1,n}$ and $g \in \mathrm{GL}(n)$. Set $e^i = E^1_i \in \mathrm{Mat}_{1,n}$ for $i = 1, \ldots, n$. Then, $(e^1, \ldots, e^n)$ is an ordered basis for $(\mathbb{C}^n)^*$, dual to $(e_1, \ldots, e_n)$.

Arbitrary tensors in $V(\mathbf{n})$ and $V(\mathbf{n})^*$ are of the form

$$\sum x^{i_1 \cdots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \in V(\mathbf{n}),$$

and

$$\sum y_{i_1 \cdots i_r} e^{i_1} \otimes \cdots \otimes e^{i_r} \in V(\mathbf{n})^*,$$

where $x^{i_1 \cdots i_r}$ and $y_{i_1 \cdots i_r}$ are complex scalars. We may view the variables $x^{i_1 \cdots i_r}$ and $y_{i_1 \cdots i_r}$ as degree 1 polynomial functions in $\mathcal{P}_{\mathbb{R}}(V(\mathbf{n}))$, where $y_{i_1 \cdots i_r}$ are the complex conjugates of $x^{i_1 \cdots i_r}$.

Let $\mathcal{G}$ be a connected simple graph with $\beta(\mathcal{G}) = r - 1$. In [22], the isomorphism classes of m-fold covers of $\mathcal{G}$ are parameterized by the orbits in $S_m^{r-1} = S_m \times \cdots \times S_m$ ($r-1$ factors) under the conjugation action of S_m. Thus, one expects to form a basis[3] element of the space of degree $2m$ invariants from a choice, up to conjugation, of $r-1$ permutations. Let

$$[\sigma_1, \ldots, \sigma_{r-1}] = \left\{\tau(\sigma_1, \ldots, \sigma_{r-1})\tau^{-1} : \tau \in S_m\right\}$$

be such a choice. We present now an invariant associated with $[\sigma_1, \ldots, \sigma_{r-1}]$.

We define $f_{[\sigma_1, \cdots, \sigma_{r-1}]}$ as the sum over the indices

$$I_1 = (i_1^{(1)} i_2^{(1)} \cdots i_r^{(1)}), \ldots, I_m = (i_1^{(m)} i_2^{(m)} \cdots i_r^{(m)})$$

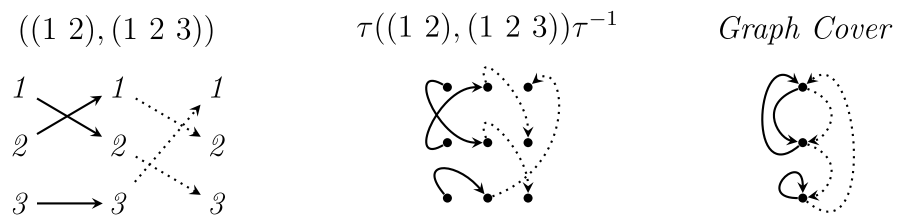
where $1 \leq i_k^{(j)} \leq n_k$ (with $j = 1, \cdots, m$) of

$$x^{I_1} \cdots x^{I_m} y_{i_1^{(\sigma_1(1))} \cdots i_{r-1}^{(\sigma_{r-1}(1))} i_r^{(1)}} \cdots y_{i_1^{(\sigma_1(m))} \cdots i_{r-1}^{(\sigma_{r-1}(m))} i_r^{(m)}}.$$

We parameterize the degree $2m$ polynomial invariants (and m-fold coverings of simple connected graphs) in the following way. Given a double coset in $\Delta\mathbf{s}\Delta$, applying θ, one obtains an orbit represented by an (r-1)-tuple of permutations $(\sigma_1, \ldots, \sigma_{r-1})$. This action "forgets" the labels of the domain and range of each permutation. The resulting combinatorial data takes the form of an unlabeled directed graph with edges colored by $r-1$ colors. For each color, both the out degree and in degree is one.

Example 7. *We will now exhibit this process in the case $r = m = 3$. Consider $\Delta(\tilde{\sigma_1}, \tilde{\sigma_2}, \tilde{\sigma_3})\Delta$ where $\tilde{\sigma_1} = (1\ 3\ 2), \tilde{\sigma_2} = (2\ 3), \tilde{\sigma_3} = (1\ 3)$. Then $\theta(\tilde{\sigma_1}, \tilde{\sigma_2}, \tilde{\sigma_3}) = (\tilde{\sigma_1}\tilde{\sigma_3}^{-1}, \tilde{\sigma_2}\tilde{\sigma_3}^{-1}) = (\sigma_1, \sigma_2) = ((1\ 2), (1\ 2\ 3))$.*

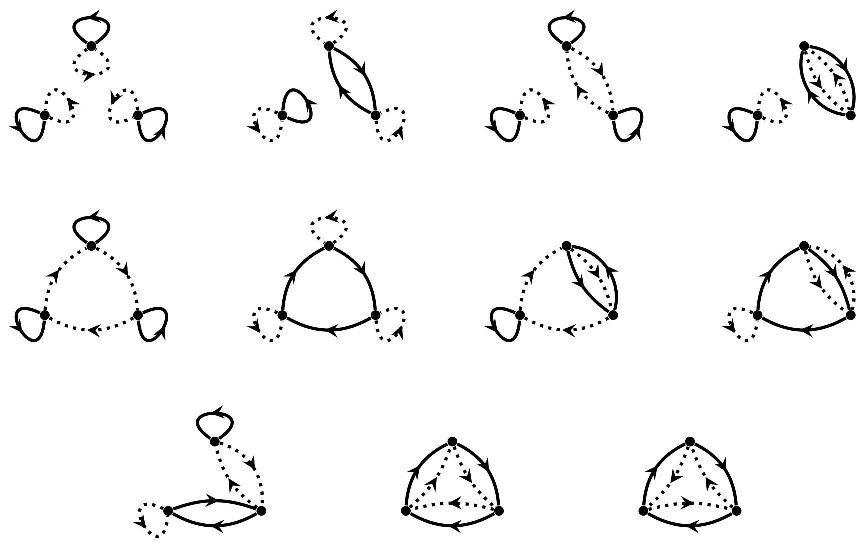
[3] In general, one obtains a spanning set for the invariants. However, if $n_i \geq m$ for all i then we have a basis for the degree $2m$ invariants.



Using the invariant defined above, we have $f_{[\sigma_1,\sigma_2]}$ is the sum of terms of the form

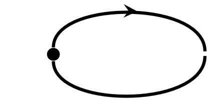
$$
\begin{aligned}
&x^{i_1^{(1)} i_2^{(1)} i_3^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)}} x^{i_1^{(3)} i_2^{(3)} i_3^{(3)}} y_{i_1^{(\sigma_1(1))} i_2^{(\sigma_2(1))} i_3^{(1)}} y_{i_1^{(\sigma_1(2))} i_2^{(\sigma_2(2))} i_3^{(2)}} y_{i_1^{(\sigma_1(3))} i_2^{(\sigma_2(3))} i_3^{(3)}} \\
&= x^{i_1^{(1)} i_2^{(1)} i_3^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)}} x^{i_1^{(3)} i_2^{(3)} i_3^{(3)}} y_{i_1^{(2)} i_2^{(2)} i_3^{(1)}} y_{i_1^{(1)} i_2^{(3)} i_3^{(2)}} y_{i_1^{(3)} i_2^{(1)} i_3^{(3)}}
\end{aligned}
$$

All possible diagrams for the $r = m = 3$ case are shown below:

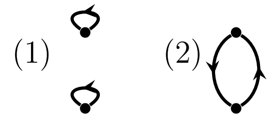


Each coloring of the directed graphs corresponds to an isomorphism class of m-fold covering of a connected simple graph G with $\beta(G) = r - 1$ (see [22]). We illustrate this correspondence for $m = 2$, and $r = 2, 3, 4$. In Table 4.1, the simple graph is homeomorphic to a bouquet of loops (on the left) and the possible graph coverings are on the right. The colors and orientations determine the covering map. The corresponding $K(\mathbf{n})$-invariants are written out explicitly following the Einstein summation convention.

$\beta(G) = 1,\ m = 2$



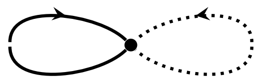
$r = 2,\ m = 2$



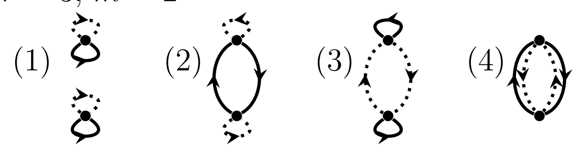
(1) $x^{i_1^{(1)} i_2^{(1)}} y_{i_1^{(1)} i_2^{(1)}} x^{i_1^{(2)} i_2^{(2)}} y_{i_1^{(2)} i_2^{(2)}}$

(2) $x^{i_1^{(1)} i_2^{(1)}} y_{i_1^{(2)} i_2^{(1)}} x^{i_1^{(2)} i_2^{(2)}} y_{i_1^{(1)} i_2^{(2)}}$

$\beta(G) = 2,\ m = 2$



$r = 3,\ m = 2$



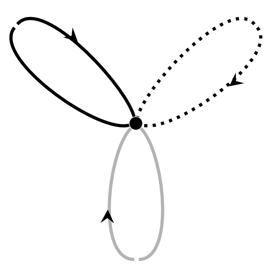
(1) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)}} y_{i_1^{(1)} i_2^{(1)} i_3^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)}} y_{i_1^{(2)} i_2^{(2)} i_3^{(2)}}$

(2) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)}} y_{i_1^{(2)} i_2^{(1)} i_3^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)}} y_{i_1^{(1)} i_2^{(2)} i_3^{(2)}}$

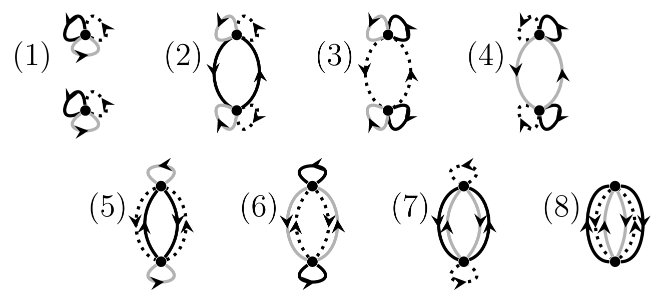
(3) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)}} y_{i_1^{(1)} i_2^{(2)} i_3^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)}} y_{i_1^{(2)} i_2^{(1)} i_3^{(2)}}$

(4) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)}} y_{i_1^{(2)} i_2^{(2)} i_3^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)}} y_{i_1^{(1)} i_2^{(1)} i_3^{(2)}}$

$\beta(G) = 3,\ m = 2$



$r = 4,\ m = 2$



(1) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} y_{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(2)}} y_{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(2)}}$

(2) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} y_{i_1^{(2)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(2)}} y_{i_1^{(1)} i_2^{(2)} i_3^{(2)} i_4^{(2)}}$

(3) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} y_{i_1^{(1)} i_2^{(2)} i_3^{(1)} i_4^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(2)}} y_{i_1^{(2)} i_2^{(1)} i_3^{(2)} i_4^{(2)}}$

(4) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} y_{i_1^{(1)} i_2^{(1)} i_3^{(2)} i_4^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(2)}} y_{i_1^{(2)} i_2^{(2)} i_3^{(1)} i_4^{(2)}}$

(5) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} y_{i_1^{(2)} i_2^{(2)} i_3^{(1)} i_4^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(2)}} y_{i_1^{(1)} i_2^{(1)} i_3^{(2)} i_4^{(2)}}$

(6) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} y_{i_1^{(1)} i_2^{(2)} i_3^{(2)} i_4^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(2)}} y_{i_1^{(2)} i_2^{(1)} i_3^{(1)} i_4^{(2)}}$

(7) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} y_{i_1^{(2)} i_2^{(1)} i_3^{(2)} i_4^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(2)}} y_{i_1^{(1)} i_2^{(2)} i_3^{(1)} i_4^{(2)}}$

(8) $x^{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(1)}} y_{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(1)}} x^{i_1^{(2)} i_2^{(2)} i_3^{(2)} i_4^{(2)}} y_{i_1^{(1)} i_2^{(1)} i_3^{(1)} i_4^{(2)}}$

Table 4.1: Invariants and corresponding graph coverings for $m = 2$ and $K(\mathbf{n}) = U(n_1) \times \cdots \times U(n_r)$

4.2 The Orthogonal Setting

In this section, we provide an algebraic description of the degree d homogeneous polynomials on $V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_r}$ invariant under the action of the product group $K(\mathbf{n}) = O_{n_1} \times O_{n_2} \times \cdots \times O_{n_r}$ in the stable range where each $n_i \geq d$. We then display a bijection between the basis of these invariant spaces and certain edge colored graphs. Finally, we explore the relationship between the invariant spaces and phylogenetic trees.

4.2.1 An Algebraic Description

We begin by setting up notation for an arbitrary tensor in $V(\mathbf{n})$. Let $e_i \in \mathbb{C}^n$ denote the vector with a 1 in row i and 0 elsewhere; note that $(e_1, e_2, \ldots, e_n)$ is an ordered basis for $\mathbb{C}^n$. An arbitrary tensor in $V(\mathbf{n})$ is of the form

$$\sum x_{i_1 i_2 \cdots i_r} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$$

where i_j ranges from 1 to n_j and $x_{i_1 i_2 \cdots i_r}$ is a complex scalar.

Recall that $\tilde{I}^r_{2m}$ denotes the set of r-tuples of matchings on $2m$ letters. The group S_{2m} acts on this set by simultaneous conjugation. We choose a representative $(\tau_1, \tau_2, \ldots, \tau_r)$ from each orbit under this action, which we denote by

$$[\tau_1, \tau_2, \ldots, \tau_r] = \{\sigma(\tau_1, \tau_2, \ldots, \tau_r)\sigma^{-1} \mid \sigma \in S_{2m}\}$$

Fix an ordering on the cycles of each τ_i, and let j^k_i denote the cycle containing k in τ_i. For instance, if $\tau_2 = (1\ 3)(2\ 4)$, then $j^4_2 = 2$, since 4 appears in the second cycle of τ_2. The invariant associated to $[\tau_1, \tau_2, \ldots, \tau_r]$ can now written as the sum of terms of the form

$$x_{a^{(1)}_{j^1_1} a^{(2)}_{j^1_2} \cdots a^{(r)}_{j^1_r}} x_{a^{(1)}_{j^2_1} a^{(2)}_{j^2_2} \cdots a^{(r)}_{j^2_r}} \cdots x_{a^{(1)}_{j^{2m}_1} a^{(2)}_{j^{2m}_2} \cdots a^{(r)}_{j^{2m}_r}}$$

where each $a_i^{(t)}$ ranges from 1 to n_k.

Example 8. *Let $r = 2$ and $m = 2$ (note $d = 2m = 4$). We are now equipped to write down basis elements of the degree 4 homogeneous polynomials on the space $V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ invariant under the action of the group $K(\mathbf{n}) = O_{n_1} \times O_{n_2}$ where $n_1, n_2 \geq 4$. We have three matchings on $d = 4$ letters:*

$$(1\ 2)(3\ 4) \qquad (1\ 3)(2\ 4) \qquad (1\ 4)(2\ 3)$$

We first determine the equivalence classes of ordered 2-tuples of these matchings under simultaneous conjugation by S_4. Observe that we have nine such tuples, with only two classes:

$$\{\{(1\ 2)(3\ 4),(1\ 2)(3\ 4)\},\{(1\ 3)(2\ 4),(1\ 3)(2\ 4)\},\{(1\ 4)(2\ 3),(1\ 4)(2\ 3)\}\}$$
$$\{\{(1\ 2)(3\ 4),(1\ 3)(2\ 4)\},\{(1\ 2)(3\ 4),(1\ 4)(2\ 3)\},\{(1\ 3)(2\ 4),(1\ 2)(3\ 4)\},$$
$$\{(1\ 3)(2\ 4),(1\ 4)(2\ 3)\},\{(1\ 4)(2\ 3),(1\ 2)(3\ 4)\},\{(1\ 4)(2\ 3),(1\ 3)(2\ 4)\}$$

Choosing a representative (τ_1, τ_2) of each class, we write the associated polynomial. For simplicity, we choose the first tuple listed in each class above. First, we have $\tau_1 = \tau_2 = (1\ 2)(3\ 4)$. The related invariant polynomial is then a sum of terms of the form

$$x_{a_1^{(1)} a_1^{(2)}} x_{a_1^{(1)} a_1^{(2)}} x_{a_2^{(1)} a_2^{(2)}} x_{a_2^{(1)} a_2^{(2)}}$$

where $a_1^{(1)}, a_2^{(1)}$ range from 1 to n_1 and $a_1^{(2)}, a_2^{(2)}$ range from 1 to n_2. Next, we have $\tau_1 = (1\ 2)(3\ 4)$ and $\tau_2 = (1\ 3)(2\ 4)$. The related invariant polynomial in this case is *a sum of terms of the form*

$$x_{a_1^{(1)} a_1^{(2)}} x_{a_1^{(1)} a_2^{(2)}} x_{a_2^{(1)} a_1^{(2)}} x_{a_2^{(1)} a_2^{(2)}}$$

where, again, $a_1^{(1)}, a_2^{(1)}$ *range from 1 to* n_1 *and* $a_1^{(2)}, a_2^{(2)}$ *range from 1 to* n_2*.*

4.2.2 Correspondence to r-regular graphs with $2m$ vertices

A *k-regular graph* on n vertices is a graph in which each of the n vertices has degree k; that is, each vertex is met by exactly k edges. Figure 4.2 shows a few examples of regular graphs on 4 vertices. Note that 1-regular graphs are already familiar, having appeared in the Introduction as a possible way of representing matchings and basis elements of the Brauer algebra [3]. A 1-regular graph is always composed of disconnected edges.

Figure 4.2: Examples of regular graphs

An *edge coloring* of a graph is a coloring of its edges so that no two adjacent edges are the same color (this is also referred to as a *1-factorization* of the graph). In this case of a k-regular graph, an edge coloring implies that each vertex is met by k edges, which are each a distinct color. Figure 4.3 displays a few examples of 3-regular edge colored graphs on 4 vertices (note, we represent different colors via solid, dashed, and dotted lines).

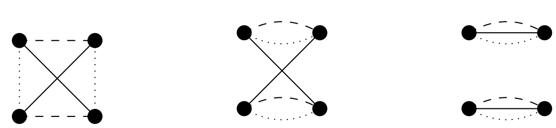


Figure 4.3: Examples of 3-regular edge colored graphs

Observe that two k-regular edge colored graphs can be isomorphic, meaning there may exist an edge-preserving bijection between the two graphs. The two graphs shown in Figure 4.4 are isomorphic, as a rearrangement of vertices of the graph on the left yields the graph on the right.

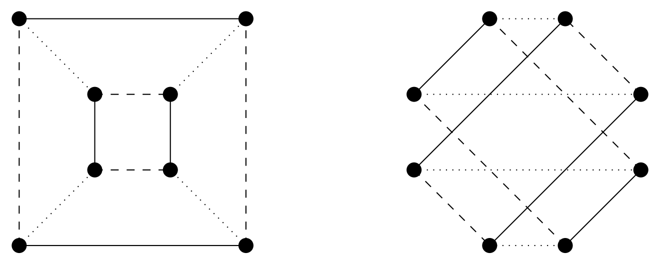


Figure 4.4: Isomorphic 3-regular edge colored graphs

We now present a bijection between the orbits of the action of S_{2m} on $\tilde{I}_{2m}^r$ and the isomorphism classes of edge colored r-regular graphs with $2m$ vertices. Recall that $\tilde{I}_{2m}^r$ denotes the set of r-tuples of matchings on $2m$ letters. S_{2m} acts on this set by simultaneous conjugation. For each equivalence class of this action, we choose a representative tuple $(\tau_1, \tau_2, \ldots, \tau_r)$. We denote the class containing this representative as $[\tau_1, \tau_2, \ldots, \tau_r]$.

To construct the graph associated to $[\tau_1, \tau_2, \ldots, \tau_r]$, we begin by numbering the vertices of the graph from 1 to $2m$. An edge of color i is drawn between vertices j and k if τ_i contains the cycle $(j\ k)$. Repeating this process for all r matchings, we obtain an undirected graph with $2m$ vertices and r colors.

To illustrate, we consider the case $r = 3$, $m = 2$ where we have chosen the 3-tuple of matchings

$$(\tau_1, \tau_2, \tau_3) = ((1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3))$$

The invariant associated to $[\tau_1, \tau_2, \tau_3]$ is then

$$\sum_{a_1^{(1)}, a_2^{(1)}}^{n_1} \sum_{a_1^{(2)}, a_2^{(2)}}^{n_2} \sum_{a_1^{(3)}, a_2^{(3)}}^{n_3} x_{a_1^{(1)} a_1^{(2)} a_1^{(3)}} x_{a_1^{(1)} a_2^{(2)} a_2^{(3)}} x_{a_2^{(1)} a_1^{(2)} a_2^{(3)}} x_{a_2^{(1)} a_2^{(2)} a_1^{(3)}}$$

To construct the graph associated to $[\tau_1, \tau_2, \tau_3]$, we encode τ_1, τ_2, τ_3 with three colors portrayed by solid, dashed, and dotted edges, respectively. Forgetting the labels of the vertices leaves us with a representative of the isomorphism class containing the graph. The result is shown in Figure 4.5.

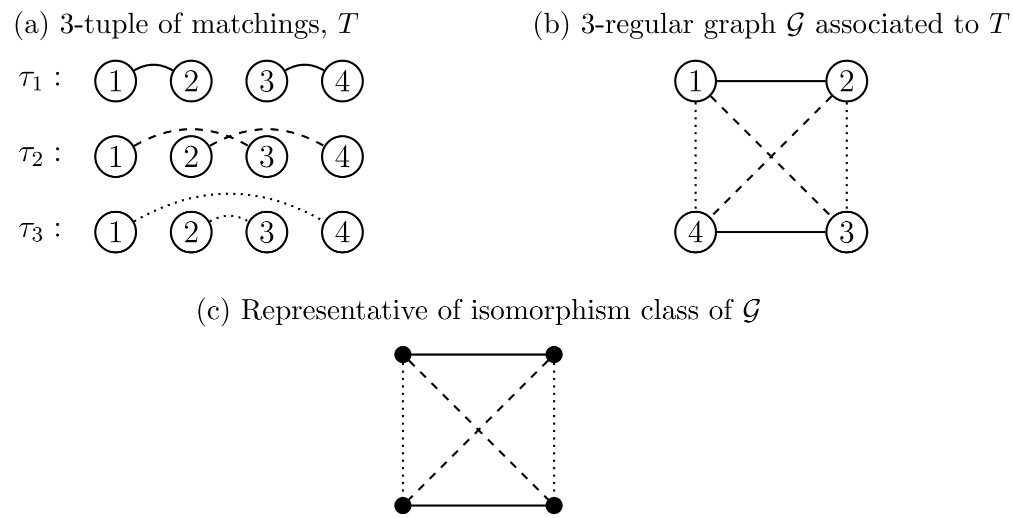


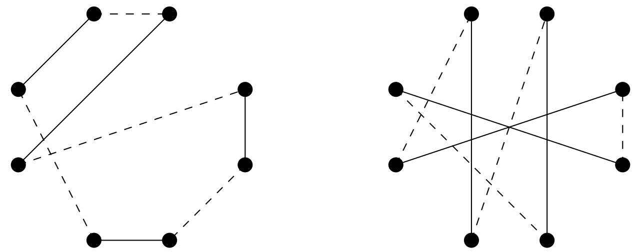
Figure 4.5: Constructing a 3-regular graph from a triple of matchings

The graph produced in this process depends on the representative chosen; however, a graph associated to any tuple will be isomorphic to a graph associated to any other tuple in the same equivalence class. Suppose $T = (\tau_1, \tau_2, \dots, \tau_r)$ and $R = (\rho_1, \rho_2, \dots, \rho_r)$ are r-tuples of matchings such that

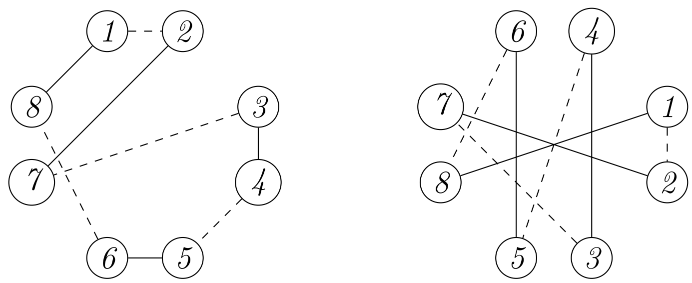
$$T = (\tau_1, \tau_2, \dots, \tau_r) = (\sigma\rho_1\sigma^{-1}, \sigma\rho_2\sigma^{-1}, \dots, \sigma\rho_2\sigma^{-1}) = \sigma R \sigma^{-1}$$

for some $\sigma \in S_{2m}$. Let $\mathcal{G}, \mathcal{H}$ be the graphs associated to T, R, respectively. Labeling the vertices of $\mathcal{G}$ and applying the permutation σ to these vertices will result in the graph $\mathcal{H}$.

Example 9. *Let* $T = ((1\ 2)(3\ 7)(4\ 5)(6\ 8), (1\ 8)(2\ 7)(3\ 4)(5\ 6))$ *and let* $R = ((1\ 7)(2\ 6)(3\ 4)(5\ 8), (1\ 6)(2\ 5)(3\ 7)(4\ 8))$. *Note that* T *and* R *belong to the same equivalence class under conjugation by* S_8*; for example,* $R = \sigma T \sigma^{-1}$ *when* $\sigma = (1\ 3\ 5\ 6)(2\ 4)(7\ 8)$. *The graph on the left is the graph obtained from* T *in the* *described manner, while the graph obtained from R is on the right.*



Labeling the vertices of the graph associated to T (shown below, left), then applying the permutation σ to these vertices (shown below, right) results in the graph associated to R:



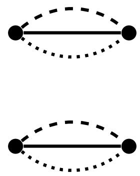
A list of representatives of all isomorphism classes of 3-regular graphs on four vertices is shown in Table 4.2, along with the corresponding invariant. We have chosen a representative (τ_1, τ_2, τ_3) of each orbit so that $\tau_1 = (1\ 2)(3\ 4)$.

4.2.3 Correspondence to Forests of Phylogenetic Trees

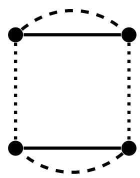
Finally, we show a way of encoding these invariants with forests of phylogenetic trees.

Trees

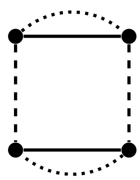
We begin by reviewing some terminology related to trees. A *tree* is an undirected, connected graph with no cycles; that is, a graph in which there exists exactly one simple path from any vertex to any other vertex. The *degree* of a vertex is the number of edges incident to that vertex. If a vertex in a tree has degree one, we say it is a *leaf*. Otherwise, we say the vertex is an *interior node*, or simply a *node*.



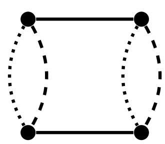
$$\sum_{a_1^{(1)},a_2^{(1)}}^{n_1} \sum_{a_1^{(2)},a_2^{(2)}}^{n_2} \sum_{a_1^{(3)},a_2^{(3)}}^{n_3} x_{a_1^{(1)}a_1^{(2)}a_1^{(3)}} x_{a_1^{(1)}a_1^{(2)}a_1^{(3)}} x_{a_2^{(1)}a_2^{(2)}a_2^{(3)}} x_{a_2^{(1)}a_2^{(2)}a_2^{(3)}}$$



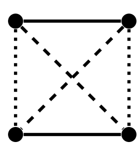
$$\sum_{a_1^{(1)},a_2^{(1)}}^{n_1} \sum_{a_1^{(2)},a_2^{(2)}}^{n_2} \sum_{a_1^{(3)},a_2^{(3)}}^{n_3} x_{a_1^{(1)}a_1^{(2)}a_1^{(3)}} x_{a_1^{(1)}a_1^{(2)}a_2^{(3)}} x_{a_2^{(1)}a_2^{(2)}a_2^{(3)}} x_{a_2^{(1)}a_2^{(2)}a_1^{(3)}}$$



$$\sum_{a_1^{(1)},a_2^{(1)}}^{n_1} \sum_{a_1^{(2)},a_2^{(2)}}^{n_2} \sum_{a_1^{(3)},a_2^{(3)}}^{n_3} x_{a_1^{(1)}a_1^{(2)}a_1^{(3)}} x_{a_1^{(1)}a_2^{(2)}a_1^{(3)}} x_{a_2^{(1)}a_2^{(2)}a_2^{(3)}} x_{a_2^{(1)}a_1^{(2)}a_2^{(3)}}$$



$$\sum_{a_1^{(1)},a_2^{(1)}}^{n_1} \sum_{a_1^{(2)},a_2^{(2)}}^{n_2} \sum_{a_1^{(3)},a_2^{(3)}}^{n_3} x_{a_1^{(1)}a_1^{(2)}a_1^{(3)}} x_{a_1^{(1)}a_2^{(2)}a_2^{(3)}} x_{a_2^{(1)}a_2^{(2)}a_2^{(3)}} x_{a_2^{(1)}a_1^{(2)}a_1^{(3)}}$$

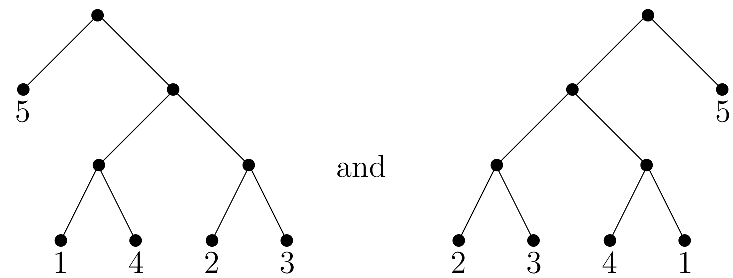


$$\sum_{a_1^{(1)},a_2^{(1)}}^{n_1} \sum_{a_1^{(2)},a_2^{(2)}}^{n_2} \sum_{a_1^{(3)},a_2^{(3)}}^{n_3} x_{a_1^{(1)}a_1^{(2)}a_1^{(3)}} x_{a_1^{(1)}a_2^{(2)}a_2^{(3)}} x_{a_2^{(1)}a_1^{(2)}a_2^{(3)}} x_{a_2^{(1)}a_2^{(2)}a_1^{(3)}}$$

Table 4.2: Representative of graph isomorphism class and corresponding invariant in the case $r = 3, m = 2$

If a tree T has a distinguised node r, called a *root*, we say the pair (T, r) is a *rooted tree* following [2]. We can define a partial ordering $\leq$ on (T, r), where $s \leq s'$ if and only if the vertex s' lies on the path from the root r to the vertex s. With this ordering, the root r is the largest element, and the leaves will be the minimal elements. If an edge of the tree lies between vertices a and b *originates* from the vertex a if $b \leq a$. In this case, we say a is a *parent* of b, and that b is a *child* of a. If two vertices have the same parent, they are called *siblings*. The parents in a tree are collectively referred to as *ancestors*.

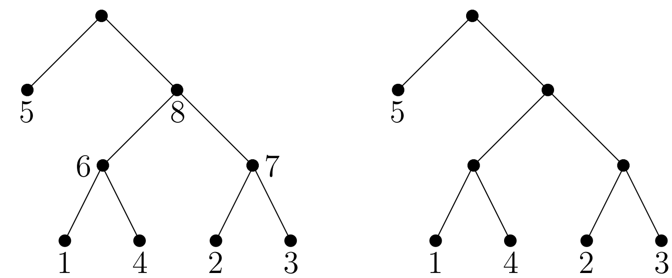
A *rooted binary tree* is a rooted tree such that the root itself has degree two and all remaining internal nodes have degree three. A *phylogenetic tree* is a rooted binary tree in which each leaf has been given a distinct label. We will identify two phylogenetic trees if there exists a graph isomorphism between them which preserves such a labeling, as in [5] and [6]. That is, the trees



are considered the same. It was shown in [33] that the number of phylogenetic trees with $m+1$ leaves is equal to the number of matchings on $2m$ letters, $(2m)!/2^m m!$. As explained in the Introduction and [5], Diaconis and Holmes [6] provide a bijection between matchings on $2m$ letters and phylogenetic trees with $m+1$ labeled leaves. We now explain this bijection in detail.

Correspondence between matchings and phylogenetic trees

We first describe the procedure for building a tree with $n+1$ labeled leaves from a matching on the set $\{1, 2, \ldots, 2n\}$, which we illustrate with the example $\tau = (1\ 4)(2\ 3)(5\ 8)(6\ 7)$. The resulting tree will have leaves labeled with the set $\{1, 2, \ldots, n+1\}$; the ancestors of the tree will be labeled $\{n+2, n+3, \ldots, 2n\}$. First, look for any pairs in the matching that contain only numbers at most $n+1$. In our specific example, we have two such cycles, $(1\ 4)$ and $(2\ 3)$. Each such cycle defines a sibling pair of leaves. Choose the pair with the smallest child, in this case, $(1\ 4)$. Label the parent of this pair with the smallest ancestor label, 6. The other sibling pair will be labeled with the next available ancestor label, 7. Since 6 and 7 appear in the same cycle of the matching, they must also be siblings in the tree. Their parent will be the last available ancestor, 8. Finally, 8 must be paired with 5, which will form its own leaf. The resulting tree is shown on the left in Figure 4.6, while the corresponding tree with labeled leaves is on the right.

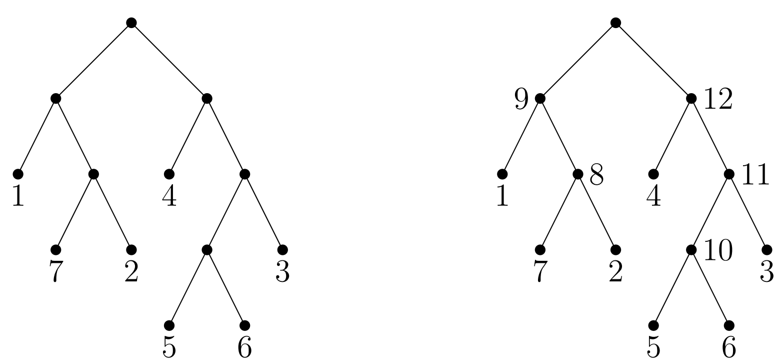


(a) Tree with all nodes labeled (b) Tree with labeled leaves

Figure 4.6: labeling non-leaf nodes of a phylogenetic tree

Next, we set out a method of finding a matching when given a tree. We start with a binary rooted tree with $n+1$ labeled leaves. We look for the pair of siblings with the smallest child, and label the parent of these children with $n+2$. Repeat this process until all nodes (except the root) have been labeled. The matching defined by this tree is formed by pairing siblings into cycles. A particular example is shown in Figure 4.7. The tree on the left is a binary tree with 7 leaves. The diagram on the right is obtained by labeling the nodes of the tree as described, and the corresponding permutation is shown below.

(a) Tree with labeled leaves (b) Tree with all nodes labeled



(c) The associated matching

(1 8)(2 7)(3 10)(4 11)(5 6)(9 12)

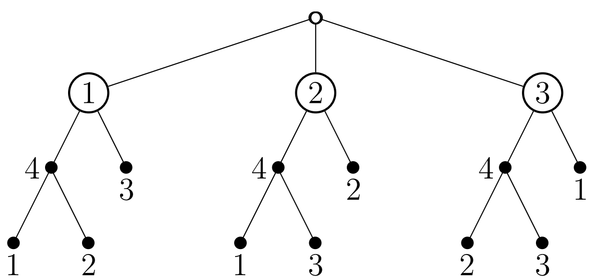
Figure 4.7: Constructing a matching from a phylogenetic tree

Correspondence between invariants and forests of phylogenetic trees

When given an r-tuple of matchings, we can define a *forest* of r trees with roots labeled $1, 2, \ldots, r$. We consider two forests to be equivalent if the individual trees within the forest are equivalent in the traditional way. Additionally, note that the order of the trees is significant, as the ith tree must correspond to the ith matching in the tuple. As an example, we again consider the 3-tuple

$$(\tau_1, \tau_2, \tau_3) = ((1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3))$$

which can be interpreted as the forest



after we have labeled all vertices following the method described in [6].

The invariant associated to a forest with r trees can again be written as a sum of terms of the form

$$x_{a^{(1)}_{j^1_1} a^{(2)}_{j^1_2} \cdots a^{(r)}_{j^1_r}} x_{a^{(1)}_{j^2_1} a^{(2)}_{j^2_2} \cdots a^{(r)}_{j^2_r}} \cdots x_{a^{(1)}_{j^{2m}_1} a^{(2)}_{j^{2m}_2} \cdots a^{(r)}_{j^{2m}_r}}$$

where the index (i) of the subscript refers to the root of an individual tree in the forest, and $j^s_i = j^t_i$ if s and t are siblings in the ith tree. To build the invariant associated with our particular forest, we begin by examining the tree labeled (1). This tree has two pairs of siblings, $(1, 2)$ and $(3, 4)$, and so our invariant begins as

$$x_{a^{(1)}_1 --}\; x_{a^{(1)}_1 --}\; x_{a^{(1)}_2 --}\; x_{a^{(1)}_2 --}$$

Next, we see that the second tree has sibling pairs $(1,3)$ and $(2,4)$, so we continue building the invariant:

$$x_{a_1^{(1)}a_1^{(2)}-}\, x_{a_1^{(1)}a_2^{(2)}-}\, x_{a_2^{(1)}a_1^{(2)}-}\, x_{a_2^{(1)}a_2^{(2)}-}$$

Finally, the last placeholders of the invariant are filled by observing that the third tree has sibling pairs $(2,3)$ and $(1,4)$. The result is identical to the invariant associated to the same tuple of matchings determined earlier.

$$x_{a_1^{(1)}a_1^{(2)}a_1^{(3)}}\, x_{a_1^{(1)}a_2^{(2)}a_2^{(3)}}\, x_{a_2^{(1)}a_1^{(2)}a_2^{(3)}}\, x_{a_2^{(1)}a_2^{(2)}a_1^{(3)}}$$

Just as S_{2m} acts on r-tuples of matchings on $2m$ letters by simultaneous conjugation, we can consider an action of S_{2m} on r-tuples of phylogenetic trees with $m+1$ leaves. Using the method described in the introduction, we write an r-tuple of trees as an r-tuple of matchings, apply the action of simultaneous conjugation by an element of S_{2m}, and then draw the list of trees associated to the resulting r-tuple of matchings. Hence, we define an action of S_{2m} on a forest of r phylogenetic trees, each with $m+1$ leaves.

To illustrate, suppose we choose a tuple of three trees, each with four leaves, shown in shown in Figure 4.8. Labeling the roots of the trees 1 through 3 and adjoining these roots to a common vertex creates a forest. By following the process outlined earlier, we associate this forest to the 3-tuple of matchings:

$$(\tau_1, \tau_2, \tau_3) = ((1\ 3)(2\ 5)(4\ 6), (1\ 3)(2\ 4)(5\ 6), (1\ 6)(2\ 4)(3\ 5))$$

We choose a permutation $g = (1\ 3\ 5)(2\ 4)(6)$ to act on this tuple:

$$(g\tau_1 g^{-1}, g\tau_2 g^{-1}, g\tau_3 g^{-1}) = ((1\ 4)(2\ 6)(3\ 5), (1\ 6)(2\ 4)(3\ 5), (1\ 5)(2\ 4)(3\ 6))$$

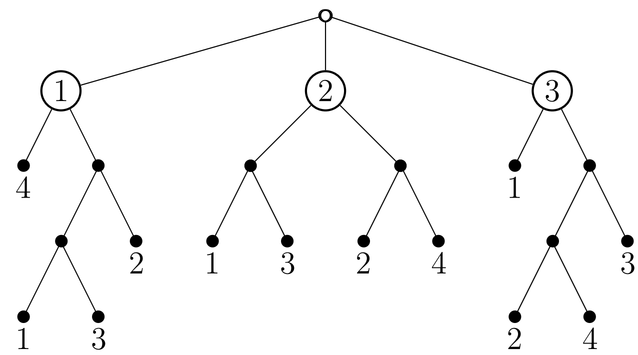


Figure 4.8: The forest corresponding to (τ_1, τ_2, τ_3)

and draw the forest associated to the result in Figure 4.9.

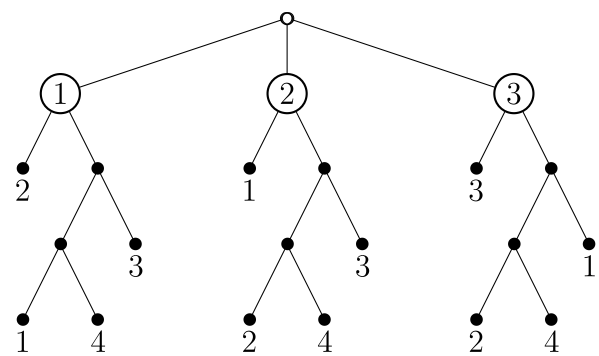


Figure 4.9: The forest corresponding to $g(\tau_1, \tau_2, \tau_3)g^{-1}$

In Figure 4.10, we display the trees of Figures 4.8 and 4.9, with additional labels on vertices following the procedure described earlier[4]. The 3-regular edge colored graph on six vertices associated to each tree is shown on the right. Trees 1, 2, and 3 are shown by solid, dashed, and dotted lines, respectively.

Note that the two graphs shown in Figure 4.10 are isomorphic, and we obtain the second graph from the first by applying the permutation g to its vertices. We may find additional isomorphic graphs by applying any element of S_6. In Figure 4.11, we display such a graph so that the solid edges correspond to the matching (1 2)(3 4)(5 6). The related invariant is a sum of terms of the form shown, as $a_i^{(j)}$

[4]Note that various methods for labeling nodes can be chosen, we refer to the algorithm specified in [6]

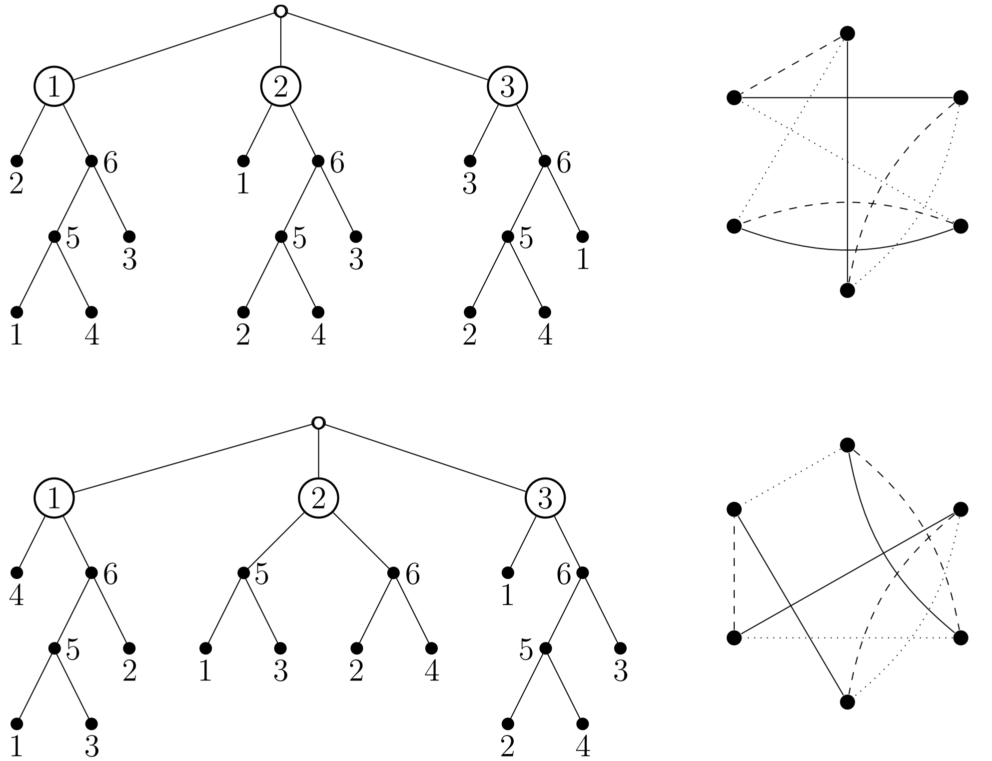
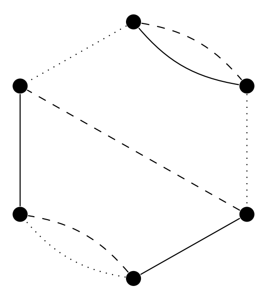


Figure 4.10: 3-regular edge colored graphs corresponding to equivalent trees

ranges from 1 to n_j.



$$x_{a_1^{(1)}a_1^{(2)}a_1^{(3)}}\,x_{a_1^{(1)}a_1^{(2)}a_2^{(3)}}\,x_{a_2^{(1)}a_2^{(2)}a_2^{(3)}}\,x_{a_2^{(1)}a_3^{(2)}a_3^{(3)}}\,x_{a_3^{(1)}a_3^{(2)}a_3^{(3)}}\,x_{a_3^{(1)}a_2^{(2)}a_1^{(3)}}$$

Figure 4.11: Graph and terms of invariant corresponding to trees in Figure 4.10

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Appendix A: Data Related to Nonstable Cases

In Chapter 3, we determined a formula for the dimension of the vector space of degree $d = 2m$ homogeneous polynomials on the tensor product

$$V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_r}$$

under the action of the product group

$$K(\mathbf{n}) = O(n_1) \times O(n_2) \times \cdots \times O(n_r)$$

in the stable range, where each $n_i \geq d$. The values generated by the formula (see Table 3.1 provide upper bounds for the nonstable case, where no restriction is placed on the values n_i.

In this Appendix, we display some data for specific cases. As in the stable range, we still assume $d = 2m$ is a positive, even integer; otherwise, the space is immediately zero dimensional. Hence, we continue using $2m$ in place of d.

.0.4 $K(\mathbf{n}) = O(2)^r$

With the notation above, suppose that $n_i = 2$ for all $1 \leq i \leq r$. For $m > 1$, we no longer satisfy the stability condition for any r. In order to generate the dimension of the spaces

$$\dim \mathcal{P}^{2m}(\otimes^r \mathbb{C}^2)^{O(2)\times O(2)\times\cdots\times O(2)}$$

for various values of r, we rely on Equation 3.2:

$$\dim\left[\mathcal{P}^d(V(\mathbf{n}))\right]^{K(\mathbf{n})} = \sum_{\substack{\lambda^{(1)},\ldots,\lambda^{(r)} \\ \lambda^{(i)}\vdash d \\ \lambda^{(i)} \text{ even}}} g_{\lambda^{(1)}\lambda^{(2)}\cdots\lambda^{(r)}}$$

Using Sage (see Appendix B for programs), we were able to compute the required sums of Kronecker coefficients, to construct Table A1.

	m=1	m=2	m=3	m=4	m=5	m=6	m=7
r=1	1	1	1	1	1	1	1
r=2	1	2	2	3	3	4	4
r=3	1	5	6	16	20	40	50
r=4	1	14	28	152	385	1335	3329
r=5	1	41	188	2464	18903	150593	978505
r=6	1	122	1573	59318	1555740	35860078	680715715
r=7	1	365	14596	1772231	162013392	12184049699	748529603427

Table A1: $\dim \mathcal{P}^{2m}(\otimes^r \mathbb{C}^2)^{\times^r O(2)}$

Aside from simply calculating the dimensions, we are interested in finding a Hilbert series $h(q)$ to generate these values. For each fixed r, we seek a series that will produce infinite sums of terms $a_m q^{2m}$, where $a_m = \dim \mathcal{P}^{2m}(\otimes^r \mathbb{C}^2)^{O(2)\times\cdots\times O(2)}$. We have found such series for three particular values of r. These series were obtained and verified experimentally using Sage.

$$r=1: \qquad h(q) = \frac{1}{1-q^2}$$

$$r=2: \qquad h(q) = \frac{1}{(1-q^2)(1-q^4)}$$

$$r=3: \quad h(q) = \frac{1+q^2+q^4+q^6+q^8+q^{10}}{(1-q^4)^4(1-q^6)}$$

For some values of r, the sequence $a_m = \dim \mathcal{P}^{2m}(\otimes^r \mathbb{C}^2)^{\times^r O(2)}$ appears in the Online Encyclopedia of Integer Sequences. In particular, the $r = 3$ row of the table is sequence A081283, describing an interleaved sequence of pyramidal and polygonal numbers [34].

.0.5 $K(\mathbf{n}) = O(3)^r$

Table A2 provides the dimensions of the space $\mathcal{P}^{2m}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3)^{O(3)\times O(3)\times\cdots\times O(3)}$.

These values were also computed using the Sage program described in Appendix B.

	m=1	m=2	m=3	m=4	m=5	m=6
r=1	1	1	1	1	1	1
r=2	1	2	3	4	5	7
r=3	1	5	16	52	168	564
r=4	1	14	132	2191	43615	854968
r=5	1	41	1439	163306	22724613	2948333693
r=6	1	122	18373	14241595	13342442956	*
r=7	1	365	254766	1284613267	*	*

Table A2: $\dim \mathcal{P}^{2m}(\otimes^r \mathbb{C}^3)^{\times^r O(3)}$

Appendix B: Sage Programs

In this Appendix, we display some programs used to support the main results. The code is written for the Python based open-source mathematical program Sage.

.0.6 Calculating Stable Dimensions Using Formula of Theorem 2

The code presented in Figure 12 calculates, for given r and m, a list of representatives of the conjugacy classes of r-tuples of matchings, under simultaneous conjugation by the symmetric group. In this case, the values of This is a brute force approach, which begins by searching the symmetric group S_{2m} for matchings, then creating a list `K` of all r-tuples. The program then selects a matching (the first in the list), finds all tuples in `K` equivalent to it, and adds these tuples to a list `H`. A representative of `H` is chosen before all of its elements are removed from `K`, at which point the program chooses the next remaining tuple (if any) and finds its class. The representatives chosen from each class are returned, and the dimension of the space $\mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})}$ is the number of these representatives.

This is a fairly slow approach to finding the data in Table 3.1. The code, as shown, calculates only a single value of the table (in this case, the entry $r = 3$, $m = 3$), yet takes approximately 15 seconds to compile. For larger values of m, the output takes considerably longer. Changing m to a value greater than 6 may take hours to produce a result.

```
m = 3
t = []
for i in range(m):
    t.append(2)
L = []
G = SymmetricGroup(2*m)
G[0]
for g in G:
    if Permutation(g).cycle_type() == t:
        L.append(g)
l = len(L)
K = []
for i in range(l):
    for j in range(l):
        for k in range(l):
           K.append((L[i],L[j],L[k]))
I = []
while K != []:
    k = K[0]
    print k
    I.append(k)
    H = []
    for g in G:
        h = (g*k[0]*g^(-1),g*k[1]*g^(-1), g*k[2]*g^(-1))
        H.append(h)
    H.sort()
    M = Set(tuple(h) for h in H)
    N = Set(tuple(k) for k in K)
    O = N.difference(M)
    K = O.list()
```

Figure 12: Program to find representatives of equivalence classes of r-tuples of matchings, under simultaneous conjugation by S_{2m}

```
((1,2)(3,4)(5,6), (1,2)(3,4)(5,6), (1,2)(3,4)(5,6))
((1,6)(2,3)(4,5), (1,6)(2,4)(3,5), (1,2)(3,4)(5,6))
((1,4)(2,3)(5,6), (1,6)(2,5)(3,4), (1,2)(3,6)(4,5))
((1,3)(2,4)(5,6), (1,2)(3,6)(4,5), (1,6)(2,5)(3,4))
((1,6)(2,5)(3,4), (1,2)(3,5)(4,6), (1,2)(3,6)(4,5))
((1,4)(2,3)(5,6), (1,2)(3,6)(4,5), (1,3)(2,4)(5,6))
((1,5)(2,4)(3,6), (1,4)(2,6)(3,5), (1,5)(2,6)(3,4))
((1,2)(3,5)(4,6), (1,2)(3,4)(5,6), (1,2)(3,4)(5,6))
((1,4)(2,6)(3,5), (1,4)(2,5)(3,6), (1,5)(2,4)(3,6))
((1,3)(2,4)(5,6), (1,3)(2,5)(4,6), (1,6)(2,4)(3,5))
((1,2)(3,5)(4,6), (1,5)(2,3)(4,6), (1,2)(3,5)(4,6))
((1,4)(2,6)(3,5), (1,3)(2,4)(5,6), (1,3)(2,4)(5,6))
((1,2)(3,4)(5,6), (1,3)(2,4)(5,6), (1,4)(2,3)(5,6))
((1,2)(3,4)(5,6), (1,2)(3,4)(5,6), (1,6)(2,3)(4,5))
((1,4)(2,6)(3,5), (1,4)(2,6)(3,5), (1,6)(2,4)(3,5))
((1,6)(2,3)(4,5), (1,2)(3,5)(4,6), (1,6)(2,3)(4,5))
```

Figure 13: Result of program in Figure 12

.0.7 Calculating Stable Dimensions Using Formulae of Theorems 2 and 3

The code in Figure 14 uses the formulae proven in Chapter 3 to calculate the data shown in Table 3.1. The result, displayed in Figure 15, is calculated approximately 12 seconds.

.0.8 Computing Stable and Nonstable Dimensions Using Kronecker Coefficients

The program shown in Figure 16 does not rely on results from Chapter 3 to calculate the dimension of the space $\mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})}$, instead using Equation 3.2 and Schur polynomials. The advantage here is flexibility over speed. While only a single column of the dimension tables is computed, and can take considerable time to do so, we are able to use this program to calculate dimensions outside the stable range by restricting the maximum length of the partitions. The code, as shown, finds the dimensions of the space

$$\mathcal{P}^{12}(\otimes^r \mathbb{C}^2)^{O_2 \times O_2 \times \cdots \times O_2}$$

for values of r up to 8; this is the $m = 6$ column of Table A1. This instance takes only a few seconds to compile, but takes much longer for larger values of m and r.

```
H = []; T = []
for r in range(1,10):
    h = []
    for m in range(1,10):
        P = Partitions(2*m).list()
        F = 0
        for q in P:
            e = Partition(q).to_exp()
            N = 1; z = 1
            for i in range(len(e)):
                a = i+1
                b = e[i]
                if is_odd(a) and is_odd(b):
                    n = 0
                if is_even(a) and is_odd(b):
                    n = sum((factorial(b)*(a^((b-k)/2)))/(factorial(k)*
                        factorial((b-k)/2)*2^((b-k)/2)) for k in range
                        (1,b+1) if (is_odd(k)))
                if is_odd(a) and is_even(b):
                    if b == 0:
                        n = 1
                    else:
                        n = (factorial(b)*a^(b/2))/(2^(b/2)*factorial(b
                            /2))
                if is_even(a) and is_even(b):
                    if b == 0:
                        n = 1
                    else:
                        n = sum((factorial(b)*(a^((b-k)/2)))/(factorial
                            (k)*factorial((b-k)/2)*2^((b-k)/2)) for k
                            in range(b+1) if (is_even(k)))
                N = n*N
                z = z*(a^b)*factorial(b)
            f = (N^r)/z
            F = f+F
        T.append(F)
        h.append(F)
    H.append(h)
print H
```

Figure 14: Program to calculate stable dimensions of $\mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})}$ using results of thesis

```
[[1, 1, 1, 1, 1, 1, 1, 1, 1], [1, 2, 3, 5, 7, 11, 15, 22, 30],
[1, 5, 16, 86, 448, 3580, 34981, 448628, 6854130],
[1, 14, 132, 4154, 234004, 24791668, 3844630928, 809199787472,
    220685007519070],
[1, 41, 1439, 343101, 208796298, 253588562985, 517068048408385,
    1635862619286705997, 7590088213330525666347],
[1, 122, 18373, 33884745, 196350215004, 2634094790963313,
    69856539937093983210, 3315428327367822097464999,
    2615230073066222218759059319 24],
[1, 365, 254766, 3505881766, 185471778824111, 27380169200102651288,
    9439895841362297019735454, 6720389536120974873107885015406,
    9011878475307752483094496672461123697],
[1, 1094, 3680582, 366831842914, 175264150734326927,
    2846158777317087601688 66,
    127565860878154690744950032 4484,136223886548361269839841302030033 45717,
     310544041361897756266513647669616309280179305],
[1, 3281, 54236989, 38485365487361, 165624048771051595711,
    2958581186394008254445689753, 172386108325455210471892241576613445,
     276129211559000936568752833099016957 44425749,
    10701168881568205559839354147831334255977752462157874]]
```

Figure 15: Result of program in Figure 14

```
s = SFASchur(QQ)
m = 6; r = 10
L = Partitions((2*m), max_length=2).list()
for l in L:
    if is_odd(l[0]): L.remove(l)
A = sum(s(l) for l in L)
print A
T = A
for i in range(r-1):
    T = A.itensor(T)
    C = T.coefficients(); C.reverse()
    print "r =", i+2, ": ", C[0]
```

Figure 16: Program to calculate dimensions of $\mathcal{P}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})}$ using Schur polynomials

```
s[6, 6] + s[8, 4] + s[10, 2] + s[12]
r = 2 :  4
r = 3 :  40
r = 4 :  1335
r = 5 :  150593
r = 6 :  35860078
r = 7 :  12184049699
r = 8 :  4884050603107
r = 9 :  2114159728092536
r = 10 :  948128664527518983
```

Figure 17: Result of program in Figure 16

Curriculum Vitae

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