COMBINATORIAL PROBLEMS RELATED TO
KOSTANT’S WEIGHT MULTIPLICITY FORMULA

by

Pamela Estephania Harris

A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY
in
MATHEMATICS

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ABSTRACT

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The University of Wisconsin-Milwaukee, 2012
Under the Supervision of Professor Jeb F. Willenbring

It is well known that the dimension of a weight space for a finite-dimensional representation of a simple Lie algebra is given by Kostant’s weight multiplicity formula, which consists of an alternation of a partition function over the Weyl group. We take a combinatorial approach to address the question of how many terms in the alternation contribute to the multiplicity of the zero weight for any semi-simple Lie algebra of rank 2 and provide diagrams associated to the contributing sets in these low rank examples.

We then consider the multiplicity of the zero weight for certain, very special, highest weights. Specifically, we consider the case where the highest weight is equal to the sum of all simple roots. This weight is dominant only in Lie types $A$ and $B$. We prove that in all such cases the number of contributing terms is a Fibonacci number. Combinatorial consequences of this fact are provided.
To Akira.
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“The road and the tale have both been long, would you not say so? The trip has been long and the cost has been high... but no great thing was ever attained easily.

A long tale, like a tall Tower, must be built a stone at a time.”

—Stephen King, The Dark Tower

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Chapter 1

Introduction

Let $G$ be a simple linear algebraic group over $\mathbb{C}$ and $T$ a maximal algebraic torus in $G$ of dimension $r$. Let $B$, $T \subseteq B \subseteq G$, be a choice of Borel subgroup. Let $U$ be the maximal unipotent subgroup of $G$ such that $B = T \cdot U$. Then let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}$, and $\mathfrak{n}^+$ denote the Lie algebras of $G$, $T$, $B$, and $U$ respectively. Let $W := \text{Norm}_G(T)/T$ denote the Weyl group corresponding to $G$ and $T$.

The weights of $\mathfrak{g}$ are the linear functionals $\xi \in \mathfrak{h}^*$. For $\alpha \in \mathfrak{h}^*$, let

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \text{ for all } H \in \mathfrak{h} \}.$$ 

If $\alpha \neq 0$ and $\mathfrak{g} \neq (0)$, then $\alpha$ is called a root and $\mathfrak{g}_\alpha$ is called a root space. Whenever $\alpha$ is a root the $\text{dim } \mathfrak{g}_\alpha = 1$. We call the set $\Phi$ of roots for $(\mathfrak{g}, \mathfrak{h})$ the root system. The Lie algebra $\mathfrak{g}$ has the following root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$ 

The choice of the Borel subgroup, $B$, gives us a decomposition $\Phi = \Phi^+ \cup -\Phi^+$, in such a way that $\mathfrak{n}^+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. We call $\Phi^+$ the set of positive roots and $-\Phi^+$ the set of negative roots. A subset $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subset \Phi^+$ is a set of simple roots if every $\gamma \in \Phi^+$ can be written uniquely as $\gamma = n_1\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r$, with $n_1, \ldots, n_r \in \mathbb{N}$. The requirement of uniqueness for the set of simple roots and the fact that $\Phi^+$ spans $\mathfrak{h}^*$, implies that $\Delta$ is a basis for $\mathfrak{h}^*$.

\footnote{As usual $\mathbb{N} = \{0, 1, 2, \ldots\}$ (the nonnegative integers).}
If $\beta = n_1\alpha_1 + \cdots + n_r\alpha_r$ with $n_1, \ldots, n_r \in \mathbb{N}$ is a root, then we define the height of $\beta$, relative to $\Delta$, as $ht(\beta) = n_1 + \cdots + n_r$. We can then describe the positive roots as the roots $\beta$ with $ht(\beta) > 0$. A root $\beta$ is called the highest root of $\Phi$, relative to $\Delta$, if $ht(\beta) > ht(\gamma)$ for all roots $\gamma \neq \beta$.

For each $\alpha \in \Phi^+$ there exist $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that the element $h_\alpha = [e_\alpha, f_\alpha] \in \mathfrak{h}$ satisfies $\alpha(h_\alpha) = 2$. We call $h_\alpha$ the coroot of $\alpha$. The Span\{ $e_\alpha, f_\alpha, h_\alpha$ \} is a three dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

For $X, Y \in \mathfrak{g}$ define the symmetric bilinear form $(X, Y) = tr_{\mathbb{C}^n}(XY)$ on $\mathfrak{g}$. Let $\mathfrak{h}_\mathbb{R}$ be the real span of the coroots, then the real dual space, $\mathfrak{h}_\mathbb{R}^*$, is the real linear span of the roots. Since the trace form is positive definite on $\mathfrak{h}_\mathbb{R}$, we can use it to identify $\mathfrak{h}_\mathbb{R}$ with $\mathfrak{h}_\mathbb{R}^*$ to obtain a positive definite inner product, $(,)$, on $\mathfrak{h}_\mathbb{R}^*$. We define the root reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by $s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$.

We can define the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ as the group $W = W(\mathfrak{g}, \mathfrak{h})$ of orthogonal transformations of $\mathfrak{h}_\mathbb{R}^*$ generated by the simple root reflections. Then each element $w \in W$ can be written as the product of generators $w = s_{\alpha_1} \cdots s_{\alpha_k}$, where $1 \leq i_j \leq r$ for all $j = 1, \ldots, k$. If $s_{\alpha_1} \cdots s_{\alpha_k}$ is minimal among all such expressions for $w$, then we call $k$ the length of $w$ and write $\ell(w) = k$. If $k = 0$, then $w$ is the empty product of simple reflections and hence $w$ is the identity element in $W$. Set $\epsilon(w) = (-1)^{\ell(w)}$.

Let $\{ \varpi_1, \ldots, \varpi_r \}$ be the corresponding set of fundamental weights of $(\mathfrak{g}, \mathfrak{h})$, which are defined by the conditions $\varpi_i(h_{\alpha_j}) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Then the set of integral and dominant integral weights are

$$P(\mathfrak{g}) = \{a_1\varpi_1 + \cdots + a_r\varpi_r | a_1, \ldots, a_r \in \mathbb{Z}\} \quad \text{and} \quad P_+(\mathfrak{g}) = \{n_1\varpi_1 + \cdots + n_r\varpi_r | n_1, \ldots, n_r \in \mathbb{N}\},$$ respectively.

The closed convex cone $C = \{ \mu \in \mathfrak{h}_\mathbb{R}^* : (\mu, \alpha_i) \geq 0, \text{ for all } 1 \leq i \leq r \}$ is called the positive Weyl chamber relative to a choice of $\Phi^+$. The dual cone is defined as $C^* = \{ \mu \in \mathfrak{h}_\mathbb{R}^* : (\mu, \varpi_i) \geq 0, \text{ for all } 1 \leq i \leq r \}$. We use the dual cone to define the root order, a partial order on $\mathfrak{h}_\mathbb{R}^*$ defined by $\mu \preceq \lambda$ if $\lambda - \mu \in C^*$.

The theorem of the highest weight asserts that any finite-dimensional complex
irreducible representation of $\mathfrak{g}$ is equivalent to a highest weight representation with dominant integral highest weight $\lambda$. We denote such a representation by $L(\lambda)$.

An area of interest in combinatorial representation theory is finding the multiplicity of a weight $\mu$ in $L(\lambda)$. One way to compute this multiplicity, which we denote by $m(\lambda, \mu)$, is by Kostant’s weight multiplicity formula, [9]:

$$m(\lambda, \mu) = \sum_{\sigma \in W} \epsilon(\sigma) \varphi(\sigma(\lambda + \rho) - (\mu + \rho)),$$

where $\varphi$ denotes Kostant’s partition function and $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. We recall that Kostant’s partition function is the nonnegative integer valued function, $\varphi$, defined on $\mathfrak{h}^*$, by $\varphi(\xi) =$ number of ways $\xi$ may be written as a nonnegative integral sum of positive roots, for $\xi \in \mathfrak{h}^*$.

One complication in using (1.1) to compute multiplicities is that closed formulas for the value of Kostant’s partition function are not known in much generality. A second complication concerns the exponential growth of the Weyl group order as $r \to \infty$. In practice, as noted in [4], most terms in Kostant’s weight multiplicity formula are zero and hence do not contribute to the overall multiplicity. With the aim of describing the contributing terms in (1.1), we give the following:

**Definition 1.0.1.** For $\lambda, \mu$ dominant integral weights of $\mathfrak{g}$ define the Weyl alternation set to be

$$\mathcal{A}(\lambda, \mu) = \{ \sigma \in W \mid \varphi(\sigma(\lambda + \rho) - (\mu + \rho)) > 0 \}.$$

Thus, $\sigma \in \mathcal{A}(\lambda, \mu)$ if and only if $\sigma(\lambda + \rho) - (\mu + \rho)$ can be written as a nonnegative integral combination of positive roots.

The purpose of this thesis is to demonstrate that the sets $\mathcal{A}(\lambda, \mu)$ are combinatorially interesting. We dedicate Chapter 2 to computations of Weyl alternation sets in some low rank examples. We then use these computations to create diagrams corresponding to the zero weight Weyl alternation sets associated to Lie algebras of rank 2.

In Chapter 3 we consider the case when the highest weight is the sum of all simple roots to prove:
Theorem. If $r \geq 1$ and $\tilde{\alpha} = \sum_{\alpha \in \Delta} \alpha$ is the highest root of $\mathfrak{sl}_{r+1}(\mathbb{C})$, then $|A(\tilde{\alpha}, 0)| = F_r$, where $F_r$ denotes the $r^{th}$ Fibonacci number.

Theorem. If $r \geq 2$ and $\varpi_1 = \sum_{\alpha \in \Delta} \alpha$ is a fundamental weight of $\mathfrak{so}_{2r+1}(\mathbb{C})$, then $|A(\varpi_1, 0)| = F_{r+1}$, where $F_{r+1}$ denotes the $(r+1)^{th}$ Fibonacci number.

We then prove some combinatorial identities related to Cartan subalgebras of $\mathfrak{sl}_{r+1}(\mathbb{C})$ and $\mathfrak{so}_{2r+1}(\mathbb{C})$. We also consider the nonzero weights, $\mu$, of $\mathfrak{sl}_{r+1}(\mathbb{C})$ and $\mathfrak{so}_{2r+1}(\mathbb{C})$, from the same point of view. In the last sections of Chapter 3 we prove that the weight defined by the sum of the simple roots is not a dominant integral weight of the Lie algebras $\mathfrak{sp}_{2r}(\mathbb{C})$ (for $r \geq 3$), $\mathfrak{so}_{2r}(\mathbb{C})$ (for $r \geq 4$), $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$. Hence it does not correspond to a finite-dimensional representation.
Chapter 2

Low Rank Examples

We begin this chapter by providing a brief introduction to the finite-dimensional complex irreducible representations of $\mathfrak{sl}_3(\mathbb{C})$. Specifically, we recall a procedure, presented in [7], used to find the set of weights of a highest weight representation and their corresponding multiplicities.

We then take a combinatorial approach to finding the set of contributing terms to Kostant’s weight multiplicity, by explicitly computing the zero weight Weyl alternation sets of Lie algebras of rank 2 and we give some diagrams associated to these sets. We conclude the chapter by giving a result regarding the convexity of Weyl alternation sets.

2.1 Finite-dimensional representations of $\mathfrak{sl}_3(\mathbb{C})$

The theorem of the highest weight asserts that finite-dimensional complex irreducible representations of a semi-simple Lie algebra are parameterized by dominant integral weights. This theorem is due to Cartan who, in [3], proved the existence of finite-dimensional irreducible representations with a given dominant integral highest weight.

In this section we will examine, as in [7], the set of weights of the finite-dimensional irreducible representations of $\mathfrak{sl}_3(\mathbb{C})$ and their corresponding multiplicities. We begin with some necessary notation and terminology to make our approach precise.
Let $G = SL_3(C) = \{g \in GL_3(C) : det(g) = 1\}$, $\mathfrak{g} = \mathfrak{sl}_3 = \mathfrak{sl}_3(C) = \{X \in M_3(C) : tr(X) = 0\}$, and let $\mathfrak{h} = \{diag[a_1, a_2, a_3]|a_1, a_2, a_3 \in C, \sum_{i=1}^3 a_i = 0\}$ be a fixed choice of Cartan subalgebra. Let $\mathfrak{b}$ be the set of $3 \times 3$ upper triangular complex matrices with trace zero. For $1 \leq i \leq 3$, define the linear functionals $\varepsilon_i : \mathfrak{h} \to C$ by $\varepsilon_i(H) = a_i$, for any $H = diag[a_1, a_2, a_3] \in \mathfrak{h}$. The Weyl group, $W$, is isomorphic to $S_3$, the symmetric group on 3 letters and acts on $\mathfrak{h}^*$ by permutations of $\varepsilon_1, \varepsilon_2$, and $\varepsilon_3$.

Let $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$. Then the set of simple and positive roots corresponding to $(\mathfrak{g}, \mathfrak{b})$ are $\Delta = \{\alpha_1, \alpha_2\}$ and $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$, respectively. Observe that $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3$. Let $Q = Z\alpha_1 \oplus Z\alpha_2$ be the root lattice of $\mathfrak{sl}_3$ and let $Q^+ = N\alpha_1 \oplus N\alpha_2$.

The fundamental weights are
\[
\varpi_1 = \frac{2}{3} \varepsilon_1 - \frac{1}{3} \varepsilon_2 - \frac{1}{3} \varepsilon_3 = \frac{2}{3} \alpha_1 + \frac{1}{3} \alpha_2 \quad \text{and} \quad (2.1)
\]
\[
\varpi_2 = \frac{1}{3} \varepsilon_1 + \frac{1}{3} \varepsilon_2 - \frac{2}{3} \varepsilon_3 = \frac{1}{3} \alpha_1 + \frac{2}{3} \alpha_2. \quad (2.2)
\]

Then the sets of integral weights and dominant integral weights are
\[
P(\mathfrak{sl}_3) = \{a_1 \varpi_1 + a_2 \varpi_2|a_1, a_2 \in Z\}
\]
\[
P_+(\mathfrak{sl}_3) = \{a_1 \varpi_1 + a_2 \varpi_2|a_1, a_2 \in N\},
\]
respectively.

The roots (indicated by $\circ$), the fundamental weights and some dominant weights (indicated by $\bullet$) of $\mathfrak{sl}_3$ are shown in Figure 2.1. Observe that the two simple roots, $\alpha_1$ and $\alpha_2$, have the same length and the angle between them is $120^\circ$. The set of dominant weights is contained in the positive Weyl chamber, a cone of opening $60^\circ$ whose walls are the $R_{\geq 0}$-span of the fundamental weights (indicated by dashed lines). The action of the Weyl group is generated by reflections across the walls of the positive Weyl chamber. Notice that the only root which is also dominant is $\rho = \alpha_1 + \alpha_2 = \varpi_1 + \varpi_2$.\footnote{\textit{diag}[a_1, a_2, a_3] is the diagonal $3 \times 3$ matrix whose entries are $a_1, a_2, a_3$.}
Remark 2.1.1. Observe, from Figure 2.1, the hexagonal symmetry of the set of positive roots and the weight lattice.

Since the highest weight of an irreducible representation occurs with multiplicity one and uniquely determines the representation, the question still remains as to what are the weights of an irreducible representation. Furthermore, we would like to know what are the multiplicities of these weights. We begin to answer this question by introducing a finite set of weights that is stable under $W$.

We call a subset $\Pi$ of $P(\mathfrak{g})$ saturated if for all $\lambda \in \Pi$, $\alpha \in \Phi$, and $i$ between 0 and $(\lambda, \alpha)$, the weight $\mu - i\alpha \in \Pi$.

Proposition 2.1.1 (Proposition 21.3 in [7]). Let $\lambda \in P_{+}(\mathfrak{g})$, and let $\Pi(\lambda)$ denote the set of all weights of the finite-dimensional irreducible representation $L(\lambda)$. Then the set $\Pi(\lambda)$ is saturated. In particular, the necessary and sufficient condition for $\mu \in P(\mathfrak{g})$ to belong to $\Pi(\lambda)$ is that $\mu$ and all of its $W$-conjugates be $\preceq \lambda$.

By Proposition 3.1.20 in [5], if $\mu \in P(\mathfrak{g})$, then there exists $w \in W$ and $\xi \in P_{+}(\mathfrak{g})$ such that $w(\xi) = \mu$. Hence it suffices to find the dominant weights in $\Pi(\lambda)$, since the
remaining weights will be acquired through the Weyl group orbits of the dominant weights belonging to $\Pi(\lambda)$.

**Corollary 2.1.1** (Corollary 3.2.12 in [5]). Let $L(\lambda)$ be the finite-dimensional irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$. Then $\Pi(\lambda) \cap P_+(\mathfrak{g})$ consists of all $\mu \in P_+(\mathfrak{g})$ such that $\mu \preceq \lambda$.

These results provide an algorithm for finding the weights of $L(\lambda)$. We take all $\beta \in \mathbb{Q}^+$ such that $||\lambda - \beta|| \leq ||\alpha||$ and write $\mu = \lambda - \beta$ in terms of the fundamental weights. If all of the coefficients are nonnegative, then $\mu \in \Pi(\lambda)$. This will give all of the dominant weights of $L(\lambda)$. The Weyl group orbit of the dominant weights of $L(\lambda)$ gives the entire set of weights of $L(\lambda)$, namely $\Pi(\lambda)$.

In computing the multiplicities of the weights of $L(\lambda)$, it will be helpful to create a weight diagram. This is a diagram on the weight lattice of $\mathfrak{sl}_3$, which we superimpose on a plane defined by the $\mathbb{R}$-span of the fundamental weights.

To create a weight diagram choose $|\Pi(\lambda) \cap P_+(\mathfrak{g})|$ distinct symbols (circle, diamond, asterisk, etc) and assign a distinct symbol to each of the dominant weights of $L(\lambda)$. Then place the symbol assigned to the dominant weight $\mu$ of $L(\lambda)$, on all integral weights in the Weyl group orbit of $\mu$. The polygons created by connecting all weights in the same Weyl group orbit through line segments are called shells.

If a weight of a representation is regular, then its Weyl group orbit will contain 6 elements and hence its corresponding shell will be a hexagon. On the other hand, if the weight is a multiple of a fundamental weight, then the Weyl group orbit will contain only 3 elements and thus its corresponding shell will be a triangle.

The multiplicity of a weight will depend on the shell it lies in. All weights lying on the outer shell, the shell corresponding to the highest weight, have multiplicity one. As we pass from outer shells to inner shells, the multiplicity of the weights (not already lying on the outer shell) increases by one provided we moved from a hexagonal shell to either a hexagonal or triangular shell. Once we reach a triangular shell, the multiplicity of the weights remains constant for all weights lying in the interior of this shell.
This weight multiplicity behavior is a special fact about \( \mathfrak{sl}_3 \), as proved in [1]. Using the above procedures we find the weights and their respective multiplicity for the following highest weight representations.

**Example 2.1.1.** Let \( \lambda = 4\varpi \). Observe that \( 4\varpi, 2\varpi + \varpi, 2\varpi, \varpi \) are the only dominant weights in \( \lambda - Q^+ \). The weight diagram for \( L(4\varpi) \) is given in Figure 2.2a. Observe that the weights \( 4\varpi, \varpi \) and \( 2\varpi \) are fixed by a simple root reflection and hence their Weyl group orbits have 3 elements, indicated by \( \circ, \bullet, \) and \( \square \), respectively. The weight \( 2\varpi + \varpi \) is regular so its Weyl group orbit contains 6 elements, we indicate these weights by \( * \). From Figure 2.2b, we can see that since the highest weight is a multiple of a fundamental weight, its corresponding shell is triangular and hence all the weights of \( L(4\varpi) \) have multiplicity 1.

![Diagram](image)

Figure 2.2: Weight diagram of the representation \( L(4\varpi) \) of \( \mathfrak{sl}_3(\mathbb{C}) \)

As noticed in Example 2.1.1, if \( \lambda \) is a multiple of a fundamental weight, we say \( L(\lambda) \) is a fundamental weight representation. In this case the highest weight is fixed by a simple root reflection and thus the Weyl group orbit consists of only 3 elements. Therefore, the outer shell corresponding to the highest weight is triangular and thus all weights of a fundamental weight representation will have multiplicity one. Notice that the zero weight will be a weight of \( L(\lambda) \), only when \( \lambda \in Q \).
Example 2.1.2. Let $\lambda = \rho = \omega_1 + \omega_2 = \alpha_1 + \alpha_2$. Observe that $\lambda$ and 0 are the only dominant weights in $\lambda - Q^+$. The highest weight, $\rho$, is regular, and its Weyl group orbit has 6 elements. The weight diagram for $L(\rho)$ is given in Figure 2.3a. Notice that $L(\lambda)$ is the adjoint representation since $\rho = \alpha_1 + \alpha_2$ is the highest root of $\mathfrak{sl}_3$. Hence the weights of $L(\rho)$ are the roots (indicated by $\circ$) each with multiplicity 1 and the zero weight (indicated by $\bullet$) with multiplicity 2, which is equal to the rank of $\mathfrak{sl}_3$. This may be observed from Figure 2.3b where the roots lie on the outer shell hence have multiplicity 1, while the zero weight lies in the inner shell and hence has multiplicity 2.

![Weight Diagram](image)

(a) Weights of $L(\rho)$  
(b) Multiplicities of the weights of $L(\rho)$

Figure 2.3: Weight diagram of the representation $L(\rho)$ of $\mathfrak{sl}_3(\mathbb{C})$

As we saw in Example 2.1.2, if the highest weight is a positive integral multiple of $\rho$, then the shells associated to nonzero weights will be regular-hexagons. Furthermore, the weight multiplicities will begin at 1, in the outer shell, and will increase steadily by 1 as we move from outer to inner shells until we get to the zero weight.

Example 2.1.3. Let $\lambda = \omega_1 + 2\omega_2$. Observe that $\omega_1 + 2\omega_2$, $2\omega_1$, and $\omega_2$ are the only dominant weights in $\lambda - Q^+$. The weight diagram for $L(\omega_1 + 2\omega_2)$ is given in Figure 2.4a. Observe that the highest weight, $\omega_1 + 2\omega_2$, is regular, and its Weyl
group orbit has 6 elements, which we indicate by $\bullet$. The weights $2\varpi_1$ and $\varpi_2$ are fixed by one of the simple root reflections, and hence their Weyl group orbit consists of 3 elements, which we indicate by $\ast$ and $\Box$, respectively. From Figure 2.4b, we can see that the weights $\varpi_1 + 2\varpi_2$, $2\varpi_1$, and their $W$-conjugates lie on the outer shell, hence their multiplicity is 1, while the weight $\varpi_2$ and its $W$-conjugates lie on the inner shell and thus have multiplicity 2.

![Diagram](image)

(a) Weights of $L(\varpi_1 + 2\varpi_2)$

(b) Multiplicities of the weights of $L(\varpi_1 + 2\varpi_2)$

Figure 2.4: Weight diagram of the representation $L(\varpi_1 + 2\varpi_2)$ of $\mathfrak{sl}_3(\mathbb{C})$

As seen in Example 2.1.3, if the highest weight is regular, but not a multiple of $\rho$, then the initial shells will be hexagonal in shape and an inner shell will be a triangle, at which point the multiplicities will stabilize, that is will remain constant for all weights of the representation lying in the interior of this triangular shell.

Figure 2.5 displays this behavior. In these figures, notice that the triangle formed by the intersection of the red line segments is the shell at which the multiplicities stabilize. Figures 2.5a and 2.5i correspond to fundamental representations, which, as we noted before, have an outer triangular shell. Thus all weights have multiplicity one. Figure 2.5e gives a representation that is a multiple of $\rho$, in this case notice that the multiplicity stabilizes only when we reach the zero weight.

We remark that there may exist hexagonal shells between the outer shell and the
(a) Highest weight is a multiple of $\varpi_2$

(b)

(c)

(d)

(e) Highest weight is a multiple of $\rho$

(f)

(g)

(h)

(i) Highest weight is a multiple of $\varpi_1$

Figure 2.5: Weight diagrams of representations of $\mathfrak{sl}_3(\mathbb{C})$
first triangular shell, which are not depicted in Figure 2.5. For concrete examples of this behavior see Figure 2.6, which considers the representations $L(\varpi_1 + 4\varpi_2)$ and $L(2\varpi_1 + 3\varpi_2)$.

Figure 2.6: Weight diagram of the representations $L(\varpi_1 + 4\varpi_2)$ and $L(2\varpi_1 + 3\varpi_2)$ of $\mathfrak{sl}_3(\mathbb{C})$

### 2.2 The case of $\mathfrak{sl}_3(\mathbb{C})$

In Section 2.1 we gave an algorithm to find the weights of a finite-dimensional irreducible highest weight representation of $\mathfrak{sl}_3$ and a procedure to find their multiplicity. From that algorithm, determining whether the zero weight is in $\Pi(\lambda)$, for $\lambda \in P_+(\mathfrak{sl}_3)$, depends on whether $\lambda$ is in the root lattice. If $\lambda \in Q$, then the zero weight is in $\Pi(\lambda)$. Computing the multiplicity of the zero weight can be involved, since, as we saw in various examples, the multiplicity depends on the shape and number of shells in the weight diagram of $L(\lambda)$.

We now take a combinatorial approach to find the number of terms contributing, through Kostant’s weight multiplicity formula, to the multiplicity of the zero weight in a given finite-dimensional irreducible highest weight representation of $\mathfrak{sl}_3$. Namely, we compute the Weyl alternation sets $A(\lambda, 0)$, for $\lambda$ a dominant integral weight of $\mathfrak{sl}_3$. 
In this process it will be helpful to have a closed formula for the value of Kostant’s partition function, which we can compute by expanding the formal power series

\[
\prod_{\alpha \in \Phi^+} \frac{1}{1 - e^{-\alpha}} = \sum_{\xi \in P_{\text{aff}}} \varphi(\xi) e^{\xi}.
\]

(2.3)

For \(1 \leq i \leq 3\), \(e^{\xi_i} : T \to \mathbb{C}^\times\) is defined by

\[
x_i \left( \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \right) = A_i,
\]

and satisfy \(A_1A_2A_3 = 1\). Now for \(1 \leq i \leq 2\), let \(x_i = e^{\alpha_i}\). Then we can simplify both sides of (2.3) to get

\[
\left( \frac{1}{1 - x_1} \right) \left( \frac{1}{1 - x_2} \right) \left( \frac{1}{1 - x_1 x_2} \right) = \sum_{a,b,c \in \mathbb{N}} x_1^{a+c} x_2^{b+c}.
\]

(2.4)

If \(\xi = x\alpha_1 + y\alpha_2 \in Q\), then (2.4) implies that \(\varphi(\xi) = \) number of nonnegative integral solutions, \((a, b, c)\), to the system of equations

\[
a + c = x
\]

(2.5)

\[
b + c = y.
\]

(2.6)

We can now give a closed formula for the value of Kostant’s partition function.

**Proposition 2.2.1.** If \(x, y \in \mathbb{Z}\), then \(\varphi(x\alpha_1 + y\alpha_2) = \begin{cases} \min(x, y) + 1 & \text{if } x, y \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}\)

*Proof.* Let \(\lambda = x\alpha_1 + y\alpha_2\) for some \(x, y \in \mathbb{Z}\). Notice if \(x < 0\), then (2.5) has no integral nonnegative solutions, since the left hand side is nonnegative and the right hand side would be positive. Similarly if \(y < 0\), then (2.6) has no integral nonnegative solutions, since the left hand side is nonnegative and the right hand side is negative. Therefore \(\varphi(\lambda) = 0\) if either \(x < 0\) or \(y < 0\). Now suppose that both \(x, y \geq 0\). In this case by (2.5) we know there exist \(x + 1\) choices for \(c\), namely \(c \in \{0, 1, 2, 3, \ldots, x\}\). Then by (2.6), we know there exist \(y + 1\) choices for \(c\), namely \(c \in \{0, 1, 2, 3, \ldots, y\}\). Since \(c\) must simultaneously satisfy both equation (1) and

\[
\left( \frac{1}{1 - x_1} \right) \left( \frac{1}{1 - x_2} \right) \left( \frac{1}{1 - x_1 x_2} \right) = \sum_{a,b,c \in \mathbb{N}} x_1^{a+c} x_2^{b+c}.
\]
(2), we know \( c \in \{0, 1, 2, 3, \ldots, \min(x, y)\} \). Then the set of solutions to the system of equations is \( \{(x, y, 0), (x - 1, y - 1, 1), (x - 2, y - 2, 2), \ldots (x - \min(x, y), y - \min(x, y))\} \). Hence \( \varphi(\lambda) = \varphi(x\alpha_1 + y\alpha_2) = \min(x, y) + 1 \), whenever \( x, y \in \mathbb{N} \).

**Remark 2.2.1.** We will compute \( A(\lambda, 0) \) for all \( \lambda \in P(\mathfrak{sl}_3) \). We do however remark that we only compute the multiplicities of the zero weight space in the case when \( \lambda \) is a dominant integral weight of \( \mathfrak{sl}_3 \), since these correspond to finite-dimensional representations.

Since \( \sigma \in A(\lambda, 0) \) if and only if \( \sigma(\lambda + \rho) - \rho \in \mathbb{Q}^+ \), we want a description of the root lattice in terms of the integral weight lattice. Hence we prove:

**Lemma 2.2.1.** Let \( \lambda = x\varpi_1 + y\varpi_2 \) with \( x, y \in \mathbb{Z} \). Then \( \lambda \in \mathbb{Q} \) if and only if \( 3 \mid 2x + y \) and \( 3 \mid x + 2y \).

**Proof.** Let \( \lambda = x\varpi_1 + y\varpi_2 \) with \( x, y \in \mathbb{Z} \). By (2.1) and (2.2) we have that

\[
\lambda = \left( \frac{2x + y}{3} \right) \varepsilon_1 + \left( \frac{x + 2y}{3} \right) \varepsilon_2. \tag{2.7}
\]

Hence \( \lambda \in \mathbb{Q} \) if and only if \( 3 \mid 2x + y \) and \( 3 \mid x + 2y \). □

We now give the following simplifications. Let \( \lambda \in P(\mathfrak{sl}_3) \), hence \( \lambda = x\varpi_1 + y\varpi_2 \) for some \( x, y \in \mathbb{Z} \). By (2.1) and (2.2) we can write

\[
\lambda = \left( \frac{2x + y}{3} \right) \varepsilon_1 + \left( \frac{-x + y}{3} \right) \varepsilon_2 + \left( \frac{-x - 2y}{3} \right) \varepsilon_3, \quad \text{and} \tag{2.8}
\]

\[
\lambda + \rho = \left( \frac{2x + y + 3}{3} \right) \varepsilon_1 + \left( \frac{-x + y}{3} \right) \varepsilon_2 + \left( \frac{-x - 2y - 3}{3} \right) \varepsilon_3. \tag{2.9}
\]
Using (2.8) and (2.9) we compute,

\[ 1(\lambda + \rho) - \rho = \left( \frac{2x + y}{3} \right) \alpha_1 + \left( \frac{x + 2y}{3} \right) \alpha_2, \quad (2.10) \]

\[ (12)(\lambda + \rho) - \rho = \left( \frac{-x + y - 3}{3} \right) \alpha_1 + \left( \frac{x + 2y}{3} \right) \alpha_2, \quad (2.11) \]

\[ (23)(\lambda + \rho) - \rho = \left( \frac{2x + y}{3} \right) \alpha_1 + \left( \frac{x - y - 3}{3} \right) \alpha_2, \quad (2.12) \]

\[ (13)(\lambda + \rho) - \rho = \left( \frac{-x - 2y - 6}{3} \right) \alpha_1 + \left( \frac{-2x - y - 6}{3} \right) \alpha_2, \quad (2.13) \]

\[ (123)(\lambda + \rho) - \rho = \left( \frac{-x - 2y - 6}{3} \right) \alpha_1 + \left( \frac{x - y - 3}{3} \right) \alpha_2, \quad (2.14) \]

\[ (132)(\lambda + \rho) - \rho = \left( \frac{-x + y - 3}{3} \right) \alpha_1 + \left( \frac{-2x - y - 6}{3} \right) \alpha_2. \quad (2.15) \]

**Proposition 2.2.2.** Let \( \lambda \in P(sl_3) \). If \( \lambda \notin Q \), then \( A(\lambda, 0) = \emptyset \).

**Proof.** Let \( \lambda = x\varpi_1 + y\varpi_2 \) with \( x, y \in \mathbb{Z} \), and assume \( \lambda \notin Q \). Lemma 2.2.1 implies \( 3 \nmid 2x + y \) or \( 3 \nmid x + 2y \). It is easy to see that if \( a \in \mathbb{Z} \) and \( 3 \nmid a \), then \( 3 \nmid a + 3b \) for any \( b \in \mathbb{Z} \).

Case 1: Suppose that \( 3 \nmid 2x + y \). Then by (2.10), and (2.12) we see that \( 1, (23) \notin A(\lambda, 0) \). Now notice if \( 3 \nmid 2x + y \), then \( 3 \nmid -x + y \). Then by (2.11), (2.15) we get that \( (12), (132) \notin A(\lambda, 0) \). Also \( 3 \nmid -x - 2y \), so (2.13) and (2.14) imply that \( (13), (123) \notin A(\lambda, 0) \). Thus \( A(\lambda, 0) = \emptyset \).

Case 2: Suppose that \( 3 \nmid x + 2y \). Then by (2.10), and (2.11) we see that \( 1, (12) \notin A(\lambda, 0) \). Now notice if \( 3 \nmid x + 2y \), then \( 3 \nmid x - y \). Then by (2.12), (2.14) we get that \( (23), (123) \notin A(\lambda, 0) \). Also \( 3 \nmid -2x - y \), so (2.13) and (2.15) imply that \( (13), (132) \notin A(\lambda, 0) \). Thus \( A(\lambda, 0) = \emptyset \).

**Remark 2.2.2.** Proposition 2.2.2 was expected and implies that we need only compute \( A(\lambda, 0) \), for \( \lambda \in Q \).

**Proposition 2.2.3.** Let \( \lambda \in P(sl_3) \). Then \( \lambda \in Q \) if and only if \( \lambda = (3x+y)\varpi_1 + y\varpi_2 \) for some \( x, y \in \mathbb{Z} \).
Proof. \(\Rightarrow\) Let \(\lambda = x'\varpi_1 + y'\varpi_2\) where \(x', y' \in \mathbb{Z}\) satisfy \(3|2x' + y'\) and \(3|x' + 2y'.\) Hence \(3|(2x' + y') - (x' + 2y') = x' - y'.\) Let \(x = \frac{x' - y'}{3}\) and let \(y = y'.\) Then \(x' = 3x + y\) and we can write \(\lambda = x'\varpi_1 + y'\varpi_2 = (3x + y)\varpi_1 + y\varpi_2.\)

\((\Leftarrow)\) This follows from Lemma 2.2.1, since \(3|2(3x + y) + y = 6x + 3y\) and \(3|(3x + y) + 2y = 3x + 3y.\)

Now we can state the main theorem of this section:

**Theorem 2.2.1.** If \(\lambda = (3x + y)\varpi_1 + y\varpi_2\) for some \(x, y \in \mathbb{Z},\) then

\[
A(\lambda, 0) = \begin{cases}
\{1\} & x = 0, y \geq 0 \\
\{(12)\} & x \leq -1, y = -2x - 1 \\
\{(23)\} & x \geq 1, y = -x - 1 \\
\{(13)\} & x = 0, y \leq -2 \\
\{(123)\} & x \geq 1, y = -2x - 1 \\
\{(132)\} & x \leq -1, y = -x - 1 \\
\{1, (12)\} & x \leq -1, y \geq -2x \\
\{1, (23)\} & x \geq 1, y \geq -x \\
\{(23), (123)\} & x \geq 2, -2x \leq y \leq -x - 2 \\
\{(13), (123)\} & x \geq 1, y \leq -2x - 2 \\
\{(13), (132)\} & x \leq -1, y \leq -x - 2 \\
\{(132), (12)\} & x \leq 2, -x \leq y \leq -2x - 2 \\
\emptyset & x = 0, y = -1.
\end{cases}
\]

Proof. Let \(\lambda = (3x + y)\varpi_1 + y\varpi_2\) for some \(x, y \in \mathbb{Z}\). By (2.10)-(2.15) we have that

\[
1(\lambda + \rho) - \rho = (2x + y)\alpha_1 + (x + y)\alpha_2, \\
(12)(\lambda + \rho) - \rho = (-x - 1)\alpha_1 + (x + y)\alpha_2, \\
(23)(\lambda + \rho) - \rho = (2x + y)\alpha_1 + (x - 1)\alpha_2, \\
(13)(\lambda + \rho) - \rho = (-x - y - 2)\alpha_1 + (-2x - y - 2)\alpha_2, \\
(123)(\lambda + \rho) - \rho = (-x - y - 2)\alpha_1 + (x - 1)\alpha_2, \\
(132)(\lambda + \rho) - \rho = (-x - 1)\alpha_1 + (-2x - y - 2)\alpha_2.
\]
By the definition of a Weyl alternation set and from each of the above equations we have that
\[ 1 \in \mathcal{A}(\lambda, 0) \iff 2x + y \geq 0 \text{ and } x + y \geq 0, \]
\[ (12) \in \mathcal{A}(\lambda, 0) \iff -x - 1 \geq 0 \text{ and } x + y \geq 0, \]
\[ (23) \in \mathcal{A}(\lambda, 0) \iff 2x + y \geq 0 \text{ and } x - 1 \geq 0, \]
\[ (13) \in \mathcal{A}(\lambda, 0) \iff -x - y - 2 \geq 0 \text{ and } -2x - y - 2 \geq 0, \]
\[ (123) \in \mathcal{A}(\lambda, 0) \iff -x - y - 2 \geq 0 \text{ and } x - 1 \geq 0, \]
\[ (132) \in \mathcal{A}(\lambda, 0) \iff -x - 1 \geq 0 \text{ and } -2x - y - 2 \geq 0. \]

We can simplify the above inequalities as follows:
\[ 1 \in \mathcal{A}(\lambda, 0) \iff y \geq -2x \text{ and } y \geq -x, \quad (2.16) \]
\[ (12) \in \mathcal{A}(\lambda, 0) \iff x \leq -1 \text{ and } y \geq -x, \quad (2.17) \]
\[ (23) \in \mathcal{A}(\lambda, 0) \iff y \geq -2x \text{ and } x \geq 1, \quad (2.18) \]
\[ (13) \in \mathcal{A}(\lambda, 0) \iff y \leq -x - 2 \text{ and } y \leq -2x - 2, \quad (2.19) \]
\[ (123) \in \mathcal{A}(\lambda, 0) \iff y \leq -x - 2 \text{ and } x \geq 1, \quad (2.20) \]
\[ (132) \in \mathcal{A}(\lambda, 0) \iff x \leq -1 \text{ and } y \leq -2x - 2. \quad (2.21) \]

We will graph the solution set to each pair of linear inequalities in (2.16)-(2.21), on the weight lattice of $\mathfrak{sl}_3$. We will superimpose the weight lattice of $\mathfrak{sl}_3$ on a plane defined by the $\mathbb{R}$-span of the fundamental weights. The bolded axes in these figures will correspond to the $\mathbb{R}$-span of the fundamental weights, with placement of simple roots and fundamental weights as in Figure 2.1.

Observe that the solution set to the inequalities in (2.16), namely $y \geq -2x$ and $y \geq -x$, is given by Figure 2.7a. We graph the inequalities as we would on $\mathbb{R}^2$, but rather than shading the solution set (since not all weights are on the root lattice), we place a colored solid circle only on the integral weights for which $1(\lambda + \rho) - \rho \in Q^+$, since these are the integral weights for which the inequalities hold.

In the remaining subfigures 2.7b-2.7c, we graph the solution set to each pair of linear inequalities in (2.17)-(2.21), respectively. Notice again that these solution sets
are given by placing a solid circle on all \( \lambda \in P(\mathfrak{sl}_3) \), for which \( \sigma \in A(\lambda, 0) \), where \( \sigma \) is the element of \( S_3 \) corresponding to the inequalities whose solution set we are graphing. The theorem then follows from the intersection of the solution sets to the linear inequalities given in (2.16)-(2.21).

**Proposition 2.2.4.** If \( n \in \mathbb{N} \), then \( m(n\rho, 0) = n + 1 \).

**Proof.** By Theorem 2.2.1 we know that the only contributing term to the multiplicity of the zero weight in \( L(n\rho) \) is given when \( \sigma = 1 \). Thus, \( m(n\rho, 0) = \varphi(1(n\rho+\rho)-\rho) = \varphi(n\rho) = \varphi(n\alpha_1 + n\alpha_2) \). By Proposition 2.2.1, \( \varphi(n\rho) = n + 1 \).

The usefulness of Theorem 2.2.1 is evident in Proposition 2.2.4. However, our main concern here is to describe the sets of contributing terms and to find any intriguing properties of these sets. This will be of importance in Chapter 3.
(a) $\lambda \in P(\mathfrak{sl}_3)$ such that $1 \in \mathcal{A}(\lambda, 0)$

(b) $\lambda \in P(\mathfrak{sl}_3)$ such that $(12) \in \mathcal{A}(\lambda, 0)$

(c) $\lambda \in P(\mathfrak{sl}_3)$ such that $(132) \in \mathcal{A}(\lambda, 0)$

(d) $\lambda \in P(\mathfrak{sl}_3)$ such that $(13) \in \mathcal{A}(\lambda, 0)$

(e) $\lambda \in P(\mathfrak{sl}_3)$ such that $(123) \in \mathcal{A}(\lambda, 0)$

(f) $\lambda \in P(\mathfrak{sl}_3)$ such that $(23) \in \mathcal{A}(\lambda, 0)$

Figure 2.7: Graphs of solution sets to linear inequalities associated to $\mathfrak{sl}_3(\mathbb{C})$
2.3 The case of $\mathfrak{so}_5(\mathbb{C})$

Let $G = SO_5(\mathbb{C}) = \{g \in GL_5(\mathbb{C}) : g^t = g^{-1}\} \cap SL_5(\mathbb{C})$, denote the special orthogonal group of rank 2 over $\mathbb{C}$. Let $\mathfrak{g} = \mathfrak{so}_5(\mathbb{C}) = \{X \in M_5(\mathbb{C}) : X^t = -X\}$ denote the special orthogonal Lie algebra of rank 2, and fix $\mathfrak{h} = \{\text{diag}[a_1, a_2, 0, -a_2, -a_1]|a_1, a_2 \in \mathbb{C}\}$ be a fixed choice of Cartan subalgebra. For $1 \leq i \leq 3$, define the linear functionals $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(H) = a_i$, for any $H = \text{diag}[a_1, a_2, 0, -a_2, -a_1] \in \mathfrak{h}$.

Let $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2$. Let $\Delta = \{\alpha_1, \alpha_2\}$, then $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. As before let $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ denote the root lattice and let $Q^+ = \mathbb{N}\alpha_1 \oplus \mathbb{N}\alpha_2$. The fundamental weights are defined by

$$\varpi_1 = \varepsilon_1 = \alpha_1 + \alpha_2 \quad \text{and} \quad (2.22)$$
$$\varpi_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) = \frac{1}{2}\alpha_1 + \alpha_2. \quad (2.23)$$

Observe that $\rho = \varpi_1 + \varpi_2 = \frac{3}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 = \frac{3}{2}\alpha_1 + 2\alpha_2$. The Weyl group, $W$, of $\mathfrak{so}_5$ acts on $\mathfrak{h}^*$ by signed permutations of $\varepsilon_1$ and $\varepsilon_2$ and is isomorphic to the group of signed permutations on two letters.

![Figure 2.8: Roots and dominant weights of $\mathfrak{so}_5(\mathbb{C})$](image)

The roots (indicated by $\circ$), the fundamental weights and some dominant weights
(indicated by •) of \( \mathfrak{so}_5 \) are shown in Figure 2.8. Observe that the two simple roots have distinct lengths, \( \alpha_1 \) is longer than \( \alpha_2 \). Also notice that the angle between them is 135°. The set of dominant weights is contained in the positive Weyl chamber, a cone of opening 45°. The action of the Weyl group is generated by reflections across the dashed lines, which are the walls of the positive Weyl chamber and are the \( \mathbb{R}_{\geq 0} \)-span of the fundamental weights. Notice that the roots \( \alpha_1 + \alpha_2 = \varpi_1 \) and \( \alpha_1 + 2\alpha_2 = 2\varpi_2 \) are the only roots which are also dominant.

**Remark 2.3.1.** Observe that in Figure 2.8 there is a square symmetry of the set of positive roots and the weight lattice.

We want to give a complete description of the Weyl alternation sets, \( A(\lambda, 0) \), in the case when \( \lambda \) is an integral weight of \( \mathfrak{so}_5 \). We remark again that we only compute the multiplicities of the zero weight in the case when \( \lambda \) is a dominant integral weight, since these correspond to finite dimensional irreducible representations.

**Lemma 2.3.1.** Let \( \lambda = x\varpi_1 + y\varpi_2 \) with \( x, y \in \mathbb{Z} \). Then \( \lambda \in Q \) if and only if \( 2|y \).

**Proof.** By (2.22) and (2.23) observe that \( \lambda = x\varpi_1 + y\varpi_2 = (x + \frac{y}{2})\alpha_1 + (x + y)\alpha_2 \), which is in \( Q \) if and only if \( 2|y \).

We now give the following simplifications. Let \( \lambda \in P(\mathfrak{so}_5) \), hence \( \lambda = x\varpi_1 + y\varpi_2 \) for some \( x, y \in \mathbb{Z} \). By (2.22) and (2.23) we can write \( \lambda \) and \( \lambda + \rho \) as a column vector whose entry in the \( i \)th-row is given by the coefficient of \( \varepsilon_i \). That is

\[
\lambda = \left( x + \frac{y}{2} \right) \varepsilon_1 + \frac{y}{2} \varepsilon_2 = \left( x + \frac{y}{2} \right) \quad \text{and} \quad \lambda + \rho = \left( x + \frac{y}{2} + 3 \right) \varepsilon_1 + \left( \frac{y}{2} + 1 \right) \varepsilon_2 = \left( x + \frac{y}{2} + \frac{3}{2} \right). \quad (2.24)
\]

Using (2.24) and (2.25) we compute
\[
[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}] (\lambda + \rho) - \rho = \left(x + \frac{y}{2}\right) \alpha_1 + (x + y) \alpha_2,
\] (2.26)

\[
[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}] (\lambda + \rho) - \rho = \left(-x - \frac{y}{2} - 3\right) \alpha_1 + (-x - 3) \alpha_2,
\] (2.27)

\[
[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}] (\lambda + \rho) - \rho = \left(x + \frac{y}{2}\right) \alpha_1 + (x - 1) \alpha_2,
\] (2.28)

\[
[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}] (\lambda + \rho) - \rho = \left(-x - \frac{y}{2} - 3\right) \alpha_1 + (-x - y - 4) \alpha_2,
\] (2.29)

\[
[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}] (\lambda + \rho) - \rho = \left(-\frac{y}{2} - 2\right) \alpha_1 + (x - 1) \alpha_2,
\] (2.30)

\[
[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}] (\lambda + \rho) - \rho = \left(-\frac{y}{2} - 2\right) \alpha_1 + (x - 3) \alpha_2,
\] (2.31)

\[
[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}] (\lambda + \rho) - \rho = \left(-\frac{y}{2} - 2\right) \alpha_1 + (-x - y - 4) \alpha_2.
\] (2.32)

\[
[\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}] (\lambda + \rho) - \rho = \left(-\frac{y}{2} - 2\right) \alpha_1 + (-x - y - 4) \alpha_2.
\] (2.33)

**Proposition 2.3.1.** Let \( \lambda \in P(\mathfrak{so}_5) \). If \( \lambda \notin Q \), then \( A(\lambda, 0) = \emptyset \).

**Proof.** Suppose \( \lambda = x\overline{w}_1 + y\overline{w}_2 \in P(\mathfrak{so}_5) \), but \( \lambda \notin Q \). Then by Lemma 2.3.1 we have that \( x, y \in \mathbb{Z} \) and \( 2 \nmid y \). Then notice by (2.26)-(2.33), the coefficient of \( \alpha_1 \) in \( \sigma(\lambda + \rho) - \rho \) is not an integer, for any \( \sigma \in W \). Hence \( A(\lambda, 0) = \emptyset \). \( \Box \)

Thus we need only compute \( A(\lambda, 0) \) for \( \lambda \in Q \). We now state the main theorem of this section.

**Theorem 2.3.1.** If \( \lambda = x\overline{w}_1 + y\overline{w}_2 \) for some \( x, y \in \mathbb{Z} \) with \( 2|y \), then
Proof. Let $\lambda = x\omega_1 + y\omega_2$ for some $x, y \in \mathbb{Z}$ with $2|y$. By (2.26)-(2.33) we have that
\[
\begin{align*}
    \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff x \geq \frac{-y}{2} \text{ and } x \geq -y, \quad (2.34) \\
    \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff x \leq \frac{-y}{2} - 3 \text{ and } x \leq -3, \quad (2.35) \\
    \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff x \geq \frac{-y}{2} \text{ and } x \geq 1, \quad (2.36) \\
    \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff x \leq \frac{-y}{2} - 3 \text{ and } x \leq -y - 4, \quad (2.37) \\
    \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \geq 2 \text{ and } x \geq -y, \quad (2.38) \\
    \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \leq -4 \text{ and } x \geq 1, \quad (2.39) \\
    \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \geq 2 \text{ and } x \leq -3, \quad (2.40) \\
    \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \leq -4 \text{ and } x \leq -y - 4. \quad (2.41)
\end{align*}
\]

In Figure 2.9, we graph the solution set to each pair of linear inequalities in (2.34)-(2.41) by placing a solid circle on all \( \lambda \in P(\mathfrak{so}_5) \), for which \( \sigma \in \mathcal{A}(\lambda, 0) \). We graph each solution set on the weight lattice of \( \mathfrak{so}_5 \), which we superimpose on a plane defined by the \( \mathbb{R} \)-span of the fundamental weights. The bolded axes correspond to the \( \mathbb{R} \)-span of the fundamental weights, with placement of simple roots and fundamental weights as in Figure 2.8. The theorem then follows from the intersection of the solution sets to the inequalities (2.34)-(2.41). \( \square \)
(a) $\lambda \in P(\mathfrak{so}_5)$ such that 
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \in A(\lambda, 0)
\]

(b) $\lambda \in P(\mathfrak{so}_5)$ such that 
\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} \in A(\lambda, 0)
\]

(c) $\lambda \in P(\mathfrak{so}_5)$ such that 
\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \in A(\lambda, 0)
\]

(d) $\lambda \in P(\mathfrak{so}_5)$ such that 
\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \in A(\lambda, 0)
\]

(e) $\lambda \in P(\mathfrak{so}_5)$ such that 
\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \in A(\lambda, 0)
\]

(f) $\lambda \in P(\mathfrak{so}_5)$ such that 
\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix} \in A(\lambda, 0)
\]

(g) $\lambda \in P(\mathfrak{so}_5)$ such that 
\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \in A(\lambda, 0)
\]

(h) $\lambda \in P(\mathfrak{so}_5)$ such that 
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \in A(\lambda, 0)
\]

Figure 2.9: Graphs of solution sets to linear inequalities associated to $\mathfrak{so}_5(\mathbb{C})$
2.4 The case of $\mathfrak{sp}_4(\mathbb{C})$

Let $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ with $I$ the $2 \times 2$ identity matrix. Let $G = SP_4(\mathbb{C}) = \{g \in M_4(\mathbb{C}) : g^tJg = J \}$ denote the symplectic group of rank 2 over $\mathbb{C}$. Let $g = \mathfrak{sp}_4(\mathbb{C}) = \{X \in M_4(\mathbb{C}) : X^tJ = -JX \}$ denote the symplectic Lie algebra of rank 2, and fix $\mathfrak{h} = \{\text{diag}[a_1, a_2, -a_2, -a_1] | a_1, a_2 \in \mathbb{C} \}$ be a fixed choice of Cartan subalgebra. For $1 \leq i \leq 3$, define the linear functionals $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(H) = a_i$, for any $H = \text{diag}[a_1, a_2, -a_2, -a_1] \in \mathfrak{h}$.

Let $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = 2\varepsilon_2$. Let $\Delta = \{\alpha_1, \alpha_2\}$, then $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$. Let $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ denote the root lattice and let $Q^+ = \mathbb{N}\alpha_1 \oplus \mathbb{N}\alpha_2$. The fundamental weights are defined by

$$\varpi_1 = \varepsilon_1 = \alpha_1 + \frac{1}{2}\alpha_2$$
$$\varpi_2 = \varepsilon_1 + \varepsilon_2 = \alpha_1 + \alpha_2.$$  \hfill (2.42)

Observe that $\rho = \varpi_1 + \varpi_2 = 2\varepsilon_1 + \varepsilon_2 = 2\alpha_1 + \frac{3}{2}\alpha_2$. The Weyl group, $W$, of $\mathfrak{sp}_4$ acts on $\mathfrak{h}^*$ by signed permutations of $\varepsilon_1$ and $\varepsilon_2$ and is isomorphic to the group of signed permutations on two letters.

The roots (indicated by $\circ$), the fundamental weights and some dominant weights (indicated by $\bullet$) of $\mathfrak{sp}_4$ are shown in Figure 2.10. Observe that the two simple roots have distinct lengths, $\alpha_2$ is longer than $\alpha_1$. Also notice that the angle between them is $135^\circ$. The set of dominant weights is contained in the positive Weyl chamber, a cone of opening $45^\circ$. The action of the Weyl group is generated by reflections across the dashed lines, which are the walls of the positive Weyl chamber and are the $\mathbb{R}_{\geq 0}$-span of the fundamental weights. Notice that the roots $\alpha_1 + \alpha_2 = \varpi_2$ and $2\alpha_1 + \alpha_2 = 2\varpi_1$ are the only roots which are also dominant.

Remark 2.4.1. Observe that in Figure 2.10 there is a square symmetry of the set of positive roots and the weight lattice. We also remark that the roots systems of $\mathfrak{so}_5$ and $\mathfrak{sp}_4$ are isomorphic. To obtain the roots and fundamental weights of $\mathfrak{so}_5$ we need only interchange the subscripts 1 and 2 in Figure 2.10. Doing so yields Figure 2.8.

Lemma 2.4.1. Let $\lambda = x\varpi_1 + y\varpi_2$ with $x, y \in \mathbb{Z}$. Then $\lambda \in Q$ if and only if $2|x$. 

Proof. By (2.42) and (2.43) observe that \( \lambda = x\varpi_1 + y\varpi_2 = (x + y)\alpha_1 + (\frac{y}{2} + y)\alpha_2 \), which is in \( Q \) if and only if \( 2|x \).

We now give the following simplifications. Let \( \lambda \in P(\mathfrak{sp}_4) \), hence \( \lambda = x\varpi_1 + y\varpi_2 \) for some \( x, y \in \mathbb{Z} \). By (2.42) and (2.43) we can write \( \lambda \) and \( \lambda + \rho \) as a column vector whose entry in the \( i^{th} \)-row is given by the coefficient of \( \varepsilon_i \). That is

\[
\lambda = (x + y)\varepsilon_1 + y\varepsilon_2 = \begin{pmatrix} x + y \\ y \end{pmatrix}
\] and

\[
\lambda + \rho = (x + y + 2)\varepsilon_1 + (y + 1)\varepsilon_2 = \begin{pmatrix} x + y + 2 \\ y + 1 \end{pmatrix}
\]

(2.44) and (2.45)

Using (2.44) and (2.45) we compute
\[
\begin{align*}
[1 \ 0] (\lambda + \rho) - \rho &= (x + y) \alpha_1 + \left(\frac{x}{2} + y\right) \alpha_2, \\
[-1 \ 0] (\lambda + \rho) - \rho &= (-x - y - 4) \alpha_1 + \left(-\frac{x}{2} - 2\right) \alpha_2, \\
[1 \ 0] (\lambda + \rho) - \rho &= (x + y) \alpha_1 + \left(\frac{x}{2} - 1\right) \alpha_2, \\
[-1 \ 0] (\lambda + \rho) - \rho &= (-x - y - 4) \alpha_1 + \left(-\frac{x}{2} - y - 3\right) \alpha_2, \\
[0 \ 1] (\lambda + \rho) - \rho &= (y - 1) \alpha_1 + \left(\frac{x}{2} + y\right) \alpha_2, \\
[1 \ 0] (\lambda + \rho) - \rho &= (y - 3) \alpha_1 + \left(\frac{x}{2} - 1\right) \alpha_2, \\
[0 \ -1] (\lambda + \rho) - \rho &= (y - 1) \alpha_1 + \left(-\frac{x}{2} - 2\right) \alpha_2, \\
[-1 \ 0] (\lambda + \rho) - \rho &= (-y - 3) \alpha_1 + \left(-\frac{x}{2} - y - 3\right) \alpha_2.
\end{align*}
\]

**Proposition 2.4.1.** Let $\lambda \in P(\mathfrak{sp}_4)$. If $\lambda \notin Q$, then $\mathcal{A}(\lambda, 0) = \emptyset$.

**Proof.** Suppose $\lambda = x \varpi_1 + y \varpi_2 \in P(\mathfrak{sp}_4)$, but $\lambda \notin Q$. Then by Lemma 2.4.1 we have that $x, y \in \mathbb{Z}$ and $2 \nmid x$. Then notice by (2.46)-(2.53), the coefficient of $\alpha_2$ in $\sigma(\lambda + \rho) - \rho$ is not an integer, for any $\sigma \in \mathcal{W}$. Hence $\mathcal{A}(\lambda, 0) = \emptyset$. \hfill \Box

Thus we need only compute $\mathcal{A}(\lambda, 0)$ for $\lambda \in Q$. We now state the main theorem of this section.

**Theorem 2.4.1.** If $\lambda = x \varpi_1 + y \varpi_2$ for some $x, y \in \mathbb{Z}$ with $2 \nmid x$, then
Proof. Let $\lambda = x\varpi_1 + y\varpi_2$ for some $x, y \in \mathbb{Z}$ with $2|x$. By (2.46)-(2.53) we have
that

\[
\begin{align*}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \geq -x \text{ and } y \geq -\frac{x}{2}, \quad (2.54) \\
\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \leq -x - 4 \text{ and } x \leq -4, \quad (2.55) \\
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \geq -x \text{ and } x \geq 2, \quad (2.56) \\
\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \leq -x - 4 \text{ and } y \leq -\frac{x}{2} - 3, \quad (2.57) \\
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \geq 1 \text{ and } y \geq -\frac{x}{2}, \quad (2.58) \\
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \leq -3 \text{ and } x \geq 2, \quad (2.59) \\
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \geq 1 \text{ and } x \leq -4, \quad (2.60) \\
\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} & \in \mathcal{A}(\lambda, 0) \iff y \leq -3 \text{ and } y \leq -\frac{x}{2} - 3. \quad (2.61)
\end{align*}
\]

In Figure 2.13, we graph the solution set to each pair of linear inequalities in (2.54)-(2.61) by placing a solid circle on all \( \lambda \in P(\mathfrak{sp}_4) \), for which \( \sigma \in \mathcal{A}(\lambda, 0) \). We graph each solution set on the weight lattice of \( \mathfrak{sp}_4 \), which we superimpose on a plane defined by the \( \mathbb{R} \)-span of the fundamental weights. The bolded axes correspond to the \( \mathbb{R} \)-span of the fundamental weights, with placement of simple roots and fundamental weights as in Figure 2.10. The theorem then follows from the intersection of the solution sets to the inequalities (2.54)-(2.61). \( \square \)
(a) $\lambda \in P(\mathfrak{sp}_4)$ such that 
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{A}(\lambda, 0)$

(b) $\lambda \in P(\mathfrak{sp}_4)$ such that 
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{A}(\lambda, 0)$

(c) $\lambda \in P(\mathfrak{sp}_4)$ such that 
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathcal{A}(\lambda, 0)$

(d) $\lambda \in P(\mathfrak{sp}_4)$ such that 
$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{A}(\lambda, 0)$

(e) $\lambda \in P(\mathfrak{sp}_4)$ such that 
$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathcal{A}(\lambda, 0)$

(f) $\lambda \in P(\mathfrak{sp}_4)$ such that 
$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in \mathcal{A}(\lambda, 0)$

(g) $\lambda \in P(\mathfrak{sp}_4)$ such that 
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathcal{A}(\lambda, 0)$

(h) $\lambda \in P(\mathfrak{sp}_4)$ such that 
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathcal{A}(\lambda, 0)$

Figure 2.11: Graphs of solution sets to linear inequalities associated to $\mathfrak{sp}_4(\mathbb{C})$
2.5 The case of $\mathfrak{so}_4(\mathbb{C})$

Let $\mathfrak{so}_4(\mathbb{C}) = \{X \in M_4(\mathbb{C}) : X^t = -X\}$ and let $\mathfrak{h} = \{ \text{diag}[a_1, a_2, -a_2, -a_1] : a_1, a_2 \in \mathbb{C}\}$ be a fixed choice of Cartan subalgebra. For $1 \leq i \leq 2$, define the linear functionals $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(H) = a_i$, for any $H = \text{diag}[a_1, a_2, -a_2, -a_1] \in \mathfrak{h}$.

Let $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_1 + \varepsilon_2$. Let $\Delta = \{\alpha_1, \alpha_2\}$, then $\Phi^+ = \Delta$. Let $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ denote the root lattice and let $Q^+ = N\alpha_1 \oplus N\alpha_2$. The fundamental weights are defined by

\[
\varpi_1 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2) = \frac{1}{2}\alpha_1 \quad \text{and} \quad (2.62)
\]
\[
\varpi_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) = \frac{1}{2}\alpha_2. \quad (2.63)
\]

Observe that $\rho = \varpi_1 + \varpi_2 = \varepsilon_1 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$. The Weyl group, $W$, of $\mathfrak{so}_4$ acts on $\mathfrak{h}^*$ by permutations of $\varepsilon_1$ and $\varepsilon_2$ with an even number of sign changes.

![Figure 2.12: Roots and dominant weights of $\mathfrak{so}_4(\mathbb{C})$](image)

The roots (indicated by $\circ$), the fundamental weights and some dominant weights (indicated by $\bullet$) of $\mathfrak{so}_4$ are shown in Figure 2.12. Observe that the two simple roots, $\alpha_1$ and $\alpha_2$, have the same length and that the angle between them is $90^\circ$. The set
of dominant weights is contained in the positive Weyl chamber, a cone of opening $90^\circ$. The action of the Weyl group is generated by reflections across the dashed lines, which are the walls of the positive Weyl chamber and are the $\mathbb{R}_{\geq 0}$-span of the fundamental weights. Notice that both simple roots of $\mathfrak{so}_4$ are dominant.

**Remark 2.5.1.** The root system and the weight lattice of $\mathfrak{so}_4$ are the product of two copies of $\mathfrak{sl}_2$, which corresponds to the isomorphism $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

**Lemma 2.5.1.** Let $\lambda = x\varpi_1 + y\varpi_2$ with $x, y \in \mathbb{Z}$. Then $\lambda \in Q$ if and only if $2|x$ and $2|y$.

**Proof.** By (2.62) and (2.63) observe that $\lambda = x\varpi_1 + y\varpi_2 = \frac{x}{2}\alpha_1 + \frac{y}{2}\alpha_2$, which is in $Q$ if and only if $2|x$ and $2|y$. \qed

We now give the following simplifications. Let $\lambda \in P(\mathfrak{so}_4)$, hence $\lambda = x\varpi_1 + y\varpi_2$ for some $x, y \in \mathbb{Z}$. By (2.62) and (2.63) we can write $\lambda$ and $\lambda + \rho$ as a column vector whose entry in the $i^{th}$-row is given by the coefficient of $\varepsilon_i$. That is

$$\lambda = \frac{1}{2}(x+y)\varepsilon_1 + \frac{1}{2}(y-x)\varepsilon_2 = \frac{1}{2}\begin{pmatrix} x+y \\ y-x \end{pmatrix}$$

and

$$\lambda + \rho = \frac{1}{2}(x+y+2)\varepsilon_1 + \frac{1}{2}(y-x)\varepsilon_2 = \frac{1}{2}\begin{pmatrix} x+y+2 \\ y-x \end{pmatrix}.$$  

Using (2.64) and (2.65) we compute

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\lambda + \rho) - \rho = \begin{pmatrix} x \\ y \end{pmatrix} \alpha_1 + \begin{pmatrix} y \\ -2 \end{pmatrix} \alpha_2,$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} (\lambda + \rho) - \rho = \begin{pmatrix} -x-2 \\ -2 \end{pmatrix} \alpha_1 + \begin{pmatrix} -y-2 \\ -2 \end{pmatrix} \alpha_2,$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (\lambda + \rho) - \rho = \begin{pmatrix} -x+2 \\ 2 \end{pmatrix} \alpha_1 + \begin{pmatrix} y \\ -2 \end{pmatrix} \alpha_2,$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} (\lambda + \rho) - \rho = \begin{pmatrix} -x-2 \\ -2 \end{pmatrix} \alpha_1 + \begin{pmatrix} y \\ -2 \end{pmatrix} \alpha_2.$$

**Proposition 2.5.1.** Let $\lambda \in P(\mathfrak{so}_4)$. If $\lambda \notin Q$, then $\mathcal{A}(\lambda, 0) = \emptyset$. 


Proof. Suppose $\lambda = x\varpi_1 + y\varpi_2 \in P(\mathfrak{so}_4)$, but $\lambda \notin Q$. Then by Lemma 2.5.1 we have that $x, y \in \mathbb{Z}$ with either $2 \mid x$ or $2 \mid y$. Then notice by (2.66)-(2.69), if $2 \mid x$, then the coefficient of $\alpha_1$ in $\sigma(\lambda + \rho) - \rho$ is not an integer, for any $\sigma \in W$. Similarly by (2.66)-(2.69), if $2 \mid y$, then the coefficient of $\alpha_2$ in $\sigma(\lambda + \rho) - \rho$ is not an integer, for any $\sigma \in W$. Hence $A(\lambda, 0) = \emptyset$. 

Thus we need only compute $A(\lambda, 0)$ for $\lambda \in Q$. We now state the main theorem of this section.

**Theorem 2.5.1.** If $\lambda = x\varpi_1 + y\varpi_2$ for some $x, y \in \mathbb{Z}$ with $2 \mid x$ and $2 \mid y$, then

$$A(\lambda, 0) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } x \leq -2 \text{ and } y \leq -2 \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{if } x \leq -2 \text{ and } y \geq 0 \\ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} & \text{if } x \geq 0 \text{ and } y \leq -2. \end{cases}$$

Proof. Let $\lambda = x\varpi_1 + y\varpi_2$ for some $x, y \in \mathbb{Z}$ with $2 \mid x$ and $2 \mid y$. By (2.66)-(2.69) we have that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in A(\lambda, 0) \iff \frac{x}{2} \geq 0 \text{ and } \frac{y}{2} \geq 0,$$

$$(2.70)$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in A(\lambda, 0) \iff \frac{-x - 2}{2} \geq 0 \text{ and } \frac{-y - 2}{2} \geq 0,$$

$$(2.71)$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in A(\lambda, 0) \iff \frac{-x - 2}{2} \geq 0 \text{ and } \frac{y}{2} \geq 0,$$

$$(2.72)$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in A(\lambda, 0) \iff \frac{x}{2} \geq 0 \text{ and } \frac{-y - 2}{2} \geq 0.$$

$$(2.73)$$

In Figure 2.13, we graph the solution set to each pair of linear inequalities in (2.70)-(2.73) by placing a solid circle on all $\lambda \in P(\mathfrak{so}_4)$, for which $\sigma \in A(\lambda, 0)$. We graph each solution set on the weight lattice of $\mathfrak{so}_4$, which we superimpose on a plane defined by the $\mathbb{R}$-span of the fundamental weights. The bolded axes correspond to the $\mathbb{R}$-span of the fundamental weights, with placement of simple roots and fundamental weights as in Figure 2.12. The theorem then follows from the intersection of the solution sets to the inequalities (2.70)-(2.73). 

(a) $\lambda \in P(so_4)$ such that $
abla \in A(\lambda,0)$

(b) $\lambda \in P(so_4)$ such that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in A(\lambda,0)$

(c) $\lambda \in P(so_4)$ such that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in A(\lambda,0)$

(d) $\lambda \in P(so_4)$ such that $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in A(\lambda,0)$

Figure 2.13: Graphs of solution sets to linear inequalities associated to $so_4(\mathbb{C})$
2.6 The case of $G_2$

As described in [11], the underlying vector space of the exceptional Lie algebra $G_2$ is $V = \{ v \in \mathbb{R}^3 | (v, e_1 + e_2 + e_3) = 0 \}$ and the root system is given by $\Phi = \{ \pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3), \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2) \}$, where $\{ e_i : 1 \leq i \leq 3 \}$ is the standard orthonormal basis of $\mathbb{R}^3$. The set of simple roots is $\Delta = \{ \alpha_1, \alpha_2 \}$, where

$$\alpha_1 = e_1 - e_2$$
$$\alpha_2 = -2e_1 + e_2 + e_3.$$  

Then the set of positive roots, in terms of the simple roots, is given by

$$\Phi^+ = \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \}.$$  

The fundamental weights, in terms of the simple roots, are

$$\varpi_1 = 2\alpha_1 + \alpha_2$$ (2.74)
$$\varpi_2 = 3\alpha_1 + 2\alpha_2.$$ (2.75)

Then $\rho = \varpi_1 + \varpi_2 = 5\alpha_1 + 2\alpha_2$. Let $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ be the root lattice and let $Q^+ = \mathbb{N}\alpha_1 \oplus \mathbb{N}\alpha_2$. Observe that for a Lie algebra of type $G_2$ we have that $P(g) = Q$. This follows from (2.74), (2.75), and the fact that $\alpha_1 = 2\varpi_1 - \varpi_2$ and $\alpha_2 = -3\varpi_1 + 2\varpi_2$.

The roots (indicated by $\circ$), the fundamental weights and some dominant weights (indicated by $\bullet$) of $G_2$ are shown in Figure 2.14. Observe that the two simple roots have distinct lengths, $\alpha_2$ is longer than $\alpha_1$. Also notice that the angle between them is $150^\circ$. The set of dominant weights is contained in the positive Weyl chamber, a cone of opening $30^\circ$. The action of the Weyl group is generated by reflections across the dashed lines, which are the walls of the positive Weyl chamber and are the $\mathbb{R}_{\geq 0}$-span of the fundamental weights. Notice that the roots $2\alpha_1 + \alpha_2 = \varpi_1$ and $3\alpha_1 + 2\alpha_2 = \varpi_2$ are dominant.

We want to give a complete description of the Weyl alternation sets, $A(\lambda, 0)$, in the case when $\lambda$ is an integral weight of $G_2$. We remark again that we only compute
Figure 2.14: Roots and dominant weights of $G_2$

The multiplicities of the zero weight in the case when $\lambda$ is a dominant integral weight, since these correspond to finite dimensional irreducible representations.

The Weyl group, $W$, is generated by the simple root reflections $s_1 := s_{\alpha_1}$ and $s_2 := s_{\alpha_2}$, and is isomorphic to $D_6$, the dihedral group of order 6. We describe the elements of $W$ in terms of their action on simple roots.

1: $\alpha_1 \mapsto \alpha_1$
2: $\alpha_2 \mapsto \alpha_2$

$s_1 : \alpha_1 \mapsto -\alpha_1$
3: $\alpha_2 \mapsto 3\alpha_1 + \alpha_2$

$s_2 : \alpha_1 \mapsto \alpha_1 + \alpha_2$
4: $\alpha_2 \mapsto -\alpha_2$

$s_2 s_1 : \alpha_1 \mapsto -(\alpha_1 + \alpha_2)$
5: $\alpha_2 \mapsto 3\alpha_1 + 2\alpha_2$

$(s_2 s_1)^2 : \alpha_1 \mapsto -(2\alpha_1 + \alpha_2)$
6: $\alpha_2 \mapsto 3\alpha_1 + \alpha_2$

$(s_2 s_1)^3 : \alpha_1 \mapsto -\alpha_1$
7: $\alpha_2 \mapsto -\alpha_2$

$(s_2 s_1)^4 : \alpha_1 \mapsto \alpha_1 + \alpha_2$
8: $\alpha_2 \mapsto -3\alpha_1 + 2\alpha_2$

$(s_2 s_1)^5 : \alpha_1 \mapsto 2\alpha_1 + \alpha_2$
9: $\alpha_2 \mapsto -(3\alpha_1 + \alpha_2)$

$(s_2 s_1)^6 : \alpha_1 \mapsto -3\alpha_1 + 2\alpha_2$
10: $\alpha_2 \mapsto -(3\alpha_1 + \alpha_2)$
Let $\lambda = x \alpha_1 + y \alpha_2$ for some $x, y \in \mathbb{Z}$, then we can give the following simplifications:

\[
\begin{align*}
s_1(s_2s_1) : \alpha_1 &\mapsto -(2\alpha_1 + \alpha_2) & s_1(s_2s_1)^2 : \alpha_1 &\mapsto -(\alpha_1 + \alpha_2) \\
\alpha_2 &\mapsto 3\alpha_1 + 2\alpha_2 & \alpha_2 &\mapsto \alpha_2 \\
s_1(s_2s_1)^3 : \alpha_1 &\mapsto \alpha_1 & s_1(s_2s_1)^4 : \alpha_1 &\mapsto 2\alpha_1 + \alpha_2 \\
\alpha_2 &\mapsto -(3\alpha_1 + \alpha_2) & \alpha_2 &\mapsto -(3\alpha_1 + 2\alpha_2)
\end{align*}
\]

We now state the main theorem of this section.

**Theorem 2.6.1.** Let $\lambda = x \alpha_1 + y \alpha_2$, with $x, y \in \mathbb{Z}$. Then

1. $\mathcal{A}(\lambda, 0) = \{1\}$ if $x = 0, y = 0$.

2. $\mathcal{A}(\lambda, 0) = \{s_1\}$ if $x = -1, y = 0$.

3. $\mathcal{A}(\lambda, 0) = \{s_2\}$ if $x = 0, y = -1$.

4. $\mathcal{A}(\lambda, 0) = \{s_2s_1\}$ if $x = -4, y = -1$. 
5. \( A(\lambda, 0) = \{(s_2s_1)^2\} \) if \( x = -9, y = -4 \).

6. \( A(\lambda, 0) = \{(s_2s_1)^3\} \) if \( x = -10, y = -6 \).

7. \( A(\lambda, 0) = \{(s_2s_1)^4\} \) if \( x = -6, y = -5 \).

8. \( A(\lambda, 0) = \{(s_2s_1)^5\} \) if \( x = -1, y = -2 \).

9. \( A(\lambda, 0) = \{s_1(s_2s_1)\} \) if \( x = -6, y = -2 \).

10. \( A(\lambda, 0) = \{s_1(s_2s_1)^2\} \) if \( x = -10, y = -5 \).

11. \( A(\lambda, 0) = \{s_1(s_2s_1)^3\} \) if \( x = -9, y = -6 \).

12. \( A(\lambda, 0) = \{s_1(s_2s_1)^4\} \) if \( x = -4, y = -4 \).

13. \( A(\lambda, 0) = \{1, s_2\} \) if \( x = 1, y = 0 \) or \( x = 3, y = 1 \).

14. \( A(\lambda, 0) = \{1, s_1\} \) if \( x = 1, y = 1 \).

15. \( A(\lambda, 0) = \{s_1, s_2s_1\} \) if \( x = -2, y = 0 \) or \( x = -1, y = 1 \).

16. \( A(\lambda, 0) = \{s_2s_1, s_1(s_2s_1)\} \) if \( x = -5, y = -1 \).

17. \( A(\lambda, 0) = \{s_1(s_2s_1), (s_2s_1)^2\}, \) if \( x = -8, y = -3 \) or \( x = -9, y = -3 \).

18. \( A(\lambda, 0) = \{(s_2s_1)^2, s_1(s_2s_1)^2\} \) if \( x = -11, y = -5 \).

19. \( A(\lambda, 0) = \{s_1(s_2s_1)^2, (s_2s_1)^3\} \) if \( x = -11, y = -6 \) or \( x = -13, y = -7 \).

20. \( A(\lambda, 0) = \{(s_2s_1)^3, s_1(s_2s_1)^3\} \) if \( x = -11, y = -7 \).

21. \( A(\lambda, 0) = \{s_1(s_2s_1)^3, (s_2s_1)^4\} \) if \( x = -8, y = -6 \) or \( x = -9, y = -7 \).

22. \( A(\lambda, 0) = \{(s_2s_1)^4, s_1(s_2s_1)^4\} \) if \( x = -5, y = -5 \).

23. \( A(\lambda, 0) = \{s_1(s_2s_1)^4, (s_2s_1)^5\} \) if \( x = -2, y = -3 \) or \( x = -1, y = -3 \).

24. \( A(\lambda, 0) = \{(s_2s_1)^5, s_2\} \) if \( x = 1, y = -1 \).
25. \( A(\lambda, 0) = \{1, s_1, s_2\} \) if \( \frac{1}{3}x + \frac{1}{3} \leq y \leq x - 1 \) and \( \frac{2}{3}x - \frac{3}{4} < y < \frac{1}{2}x + 1 \).

26. \( A(\lambda, 0) = \{1, s_1, s_2s_1\} \) if \( x \geq 0, y \geq \frac{1}{2}x + 1, x - 1 < y < \frac{2}{3}x + 2 \).

27. \( A(\lambda, 0) = \{s_1, s_2s_1, s_1(s_2s_1)\} \) if \( x < 0, y \geq 0, \frac{2}{3}x + 2 \leq y < x + 5 \).

28. \( A(\lambda, 0) = \{s_2s_1, (s_2s_1)^2, s_1(s_2s_1)\} \) if \( x > -10, y < 0, y \geq x + 5, y \geq \frac{1}{3}x + \frac{1}{3} \).

29. \( A(\lambda, 0) = \{(s_2s_1)^2, s_1(s_2s_1), s_1(s_2s_1)^2\} \) if \( x \leq -10, y > -6, \frac{1}{2}x + 1 \leq y < \frac{1}{3}x + \frac{1}{3} \).

30. \( A(\lambda, 0) = \{(s_2s_1)^2, (s_2s_1)^3, s_1(s_2s_1)^2\} \) if \( y \leq -6, y \geq \frac{2}{3}x + 2, \frac{1}{3}x - 3 < y < \frac{1}{3}x + 1 \).

31. \( A(\lambda, 0) = \{(s_2s_1)^3, s_1(s_2s_1)^2, s_1(s_2s_1)^3\} \) if \( x + 5 \leq y \leq \frac{1}{3}x - 3, \frac{1}{2}x - 2 < y < \frac{2}{3}x + 2 \).

32. \( A(\lambda, 0) = \{(s_2s_1)^3, (s_2s_1)^4, s_1(s_2s_1)^3\} \) if \( x \leq -10, y \leq \frac{1}{2}x - 2, \frac{2}{3}x - \frac{4}{3} < y < x + 5 \).

33. \( A(\lambda, 0) = \{(s_2s_1)^4, s_1(s_2s_1)^3, s_1(s_2s_1)^4\} \) if \( x > -10, y \leq -6, y \leq \frac{2}{3}x - \frac{4}{3}, y < -x - 1 \).

34. \( A(\lambda, 0) = \{(s_2s_1)^4, (s_2s_1)^5, s_1(s_2s_1)^4\} \) if \( x < 0, y > -6, y \leq \frac{1}{3}x - 3, y \leq x - 1 \).

35. \( A(\lambda, 0) = \{s_2, (s_2s_1)^5, s_1(s_2s_1)^4\} \) if \( x \geq 0, y \leq \frac{1}{2}x - 2, \frac{1}{3}x - 3 < y < 0 \).

36. \( A(\lambda, 0) = \{1, s_2, (s_2s_1)^5\} \) if \( 0 \leq y \leq \frac{2}{3}x - \frac{4}{3}, \frac{1}{2}x - 2 < y < \frac{1}{3}x + \frac{1}{3} \).

37. \( A(\lambda, 0) = \{1, s_1, s_2, (s_2s_1)^5\} \) if \( \frac{1}{3}x + \frac{1}{3} \leq y \leq \frac{2}{3}x - \frac{4}{3}, \frac{1}{2}x - 2 < y < \frac{1}{2}x + 1 \).

38. \( A(\lambda, 0) = \{1, s_1, s_2, s_2s_1\} \) if \( \frac{1}{2}x + 1 \leq y \leq x - 1, \frac{2}{3}x - \frac{4}{3} < y < \frac{2}{3}x + 2 \).

39. \( A(\lambda, 0) = \{1, s_1, s_2s_1, s_1(s_2s_1)\} \) if \( x \geq 0, y \geq \frac{2}{3}x + 2, x - 1 < y < x + 5 \).

40. \( A(\lambda, 0) = \{s_1, s_2s_1, (s_2s_1)^2, s_1(s_2s_1)\} \) if \( -10 < x < 0, y \geq 0, y \geq x + 5 \).

41. \( A(\lambda, 0) = \{s_2s_1, (s_2s_1)^2, s_1(s_2s_1), s_1(s_2s_1)^2\} \) if \( x \leq -10, y \leq \frac{1}{3}x + \frac{1}{3}, -6 < y < 0 \).
42. $A(\lambda, 0) = \{(s_2 s_1)^2, (s_2 s_1)^3, s_1(s_2 s_1), s_1(s_2 s_1)^2\}$ if $\frac{1}{2} x + 1 \leq y \leq -6$, $\frac{1}{3} x - 3 < y < \frac{1}{3} x + \frac{1}{3}$.

43. $A(\lambda, 0) = \{(s_2 s_1)^2, (s_2 s_1)^3, s_1(s_2 s_1)^2, s_1(s_2 s_1)^3\}$ if $\frac{2}{3} x + 2 \leq y \leq \frac{1}{3} x - 3$, $\frac{1}{2} x - 2 < y < \frac{1}{2} x + 1$.

44. $A(\lambda, 0) = \{(s_2 s_1)^3, (s_2 s_1)^4, s_1(s_2 s_1)^2, s_1(s_2 s_1)^3\}$ if $x + 5 \leq y \leq \frac{1}{2} x - 2$, $\frac{2}{3} x - \frac{4}{3} < y < \frac{2}{3} + 2$.

45. $A(\lambda, 0) = \{(s_2 s_1)^3, (s_2 s_1)^4, s_1(s_2 s_1)^3, s_1(s_2 s_1)^4\}$ if $x \leq -10$, $y \leq \frac{2}{3} x - \frac{4}{3} x - 1 < y < x + 5$.

46. $A(\lambda, 0) = \{(s_2 s_1)^4, (s_2 s_1)^5, s_1(s_2 s_1)^3, s_1(s_2 s_1)^4\}$ if $-10 < x < 0$, $y \leq -6$, $y \leq x - 1$.

47. $A(\lambda, 0) = \{s_2, (s_2 s_1)^4, (s_2 s_1)^5, s_1(s_2 s_1)^4\}$ if $x \geq 0$, $y \leq \frac{1}{3} x - 3$, $-6 < y < 0$.

48. $A(\lambda, 0) = \{1, s_2, (s_2 s_1)^5, s_1(s_2 s_1)^4\}$ if $0 \leq y \leq \frac{1}{2} x - 2$, $\frac{1}{3} x - 3 < y < \frac{2}{3} x + \frac{1}{3}$.

49. $A(\lambda, 0) = \{1, s_1, s_2, s_2 s_1, (s_2 s_1)^5\}$ if $\frac{1}{2} x + 1 \leq y \leq \frac{2}{3} x - \frac{4}{3}$.

50. $A(\lambda, 0) = \{1, s_1, s_2, s_2 s_1, s_1(s_2 s_1)\}$ if $\frac{2}{3} x + 2 \leq y \leq x - 1$.

51. $A(\lambda, 0) = \{1, s_1, s_2, s_1, (s_2 s_1)^2\}$ if $x \geq 0$, $y \geq x + 5$.

52. $A(\lambda, 0) = \{s_1, s_2 s_1, (s_2 s_1)^2, s_1(s_2 s_1), s_1(s_2 s_1)^2\}$ if $x \leq -10$, $y \geq 0$.

53. $A(\lambda, 0) = \{s_2 s_1, (s_2 s_1)^2, (s_2 s_1)^3, s_1(s_2 s_1), s_1(s_2 s_1)^2\}$ if $\frac{1}{3} x + \frac{1}{3} \leq y \leq -6$.

54. $A(\lambda, 0) = \{(s_2 s_1)^2, (s_2 s_1)^3, s_1(s_2 s_1), s_1(s_2 s_1)^2, s_1(s_2 s_1)^3\}$ if $\frac{1}{2} x + 1 \leq y \leq \frac{1}{2} x - 3$.

55. $A(\lambda, 0) = \{(s_2 s_1)^2, (s_2 s_1)^3, (s_2 s_1)^4, s_1(s_2 s_1)^2, s_1(s_2 s_1)^3\}$ if $\frac{2}{3} x + 2 \leq y \leq \frac{1}{2} x - 2$.

56. $A(\lambda, 0) = \{(s_2 s_1)^3, (s_2 s_1)^4, s_1(s_2 s_1)^2, s_1(s_2 s_1)^3, s_1(s_2 s_1)^4\}$ if $x + 5 \leq y \leq \frac{2}{3} x - \frac{4}{3}$.

57. $A(\lambda, 0) = \{(s_2 s_1)^3, (s_2 s_1)^4, (s_2 s_1)^5, s_1(s_2 s_1)^3, s_1(s_2 s_1)^4\}$ if $x \leq -10$, $y \leq x - 1$.

58. $A(\lambda, 0) = \{s_2, (s_2 s_1)^4, (s_2 s_1)^5, s_1(s_2 s_1)^3, s_1(s_2 s_1)^4\}$ if $x \geq 0$, $y \leq -6$. 
59. \( A(\lambda, 0) = \{1, s_2, (s_2s_1)^4, (s_2s_1)^5, s_1(s_2s_1)^4\} \) if \( 0 \leq y \leq \frac{1}{3}x - 3 \).

60. \( A(\lambda, 0) = \{1, s_1, s_2, (s_2s_1)^5, s_1(s_2s_1)^4\} \) if \( \frac{1}{3}x + \frac{1}{3} \leq y \leq \frac{1}{2}x - 2 \).

61. \( A(\lambda, 0) = \emptyset \) if \( x, y \) do not satisfy any of the above statements.

Proof. Let \( \lambda = x\alpha_1 + y\alpha_2 \) with \( x, y \in \mathbb{Z} \). By (2.76)-(2.87) we have that

\[
1 \in A(\lambda, 0) \iff x \geq 0, y \geq 0 \tag{2.88}
\]
\[
s_1 \in A(\lambda, 0) \iff y \geq \frac{1}{3}x + \frac{1}{3}, y \geq 0 \tag{2.89}
\]
\[
s_2 \in A(\lambda, 0) \iff x \geq 0, y \leq x - 1 \tag{2.90}
\]
\[
s_2s_1 \in A(\lambda, 0) \iff y \geq \frac{1}{3}x + \frac{1}{3}, y \geq \frac{1}{2}x + 1 \tag{2.91}
\]
\[
(s_2s_1)^2 \in A(\lambda, 0) \iff y \geq \frac{2}{3}x + 2, y \geq x + 5 \tag{2.92}
\]
\[
(s_2s_1)^3 \in A(\lambda, 0) \iff x \leq -10, y \leq -6 \tag{2.93}
\]
\[
(s_2s_1)^4 \in A(\lambda, 0) \iff y \leq \frac{1}{3}x - 3, y \leq \frac{1}{2}x - 2 \tag{2.94}
\]
\[
(s_2s_1)^5 \in A(\lambda, 0) \iff y \leq \frac{2}{3}x - \frac{4}{3}, y \leq x - 1 \tag{2.95}
\]
\[
s_1(s_2s_1) \in A(\lambda, 0) \iff y \geq \frac{2}{3}x + 2, y \geq \frac{1}{2}x + 1 \tag{2.96}
\]
\[
s_1(s_2s_1)^2 \in A(\lambda, 0) \iff x \leq -10, y \geq x + 5 \tag{2.97}
\]
\[
s_1(s_2s_1)^3 \in A(\lambda, 0) \iff y \leq \frac{1}{3}x - 3, y \leq -6 \tag{2.98}
\]
\[
s_1(s_2s_1)^4 \in A(\lambda, 0) \iff y \leq \frac{2}{3}x - \frac{4}{3}, y \leq \frac{1}{2}x - 2 \tag{2.99}
\]

In Figure 2.15, we graph the solution set to each pair of linear inequalities in (2.88)-(2.99) by placing a solid circle on all \( \lambda \in P(G_2) \), for which \( \sigma \in A(\lambda, 0) \). We graph each solution set on the weight lattice of \( G_2 \), which we superimpose on a plane defined by the \( \mathbb{R} \)-span of the fundamental weights. The bolded axes correspond to the \( \mathbb{R} \)-span of the fundamental weights, with placement of simple roots and fundamental weights as in Figure 2.14. The theorem then follows from the intersection of the solution sets to the inequalities (2.88)-(2.99). \( \square \)
Figure 2.15: Graphs of solution sets to linear inequalities associated to $G_2$
2.7 Diagrams associated to Weyl alternation sets

For each rank 2 Lie algebra \( g \) we create the zero weight Weyl alternation diagram of \( g \). This is a multi-colored diagram on the weight lattice of \( g \), which we superimpose on a plane defined by the \( \mathbb{R} \)-span of the fundamental weights.

Let \( \mathcal{A} := \{ A(\lambda,0) : \lambda \in P(g) \} \backslash \{ \emptyset \} \). Choose \( |\mathcal{A}| \) distinct colors and assign a distinct color to each of the non-empty Weyl alternation sets in \( \mathcal{A} \). Then to all integral weights with the same non-empty Weyl alternation set, place a small solid circle of the color assigned to that particular Weyl alternation set.

Applying the above procedure to the Weyl alternation sets in Theorems 2.2.1, 2.3.1, 2.4.1, 2.5.1, and 2.6.1 we create the zero weight Weyl alternation diagrams of \( \mathfrak{sl}_3 \), \( \mathfrak{so}_5 \), \( \mathfrak{sp}_4 \), \( \mathfrak{so}_4 \) and \( G_2 \), respectively.

![Figure 2.16: Zero weight Weyl alternation diagram of \( \mathfrak{sl}_3(\mathbb{C}) \)](image)
Figure 2.17: Zero weight Weyl alternation diagram of $\mathfrak{so}_5(\mathbb{C})$
Figure 2.18: Zero weight Weyl alternation diagram of $\mathfrak{sp}_4(\mathbb{C})$
Figure 2.19: Zero weight Weyl alternation diagram of $\mathfrak{so}_4(\mathbb{C})$
Figure 2.20: Zero weight Weyl alternation diagram of $G_2$
Figures 2.21, 2.22, 2.23, 2.24, and 2.25 are Hasse diagrams depicting the containment of the zero weight Weyl alternation sets associated to the Lie algebras $\mathfrak{sl}_3$, $\mathfrak{so}_5$, $\mathfrak{sp}_4$, $\mathfrak{so}_4$ and $G_2$, respectively.

Figure 2.21: Hasse diagram associated to $\mathfrak{sl}_3(\mathbb{C})$
Figure 2.22: Hasse diagram associated to $\mathfrak{so}_5(\mathbb{C})$
Figure 2.23: Hasse diagram associated to $sp_4(\mathbb{C})$
Figure 2.24: Hasse diagram associated to $\mathfrak{so}_4(\mathbb{C})$
Figure 2.25: Hasse diagram associated to $G_2$
In Figure 2.25 the numbers 1 – 12 correspond to the elements of the Weyl group of $G_2$ in the following way:

1 corresponds to 1 (identity),
2 corresponds to $s_1$,
3 corresponds to $s_2 s_1$,
4 corresponds to $s_1 (s_2 s_1)$,
5 corresponds to $(s_2 s_1)^2$,
6 corresponds to $s_1 (s_2 s_1)^2$,
7 corresponds to $(s_2 s_1)^3$,
8 corresponds to $s_1 (s_2 s_1)^3$,
9 corresponds to $(s_2 s_1)^4$,
10 corresponds to $s_1 (s_2 s_1)^4$,
11 corresponds to $(s_2 s_1)^5$, and
12 corresponds to $s_2$.

### 2.8 Convexity of Weyl alternation sets

Let $\mathfrak{g}$ be a simple Lie algebra of rank $r \geq 1$, with $\mathfrak{h}$ a fixed choice of Cartan subalgebra. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ denote the set of simple roots and let $\{\varpi_1, \ldots, \varpi_r\}$ be the set of fundamental weights of $\mathfrak{g}$. Let $Q = \mathbb{Z} \alpha_1 \oplus \cdots \oplus \mathbb{Z} \alpha_r$ be the root lattice of $\mathfrak{g}$, and let $Q^+ = \mathbb{N} \alpha_1 \oplus \cdots \oplus \mathbb{N} \alpha_r$ denote the positive chamber of the root lattice.

**Proposition 2.8.1.** If $\rho = \frac{1}{2} \sum_{\alpha \in \varPhi^+} \alpha$, then $\mathcal{A}(\rho, 0) = \emptyset$.

**Proof.** Observe that for any $\sigma \in W$, we have that $\sigma(-\rho + \rho) - \rho = -\rho \notin Q^+$. Thus $\mathcal{A}(-\rho, 0) = \emptyset$. \hfill $\square$

We now prove a convexity result for Weyl alternation sets.

Fix $S \subset W$ and let $\mathcal{A}(S) = \{(\lambda, \mu) \in P(\mathfrak{g}) \times P(\mathfrak{g}) \mid S \subseteq \mathcal{A}(\lambda, \mu)\}$. 
Theorem 2.8.1. For any non-empty $S \subseteq W$ there exists a convex subset $C \subseteq h^* \times h^*$, such that $C \cap Q = A(S)$.

Proof. Let $\lambda, \mu \in P(g)$. For each $\sigma \in W$ we can write

$$\sigma(\lambda + \rho) - \rho - \mu = \sum_{i=1}^{r} (\varpi_i, \sigma(\lambda + \rho) - \rho - \mu) \alpha_i.$$ 

Notice $\sigma \in A(\lambda, \mu)$ if and only if the following conditions are satisfied:

1. $(\varpi_i, \sigma(\lambda + \rho) - \rho - \mu) \geq 0$, for all $1 \leq i \leq r$, and
2. $(\varpi_i, \sigma(\lambda + \rho) - \rho - \mu) \in \mathbb{N}$, for all $1 \leq i \leq r$.

Observe that the first condition is a convexity condition. For $\sigma \in W$, let

$$C_{\sigma} = \{(\lambda, \mu) \in P(g) \times P(g) \mid (\varpi_i, \sigma(\lambda + \rho) - \rho - \mu) \geq 0, \forall 1 \leq i \leq r\}.$$ 

Notice that for any $\sigma \in W$, $C_{\sigma} \subseteq h^* \times h^*$ is convex since it is an affine half space. Fix a nonempty subset $S \subseteq W$, and let

$$C = \{(\lambda, \mu) \in P(g) \times P(g) \mid (\lambda, \mu) \in C_{\sigma}, \forall \sigma \in S\}.$$ 

Notice that $C = \bigcap_{\sigma \in S} C_{\sigma}$ is a convex subset of $h^* \times h^*$, since the intersection of convex half-spaces is convex. Thus

$$C \cap Q = \{(\lambda, \mu) \in P(g) \times P(g) \mid (\lambda, \mu) \in C_{\sigma}, \forall \sigma \in S\} \cap Q$$

$$= \{(\lambda, \mu) \in P(g) \times P(g) \mid (\varpi_i, \sigma(\lambda + \rho) - \rho - \mu) \in \mathbb{N}, \forall 1 \leq i \leq r, \forall \sigma \in S\}$$

$$= \{(\lambda, \mu) \in P(g) \times P(g) \mid \varphi(\sigma(\lambda + \rho) - \rho - \mu) > 0, \forall \sigma \in S\}$$

$$= \{(\lambda, \mu) \in P(g) \times P(g) \mid S \subseteq A(\lambda, \mu)\}$$

$$= A(S).$$

$\square$
Chapter 3

Kostant’s weight multiplicity formula and the Fibonacci numbers

In this chapter we address the question of how many terms contribute to the multiplicity of the zero weight for certain, very special, highest weights. Specifically, we consider the case where the highest weight is equal to the sum of all simple roots. We prove that this weight is dominant only in Lie types $A$ and $B$, and in these cases we show that the number of contributing terms is a Fibonacci number. We then provide some combinatorial consequences of this fact.

3.1 Lie algebra $\mathfrak{sl}_{r+1}(\mathbb{C})$

Let $r \geq 1$ and let $n = r + 1$. Let $G = SL_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_{r+1} = \mathfrak{sl}_n(\mathbb{C})$, and let

$$\mathfrak{h} = \{diag[a_1, \ldots, a_n]|a_1, \ldots, a_n \in \mathbb{C}, \sum_{i=1}^{n} a_i = 0\}$$

be a fixed choice of Cartan subalgebra. Let $\mathfrak{b}$ denote the set of $n \times n$ upper triangular complex matrices with trace zero. For $1 \leq i \leq n$, define the linear functionals $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(H) = a_i$, for any $H = diag[a_1, \ldots, a_n] \in \mathfrak{h}$. The Weyl group, $W$, is isomorphic to $S_n$, the symmetric group on $n$ letters, and acts on $\mathfrak{h}^*$ by permutations of $\varepsilon_1, \ldots, \varepsilon_n$.

For each $1 \leq i \leq r$, let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then the set of simple and positive roots corresponding to $(\mathfrak{g}, \mathfrak{b})$ are $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ and $\Phi^+ = \{\varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n\}$,
respectively. The highest root is \( \tilde{\alpha} = \varepsilon_1 - \varepsilon_n = \alpha_1 + \cdots + \alpha_r \). The fundamental weights are defined by \( \varpi_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{n}(\varepsilon_1 + \cdots + \varepsilon_n) \), where \( 1 \leq i \leq r \). Then the sets of integral and dominant integral weights are

\[
P(g) = \{ a_1 \varpi_1 + \cdots + a_r \varpi_r \mid a_i \in \mathbb{Z}, \text{ for all } i = 1, \ldots, r \} \quad \text{and} \quad P_+(g) = \{ a_1 \varpi_1 + \cdots + a_r \varpi_r \mid a_i \in \mathbb{N}, \text{ for all } i = 1, \ldots, r \}, \]

respectively.

Let \( (, ) \) be the symmetric bilinear form on \( \mathfrak{h}^* \) corresponding to the trace form as in [5]. Observe that if \( H = \text{diag}[a_1, \ldots, a_n] \in \mathfrak{h} \), then \( (\varepsilon_1 + \cdots + \varepsilon_n)(H) = a_1 + \cdots + a_n = 0 \). Thus, for any \( 1 \leq i \leq r \), we can write \( \varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \).

As a simplification we will write \( \xi \in \mathfrak{h}^* \) as an \( n \)-tuple whose \( i \)-th coordinate is given by \( (\varepsilon_i, \xi) \). Hence, if \( \xi = (\xi_1, \ldots, \xi_n) \in \mathfrak{h}^* \) and \( 1 \leq i \leq r \), then \( (\varpi_i, \xi) = \xi_1 + \cdots + \xi_i \). Therefore

\[
\tilde{\alpha} = (1, 0, 0, \ldots, 0, -1) \\
\rho = (n - 1, n - 2, n - 3, \ldots, 2, 1, 0), \quad \text{and} \\
\tilde{\alpha} + \rho = (n, n - 2, n - 3, \ldots, 2, 1, -1).
\]

It will be useful to relabel \( \tilde{\alpha} + \rho = (a_1, a_2, \ldots, a_n) \), where

\[
a_j = \begin{cases} 
  n & \text{if } j = 1 \\
-1 & \text{if } j = n \\
  n - j & \text{otherwise.} 
\end{cases} \tag{3.1}
\]

Then for any \( \sigma \in W \),

\[
\sigma(\tilde{\alpha} + \rho) - \rho = (a_{\sigma^{-1}(1)} - n + 1, a_{\sigma^{-1}(2)} - n + 2, \ldots, a_{\sigma^{-1}(n-1)} - 1, a_{\sigma^{-1}(n)}). \tag{3.2}
\]

### 3.1.1 The zero weight space of \( \mathfrak{sl}_{r+1}(\mathbb{C}) \)

In this section we present results regarding the Weyl alternation set corresponding to the highest root and the zero weight of \( \mathfrak{sl}_{r+1} \).

**Theorem 3.1.1.** Let \( \sigma \in S_n \). Then \( \sigma \in A(\tilde{\alpha}, 0) \) if and only if \( \sigma(1) = 1 \), \( \sigma(n) = n \), and \( |\sigma(i) - i| \leq 1 \), for all \( 1 \leq i \leq n \).
The following technical propositions will be used in the proof of Theorem 3.1.1.

**Proposition 3.1.1.** Let \( \sigma \in S_n \). Then \( \sigma \) is a product of commuting neighboring transpositions if and only if \( |\sigma(i) - i| \leq 1 \), for all \( 1 \leq i \leq n \).

**Proof.** (\( \Rightarrow \)) Assume \( \sigma \in S_n \) is a product of commuting neighboring transpositions. Hence \( \sigma = (i_1 \ i_1 + 1)(i_2 \ i_2 + 1) \cdots (i_k \ i_k + 1) \), for some nonconsecutive integers \( i_1, i_2, \ldots, i_k \) between 1 and \( n - 1 \). If we let \( A = \{i \mid \sigma(i) \neq i\} \), then

\[
|\sigma(j) - j| = \begin{cases} 
1 & \text{if } j \in A \\
0 & \text{if } j \notin A.
\end{cases}
\]

Thus \( |\sigma(i) - i| \leq 1 \) for all \( i = 1, \ldots, n \).

(\( \Leftarrow \)) We can always write \( \sigma \) as a product of transpositions. Since \( \sigma \in S_n \) with \( |\sigma(i) - i| \leq 1 \), for all \( 1 \leq i \leq n \), we must have that \( \sigma = (i_1 \ i_1 + 1)(i_2 \ i_2 + 1) \cdots (i_k \ i_k + 1) \), for some nonconsecutive integers \( i_1, i_2, \ldots, i_k \) between 1 and \( n - 1 \). Thus \( \sigma \) is a product of commuting neighboring transpositions.

For \( 1 \leq i \leq r \), let \( s_i \) denote the simple root reflection corresponding to \( \alpha_i \in \Delta \). Then \( s_i(\varepsilon_k) = \varepsilon_{\sigma(k)} \), where \( \sigma \) is the neighboring transposition \( (i \ i + 1) \in S_n \).

**Proposition 3.1.2.** Let \( \sigma = s_{i_1}s_{i_2} \cdots s_{i_k} \in W \), where \( i_1, \ldots, i_k \) are nonconsecutive integers satisfying \( 2 \leq i_1 < i_2 < \cdots < i_k \leq r - 1 \). Then \( \sigma(\tilde{\alpha} + \rho) = \rho = (\varepsilon_1 - \varepsilon_{i_1}) + (\varepsilon_{i_1 + 1} - \varepsilon_{i_2}) + \cdots + (\varepsilon_{i_k + 1} - \varepsilon_n) \) is a nonnegative integral combination of positive roots.

**Proof.** We proceed by induction on the length of \( \sigma \). If \( \ell(\sigma) = 0 \), then \( \sigma = 1 \), and \( \sigma(\tilde{\alpha} + \rho) = \rho = \tilde{\alpha} = \varepsilon_1 - \varepsilon_n \), which is a nonnegative integral combination of positive roots. Suppose that for any \( m \leq k \), if \( \sigma = s_{i_1}s_{i_2} \cdots s_{i_m} \), with \( i_1, \ldots, i_m \) nonconsecutive integers satisfying \( 2 \leq i_1 < i_2 < \cdots < i_m \leq r - 1 \), then \( \sigma(\tilde{\alpha} + \rho) - \rho = (\varepsilon_1 - \varepsilon_{i_1}) + (\varepsilon_{i_1 + 1} - \varepsilon_{i_2}) + \cdots + (\varepsilon_{i_m + 1} - \varepsilon_n) \) is a nonnegative integral combination of positive roots.

Let \( \sigma = s_{i_1}s_{i_2} \cdots s_{i_{k+1}} \in W \), where \( i_1, \ldots, i_{k+1} \) are nonconsecutive integers satisfying \( 2 \leq i_1 < i_2 < \cdots < i_{k+1} \leq r - 1 \). Observe that \( \sigma = s_i \pi \), where \( \pi = s_{i_2} \cdots s_{i_{k+1}} \).
with $\ell(\pi) = k$. Then, by induction hypothesis,
\[ \pi(\tilde{\alpha} + \rho) - \rho = (\varepsilon_1 - \varepsilon_{i_2}) + (\varepsilon_{i_2+1} - \varepsilon_{i_3}) + \cdots + (\varepsilon_{i_{k+1}+1} - \varepsilon_n). \]

Notice $s_i(\alpha_j) = \alpha_j$, whenever $i$ and $j$ are nonconsecutive integers. Also, recall that $s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$ and $\rho = \omega_1 + \cdots + \omega_r$. Therefore $s_i(\rho) = \rho - \alpha_i$ and hence
\[ \sigma(\tilde{\alpha} + \rho) - \rho = s_i\pi(\tilde{\alpha} + \rho) - \rho = s_i((\varepsilon_1 - \varepsilon_{i_2}) + (\varepsilon_{i_2+1} - \varepsilon_{i_3}) + \cdots + (\varepsilon_{i_{k+1}+1} - \varepsilon_n) + \rho) - \rho = (\varepsilon_1 - \varepsilon_{i_2}) + (\varepsilon_{i_2+1} - \varepsilon_{i_3}) + \cdots + (\varepsilon_{i_{k+1}+1} - \varepsilon_n) + \rho - \alpha_i - \rho = (\varepsilon_1 - \varepsilon_{i_1}) + (\varepsilon_{i_1+1} - \varepsilon_{i_2}) + \cdots + (\varepsilon_{i_{k+1}+1} - \varepsilon_n). \]

Thus $\sigma(\tilde{\alpha} + \rho) - \rho$ is a nonnegative integral combination of positive roots. \hfill $\Box$

**Proposition 3.1.3.** Let $\sigma \in S_n$ such that $|\sigma(i) - i| \leq 1$, for all $1 \leq i \leq n$. Then $\sigma(1) = 1$ and $\sigma(n) = n$ if and only if for any $1 \leq i \leq r$,
\[
(\omega_i, \sigma(\tilde{\alpha} + \rho) - \rho) = \begin{cases} 
1 & \text{if } \{\sigma(1), \sigma(2), \ldots, \sigma(i)\} = \{1, 2, \ldots, i\} \\
0 & \text{if } \sigma(i) = i + 1.
\end{cases}
\]

**Proof.** ($\Rightarrow$) Assume $\sigma \in S_n$ such that $\sigma(1) = 1$, $\sigma(n) = n$, and $|\sigma(i) - i| \leq 1$, for all $1 \leq i \leq n$. By Proposition 3.1.1, $\sigma$ is a product of commuting neighboring transpositions and hence $\sigma = \sigma^{-1}$. Then $\sigma(\tilde{\alpha} + \rho) - \rho = (1, 2 - \sigma(2), \ldots, (n-1) - \sigma(n-1), -1)$.

We proceed by induction on $i$. If $i = 1$, then $\sigma(1) = 1$ and $(\omega_1, \sigma(\tilde{\alpha} + \rho) - \rho) = 1$.

Now let $1 < i \leq r$ and assume that for any $1 \leq j \leq i - 1$,
\[
(\omega_j, \sigma(\tilde{\alpha} + \rho) - \rho) = \begin{cases} 
1 & \text{if } \{\sigma(1), \sigma(2), \ldots, \sigma(j)\} = \{1, 2, \ldots, j\} \\
0 & \text{if } \sigma(j) = j + 1.
\end{cases}
\]
Suppose that $j = i$. By induction hypothesis, since $j - 1 \leq i - 1$, we have that
\[
(\omega_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = \begin{cases} 
1 & \text{if } \{\sigma(1), \sigma(2), \ldots, \sigma(j-1)\} = \{1, 2, \ldots, j-1\} \\
0 & \text{if } \sigma(j-1) = j.
\end{cases}
\]

Case 1: Assume $(\omega_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = 1$. So $\{\sigma(1), \ldots, \sigma(j-1)\} = \{1, \ldots, j-1\}$ and so $\sigma(j) = j$ or $\sigma(j) = j + 1$. Hence,
\[
(\omega_j, \sigma(\tilde{\alpha} + \rho) - \rho) = 1 + (j - \sigma(j)) = \begin{cases} 
1 & \text{if } \{\sigma(1), \ldots, \sigma(j)\} = \{1, \ldots, j\} \\
0 & \text{if } \sigma(j) = j + 1.
\end{cases}
\]
Case 2: Assume \((\varpi_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = 0\). So \(\sigma(j - 1) = j\). Then observe that

\[
(\varpi_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = (\varpi_{j-2}, \sigma(\tilde{\alpha} + \rho) - \rho) + ((j - 1) - \sigma(j - 1)) = (\varpi_{j-2}, \sigma(\tilde{\alpha} + \rho) - \rho) - 1 = 0.
\]

Hence \((\varpi_{j-2}, \sigma(\tilde{\alpha} + \rho) - \rho) = 1\) and by induction hypothesis we have that 
\(
\{\sigma(1), \sigma(2), \ldots, \sigma(j - 2)\} = \{1, 2, \ldots, j - 2\}. \)
So \(\sigma(j) = j + 1\) or \(\sigma(j) = j - 1\). If \(\sigma(j) = j + 1\), then \(\sigma(k) = j - 1\), for some integer \(k \geq j + 1\). This implies that 
\(|\sigma(k) - k| \geq 2\), a contradiction. Thus \(\sigma(j) = j - 1\), \(\{\sigma(1), \ldots, \sigma(j)\} = \{1, \ldots, j\}\), and \((\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = j - \sigma(j) = 1\).

\((\Leftarrow)\) Let \(\sigma \in S_n\) such that \(|\sigma(i) - i| \leq 1\), for all \(1 \leq i \leq n\). Suppose that

\[
(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = \begin{cases} 
1 & \text{if } \{\sigma(1), \sigma(2), \ldots, \sigma(j)\} = \{1, 2, \ldots, j\} \\
0 & \text{if } \sigma(j) = j + 1,
\end{cases}
\]
holds for any \(1 \leq j \leq r\). Proposition 3.1.1 implies that \(\sigma = \sigma^{-1}\), hence (3.2) simplifies to

\[
\sigma(\tilde{\alpha} + \rho) - \rho = (a_{\sigma(1)} - n + 1, a_{\sigma(2)} - n + 2, \ldots, a_{\sigma(n-1)} - 1, a_{\sigma(n)}),
\]
where \(a_i\) is defined by (3.1). If \(\sigma(1) \neq 1\), then \(\sigma(1) = 2\) and \((\varpi_1, \sigma(\tilde{\alpha} + \rho) - \rho) = -1\), a contradiction. Thus \(\sigma(1) = 1\). If \(\sigma(n) \neq n\), then \(\sigma(n) = n - 1\) and \((\varpi_r, \sigma(\tilde{\alpha} + \rho) - \rho) = -a_{\sigma(n)} = -1\), another contradiction. Thus \(\sigma(n) = n\).

\(\Box\)

Proof of Theorem 3.1.1. \((\Rightarrow)\) Let \(\sigma \in \mathcal{A}(\tilde{\alpha}, 0)\). Hence \(\varphi(\sigma(\tilde{\alpha} + \rho) - \rho) > 0\) and 
\((\varpi_i, \sigma(\tilde{\alpha} + \rho) - \rho) \in \mathbb{N}\), for all \(1 \leq i \leq r\). By (3.2) we have that

\[
\sigma(\tilde{\alpha} + \rho) - \rho = (a_{\sigma^{-1}(1)} - n + 1, a_{\sigma^{-1}(2)} - n + 2, \ldots, a_{\sigma^{-1}(n-1)} - 1, a_{\sigma^{-1}(n)}),
\]
where \(a_i\) is defined by (3.1). We want to prove \(\sigma(1) = 1\), \(\sigma(n) = n\), and \(|i - \sigma(i)| \leq 1\), for all \(1 \leq i \leq n\).

If \(1 < \sigma^{-1}(1) \leq n\), then \((\varpi_1, \sigma(\tilde{\alpha} + \rho) - \rho) = a_{\sigma^{-1}(1)} - n + 1 < 0\), a contradiction.
So $\sigma^{-1}(1) = 1$ and hence $\sigma(1) = 1$. If $1 < \sigma^{-1}(n) < n$, then

$$(\varpi, \sigma(\tilde{\alpha} + \rho) - \rho) = \sum_{i=1}^{n-1}[a_{\sigma^{-1}(i)} - n + i]$$

$$= \sum_{i=1}^{n} a_{\sigma^{-1}(i)} - n(n-1) + \frac{(n-1)n}{2} - a_{\sigma^{-1}(n)}$$

$$= n - 1 + \frac{(n-2)(n-1)}{2} - n(n-1) + \frac{(n-1)n}{2} - a_{\sigma^{-1}(n)}$$

$$= -a_{\sigma^{-1}(n)} = \sigma^{-1}(n) - n < 0$$

a contradiction.

So $\sigma^{-1}(n) = n$ and hence $\sigma(n) = n$.

Hence (3.2) simplifies to

$$\sigma(\tilde{\alpha} + \rho) - \rho = (1, 2 - \sigma^{-1}(2), 3 - \sigma^{-1}(3), \ldots, n - \sigma^{-1}(n-1), -1).$$

Observe that if $|i - \sigma^{-1}(i)| \leq 1$, for all $1 < i < n$, then Proposition 3.1.1 implies $\sigma^{-1} = \sigma$ and thus $|i - \sigma(i)| \leq 1$, for all $1 < i < n$. So it suffices to show that $|i - \sigma^{-1}(i)| \leq 1$, for all $1 < i < n$.

We proceed by induction on $i$. If $i = 2$, then $(\varpi_2, \sigma(\tilde{\alpha} + \rho) - \rho) = 1 + 2 - \sigma^{-1}(2) \geq 0$ if and only if $\sigma^{-1}(2) \leq 3$. Since $\sigma^{-1}(1) = 1$, we have that $\sigma^{-1}(2) = 2$ or $\sigma^{-1}(2) = 3$ and in either case $|2 - \sigma^{-1}(2)| \leq 1$.

Now let $2 \leq i \leq n - 1$ and assume that $|j - \sigma^{-1}(j)| \leq 1$ holds for any $j < i$.

Suppose that $j = i$. Since $j - 1 < i$, we have that $|(j - 1) - \sigma^{-1}(j - 1)| \leq 1$. Thus $\sigma^{-1}(j - 1) = j - 2, j - 1$ or $j$. If $\sigma^{-1}(j - 1) = j - 2$ or $j - 1$, then $\{\sigma^{-1}(1), \ldots, \sigma^{-1}(j - 1)\} = \{1, \ldots, j - 1\}$ and $(\varpi_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = 1$. Hence $(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = 1 + j - \sigma^{-1}(j) \geq 0$ if and only if $\sigma^{-1}(j) \leq j + 1$. Thus $\sigma^{-1}(j) = j$ or $j + 1$ and in either case $|j - \sigma^{-1}(j)| \leq 1$.

Now suppose that $\sigma^{-1}(j - 1) = j$. Since $j - 2 < i$, we have that $|(j - 2) - \sigma^{-1}(j - 2)| \leq 1$. Hence $\{\sigma^{-1}(1), \ldots, \sigma^{-1}(j - 2)\} = \{1, \ldots, j - 2\}$ and $(\varpi_{j-2}, \sigma(\tilde{\alpha} + \rho) - \rho) = 0$. Then $(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = j - \sigma^{-1}(j) \geq 0$ if and only if $\sigma^{-1}(j) \leq j$. Thus $\sigma^{-1}(j) = j - 1$ and $|j - \sigma^{-1}(j)| \leq 1$, which completes our induction step.

$(\leq)$ Let $\sigma \in S_n$ such that $\sigma(1) = 1, \sigma(n) = n,$ and $|i - \sigma(i)| \leq 1$, for all $1 \leq i \leq n$. Then, by Proposition 3.1.3, $(\varpi_i, \sigma(\tilde{\alpha} + \rho) - \rho)$ is either 1 or 0. Hence $\sigma(\tilde{\alpha} + \rho) - \rho$ is a nonnegative integral combination of simple roots, thus $\sigma \in \mathcal{A}(\tilde{\alpha}, 0).$
Remark 3.1.1. The converse of Theorem 3.1.1 also follows from the fact that if \( \sigma \in S_n \) with \( \sigma(1) = 1, \sigma(n) = n, \) and \( |i - \sigma(i)| \leq 1, \) for all \( 1 \leq i \leq n, \) then either \( \sigma = 1 \) or there exist nonconsecutive integers \( i_1, \ldots, i_k \) between 2 and \( r - 1 \) such that \( \sigma = s_{i_1}s_{i_2}\cdots s_{i_k}. \) Then Proposition 3.1.2 implies \( \sigma(\bar{\alpha} + \rho) - \rho \) is a nonnegative integral combination of positive roots and therefore \( \sigma \in A(\bar{\alpha}, 0). \)

Definition 3.1.1. The Fibonacci numbers are the sequence of numbers, \( \{ F_n \}_{n=1}^{\infty}, \) defined by the recurrence relation

\[
F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3, \text{ and } F_1 = F_2 = 1.
\]

Remark 3.1.2. The Fibonacci numbers are well known for their prevalence throughout mathematics. We refer the reader to [13].

Lemma 3.1.1. If \( m \geq 1, \) then \( \{ \sigma \in S_m : |\sigma(i) - i| \leq 1, \text{ for all } 1 \leq i \leq m \} = F_{m+1}. \)

Proof. Let \( m \geq 1, \) and let \( A_m := \{ \sigma \in S_m : |\sigma(i) - i| \leq 1 \forall i = 1, \ldots, m \}. \) We proceed by induction on \( m. \) If \( m = 1, \) then \( |A_1| = |\{ 1 \}| = F_2. \) If \( m = 2, \) then \( |A_2| = |\{ 1, (1 \ 2) \}| = 2 = F_3. \)

Assume that \( |A_k| = F_{k+1} \) holds for all \( k \leq m - 1. \)

Now let \( k = m. \) If \( \sigma \in A_m, \) then \( \sigma(m) = m \) or \( \sigma(m) = m - 1. \) If \( \sigma(m) = m, \) then by induction hypothesis there exist \( F_m \) permutations belonging to \( A_m. \) Now suppose \( \sigma(m) = m - 1. \) Since \( |\sigma(m - 1) - (m - 1)| \leq 1, \) we have that \( \sigma(m - 1) = m \) or \( \sigma(m - 1) = m - 2. \) If \( \sigma(m - 1) = m - 2, \) then \( \exists k \in \mathbb{N} \) with \( 1 \leq k \leq m - 2 \) such that \( \sigma(k) = m. \) In which case \( |\sigma(k) - k| = m - k \geq m - (m - 2) = 2, \) which contradicts \( |\sigma(k) - k| \leq 1. \) Hence \( \sigma(m - 1) = m, \) and by induction hypothesis there exists \( F_{m-1} \) permutations belonging to \( A_m. \) Thus \( |A_m| = F_m + F_{m-1} = F_{m+1}. \)

We can now give a proof of the main result of this section:

Theorem 3.1.2. If \( r \geq 1 \) and \( \bar{\alpha} \) is the highest root of \( \mathfrak{sl}_{r+1}, \) then \( |A(\bar{\alpha}, 0)| = F_r. \)

Proof. By Theorem 3.1.1 we know that

\[
A(\bar{\alpha}, 0) = \{ \sigma \in S_{r+1} : \sigma(1) = 1, \sigma(r + 1) = r + 1, \text{ and } |\sigma(i) - i| \leq 1, \forall 1 \leq i \leq r + 1 \}.
\]
Notice that the sets $\mathcal{A}(\tilde{\alpha}, 0)$ and $\{\sigma \in S_{r-1} : |\sigma(i) - i| \leq 1, \text{ for all } 1 \leq i \leq r - 1\}$ have the same cardinality. Therefore, by Lemma 3.1.1, $|\mathcal{A}(\tilde{\alpha}, 0)| = F_r$. 

Notice $s_is_j = s_js_i$ if and only if $i$ and $j$ are nonconsecutive integers between 1 and $r$. Hence we may describe the elements of $\mathcal{A}(\tilde{\alpha}, 0)$ as products of commuting simple root reflections.

**Corollary 3.1.1.** Let $\sigma \in S_n$. Then $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$ if and only if $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$, for some nonconsecutive integers $2 \leq i_1 < \cdots < i_k \leq r - 1$.

**Proof.** The corollary follows from Theorem 3.1.1, and Proposition 3.1.1. 

**3.1.2 $\mathfrak{sl}_{r+1}(\mathbb{C})$ and a $q$-analog**

The $q$-analog of Kostant’s partition function is the polynomial valued function, $\varphi_q$, defined on $\mathfrak{h}^*$ by

$$\varphi_q(\xi) = c_0 + c_1q + \cdots + c_kq^k,$$

where $c_j$ = number of ways to write $\xi$ as a nonnegative integral sum of exactly $j$ positive roots, for $\xi \in \mathfrak{h}^*$. In [12], Lusztig introduced the $q$-analog of Kostant’s weight multiplicity formula by defining a polynomial, which when evaluated at 1 gives the multiplicity of the dominant weight $\mu$ in the irreducible module $L(\lambda)$. This formula is given by

$$m_q(\lambda, \mu) = \sum_{\sigma \in W} e(\sigma)\varphi_q(\sigma(\lambda + \rho) - (\mu + \rho)).$$

Let $\mathfrak{g}$ be a simple Lie algebra of rank $r$. In the case when $\tilde{\alpha}$ is the highest root of $\mathfrak{g}$, it is a well known result of Kostant, [10], that $m_q(\tilde{\alpha}, 0) = \sum_{i=1}^r q^{e_i}$, where $e_1, \ldots, e_r$ are the exponents of $\mathfrak{g}$. From [8], we know that for $r \geq 1$ the exponents of $\mathfrak{sl}_{r+1}$ are $1, 2, \ldots, r$. In this section we use the Weyl alternation set $\mathcal{A}(\tilde{\alpha}, 0)$, as described in Theorem 3.1.1, to give a combinatorial proof of:

**Theorem 3.1.3.** If $r \geq 1$ and $\tilde{\alpha}$ is the highest root of $\mathfrak{sl}_{r+1}$, then $m_q(\tilde{\alpha}, 0) = q + q^2 + \cdots + q^r$. 

Corollary 3.1.2. If $r \geq 1$ and $\tilde{\alpha}$ is the highest root of $\mathfrak{sl}_{r+1}$, then $m(\tilde{\alpha},0) = r$.

Proof. This follows from Theorem 3.1.3 and the fact that $m_q(\tilde{\alpha},0)|_{q=1} = m(\tilde{\alpha},0)$.

Throughout the remainder of this section we let $r \geq 1$, $n = r + 1$, and let $\tilde{\alpha}$ denote the highest root of $\mathfrak{sl}_{r+1}$.

For any $n \geq 1$ we want to know the number of $\sigma \in A(\tilde{\alpha},0)$ of a given length. By Corollary 3.1.1, we know that if $\sigma \in A(\tilde{\alpha},0)$ with $\ell(\sigma) = k$, then $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$, for some nonconsecutive integers $i_1, \ldots, i_k$ satisfying $2 \leq i_1 < i_2 < \cdots < i_k \leq r - 1$. Thus, to find the number of $\sigma \in A(\tilde{\alpha},0)$ with $\ell(\sigma) = k$, it suffices to know the number of ways of choosing $k$ nonconsecutive integers from the set $\{2, 3, 4, \ldots, r - 1\}$. In light of this we prove:

Proposition 3.1.4. Let $m, k \in \mathbb{N}$ with $k \leq m$. The number of ways to choose $k$ nonconsecutive integers from the set $\{1, 2, 3, \ldots, m\}$ is $\binom{m+1-k}{k}$.

Proof. We proceed by induction on $m$. If $m = 1$, then $k = 0$ or $k = 1$. If $k = 0$, then there is only one way to choose 0 nonconsecutive integers from the set $\{1\}$, namely choosing none, and $\binom{1+1-0}{0} = 1$, as claimed. If $k = 1$, then there is one way to choose 1 nonconsecutive integer from the set $\{1\}$, namely choose 1, and notice $\binom{1+1-1}{1} = 1$. Assume that the number of ways to choose $k$ nonconsecutive integers from the set $\{1, 2, 3, \ldots, m - 1\}$ is $\binom{m-k}{k}$.

Given a choice of $k$ nonconsecutive integers from the set $\{1, 2, \ldots, m\}$ we have two possibilities: one of the numbers chosen is $m$, or all of the numbers chosen where from the set $\{1, 2, \ldots, m - 1\}$. In the case one of the numbers chosen was $m$, then the number of ways to choose $k$ nonconsecutive integers from the set $\{1, 2, 3, \ldots, m\}$ is equal to the number of ways to choose $k - 1$ nonconsecutive integers from the set $\{1, 2, 3, \ldots, m - 2\}$. Which, by induction hypothesis, is $\binom{(m-2)+1-(k-1)}{k-1} = \binom{m-k}{k-1}$. If initially all of the numbers chosen where from the set $\{1, 2, \ldots, m - 1\}$, then, by induction hypothesis, the number of ways to choose $k$ nonconsecutive integers from the set $\{1, 2, 3, \ldots, m - 1\}$ is $\binom{(m-1)+1-k}{k} = \binom{m-k}{k}$. Therefore, the number of ways
to choose \(k\) nonconsecutive integers from the set \(\{1, 2, 3, \ldots, m\}\) is \(\binom{m-k}{k-1} + \binom{m-k}{k}\).

**Lemma 3.1.2.** Let \(r \geq 1\) and let \(\tilde{\alpha}\) denote the highest root of \(\mathfrak{a}_{r+1}\). Then the cardinality of the set \(\{\sigma \in \mathcal{A}(\tilde{\alpha}, 0) \mid \ell(\sigma) = k\}\) is \(\binom{r-1-k}{k-1}\) and \(\max\{\ell(\sigma) \mid \sigma \in \mathcal{A}(\tilde{\alpha}, 0)\}\) is \(\lfloor \frac{r-1}{2} \rfloor\).

**Proof.** By Proposition 3.1.4 the number of elements in \(\mathcal{A}(\tilde{\alpha}, 0)\) of length \(k\) is given by \(\binom{r-1-k}{k}\). Now notice that if \(r\) is odd, then we can choose at most \(\frac{r-1}{2}\) many nonconsecutive integers from the set \(\{2, 3, 4, \ldots, r-1\}\), namely the even numbers. If \(r\) is even, then \(r-1\) is odd and we can choose at most \(\frac{r-2}{2}\) many nonconsecutive integers from the set \(\{2, 3, 4, \ldots, r-1\}\), either all the even or all the odd numbers. Observe that when \(r\) is odd, \(\frac{r-1}{2} = \lfloor \frac{r-1}{2} \rfloor\) and when \(r\) is even, \(\frac{r-2}{2} = \lfloor \frac{r-1}{2} \rfloor\). Therefore the \(\max\{\ell(\sigma) \mid \sigma \in \mathcal{A}(\tilde{\alpha}, 0)\}\) is \(\lfloor \frac{r-1}{2} \rfloor\).

We now prove the following combinatorial identity:

**Proposition 3.1.5.** If \(\sigma \in \mathcal{A}(\tilde{\alpha}, 0)\), then \(\varphi_q(\sigma(\tilde{\alpha} + \rho) - \rho) = q^{1+\ell(\sigma)}(1+q)^{r-1-2\ell(\sigma)}\).

**Proof.** If \(\sigma \in \mathcal{A}(\tilde{\alpha}, 0)\) with \(\ell(\sigma) = 0\), then \(\sigma = 1\) and \(\sigma(\tilde{\alpha} + \rho) - \rho = \tilde{\alpha} = \alpha_1 + \cdots + \alpha_r\). Since \(\Phi^+ = \{\alpha_i : 1 \leq i \leq r\} \cup \{\alpha_i + \cdots + \alpha_j : 1 \leq i < j \leq r\}\), for any \(i \geq 0\), we can think of \(c_{i+1}\), the coefficient of \(q^{i+1}\) in \(\varphi_q(\alpha_1 + \cdots + \alpha_r)\), as the number of ways to place \(i\) lines in \(r-1\) slots. Hence \(c_{i+1} = \binom{r-1}{i}\) and \(\varphi_q(\tilde{\alpha}) = \sum_{i=0}^{r-1} \binom{r-1}{i} q^i = q(1+q)^{r-1}\).

If \(\sigma \in \mathcal{A}(\tilde{\alpha}, 0)\) with \(\ell(\sigma) = k > 0\), then Corollary 3.1.1 implies that \(\sigma = s_1s_2\cdots s_k\), for some nonconsecutive integers \(2 \leq i_1 < i_2 < \cdots < i_k \leq r - 1\). Then by Proposition 3.1.2, \(\sigma(\tilde{\alpha} + \rho) - \rho = (\varepsilon_1 - \varepsilon_{i_1}) + (\varepsilon_{i_1+1} - \varepsilon_{i_2}) + \cdots + (\varepsilon_{i_k+1} - \varepsilon_n) = \tilde{\alpha} - \sum_{j=1}^{k} \alpha_{i_j}\). Let \(c_j\) denote the coefficient of \(q^j\) in \(\varphi_q(\sigma(\tilde{\alpha} + \rho) - \rho)\). Since \(\sigma\) subtracts \(k\) many nonconsecutive simple roots from \(\tilde{\alpha}\), we will at a minimum need \(k+1\) positive roots to write \(\tilde{\alpha} - \sum_{j=1}^{k} \alpha_{i_j}\). So \(c_j = 0\), whenever \(j < k + 1\). Also observe that \(\tilde{\alpha} - \sum_{j=1}^{k} \alpha_{i_j}\) can be written with at most \(r - k\) positive roots. Hence \(c_j = 0\), whenever \(j > n - k\).

For \(i \geq 0\), we can think of \(c_{k+1+i}\) as the number of ways to place \(i\) lines in \(r-1-2k\) slots. This is because for each simple root removed from \(\tilde{\alpha}\) to get \(\sigma(\tilde{\alpha} + \rho) - \rho\), we
lose 2 slots in which to place a line, one before and one after. So
\[ e_{k+1+i} = \binom{r-1-2k}{i}, \]
whenever \( 0 \leq i \leq r - 1 - 2k \).

Therefore \( \varphi_q(\sigma(\tilde{\alpha} + \rho) - \rho) = \sum_{i=0}^{r-1-2k} (1 + q)^{k+1+i} = q^{1+k}(1 + q)^{r-1-2k}. \)

The following closed formula of Kostant’s partition function follows directly from Proposition 3.1.5.

**Corollary 3.1.3.** If \( \sigma \in A(\tilde{\alpha}, 0) \), then \( \varphi(\sigma(\tilde{\alpha} + \rho) - \rho) = 2^{r-1-2\ell(\sigma)}. \)

We provide an alternate proof of Corollary 3.1.3 that does not depend on the use the \( q \)-analog of Kostant’s weight multiplicity formula. We begin by proving:

**Proposition 3.1.6.** If \( r \geq 1 \) and \( \tilde{\alpha} \) is the highest root of \( \mathfrak{sl}_{r+1} \), then \( \varphi(\tilde{\alpha}) = 2^{r-1}. \)

**Proof.** We proceed by induction on \( r \). If \( r = 1 \), then \( \Phi^+ = \{ \alpha_1 \} \), so \( \varphi(\alpha_1) = 1 = 2^0 \).
If \( r = 2 \), then \( \Phi^+ = \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \} \), and \( \tilde{\alpha} = \alpha_1 + \alpha_2 \) can be written using the simple roots \( \alpha_1 \) and \( \alpha_2 \), or we can use the positive root \( \alpha_1 + \alpha_2 \). Hence \( \varphi(\tilde{\alpha}) = 2 = 2^2 \).

Assume that for any \( k \leq r \), we have that \( \varphi(\tilde{\alpha}) = 2^{k-1} \). If \( k = r + 1 \), then let \( \Phi^+_{r+1} \) be the set of positive roots of \( \mathfrak{sl}_{r+1} \) and \( \Phi^+_{r} \) be the set of positive roots of \( \mathfrak{sl}_{r+1} \). Observe that \( \Phi^+_{r+1} \setminus \Phi^+_{r} = \{ \varepsilon_i - \varepsilon_{r+2} : 1 \leq i \leq r + 1 \} \). For \( 1 \leq i \leq r \), let
\[ a_i = \alpha_1 + \cdots + \alpha_{r+1} \text{ and let } a_{r+1} = \alpha_{r+1}. \]

To find the number of ways to write \( \tilde{\alpha} = \alpha_1 + \cdots + \alpha_{r+1} \) as a nonnegative integral sum of the positive roots in \( \Phi^+_{r+1} \), we add the number of ways to write \( \tilde{\alpha} \) when we use one of the positive roots in \( \Phi^+_{r+1} \setminus \Phi^+_{r} \). If we use \( a_1 \), then we can write \( \tilde{\alpha} = \alpha_1 + \cdots + \alpha_{r+1} \) in exactly 1 way. Observe that for \( 2 \leq i \leq r + 1 \), if we use \( a_i \), by induction hypothesis, we can write \( \tilde{\alpha} = \alpha_1 + \cdots + \alpha_{r+1} \) in \( 2^{i-2} \) ways. Therefore
\[
\varphi(\tilde{\alpha}) = 1 + 1 + 2 + \cdots + 2^{r-3} + 2^{r-2} + 2^{r-1}
= 1 + \sum_{i=0}^{r-1} 2^i
= 1 + \frac{1 - 2^r}{1 - 2}
= 2^r.
\]

\[ \square \]
Proposition 3.1.7. Let \( r \geq 2 \) and \( i, j \in \mathbb{N} \). If \( 1 \leq i < j \leq r \), then
\[
\wp(\alpha_i + \alpha_{i+1} + \cdots + \alpha_j) = 2^{j-i}.
\]

Proof. We proceed by induction on \( r \). If \( r = 2 \), then \( i = 1, j = 2 \) and by Proposition 3.1.6, \( \wp(\alpha_1 + \alpha_2) = 2 = 2^{2-1} \). Assume that for any \( k \leq r \), \( \wp(\alpha_i + \alpha_{i+1} + \cdots + \alpha_j) = 2^{j-i} \), whenever \( i, j \in \mathbb{N} \) satisfy \( 1 \leq i < j \leq k \).

Let \( k = r + 1 \), and let \( i, j \) be integers such that \( 1 \leq i < j \leq r + 1 \). If \( j < r + 1 \), then by induction hypothesis \( \wp(\alpha_i + \cdots + \alpha_j) = 2^{j-i} \). Now assume that \( j = r + 1 \). Let \( \Phi_{r+1}^+ \) be the set of positive roots of \( \mathfrak{sl}_{r+2} \) and \( \Phi_r^+ \) be the set of positive roots of \( \mathfrak{sl}_{r+1} \). Observe that \( \Phi_{r+1}^+ \setminus \Phi_r^+ = \{ \varepsilon_i - \varepsilon_{r+2} : 1 \leq i \leq r + 1 \} \). For \( 1 \leq i \leq r \), let \( a_i = \alpha_i + \cdots + \alpha_{r+1} \) and let \( a_{r+1} = \alpha_{r+1} \).

We want to know the number of ways to write \( \alpha_i + \cdots + \alpha_{r+1} \) as a nonnegative integral sum of the positive roots in \( \Phi_{r+1}^+ \). We can compute this number by adding the number of ways to write \( \alpha_i + \cdots + \alpha_{n+1} \) when we use one of the positive roots in \( \Phi_{r+1}^+ \setminus \Phi_r^+ \). Notice that to write \( \alpha_i + \cdots + \alpha_{r+1} \) we will not use any \( a_m \) for \( 1 \leq m \leq i - 1 \).

If we use \( a_i \), then we can write \( \alpha_i + \cdots + \alpha_{r+1} \) in exactly 1 way. Observe that for \( i + 1 \leq m \leq r + 1 \), if we use \( a_m \), then, by induction hypothesis, we can write \( \alpha_i + \cdots + \alpha_{r+1} \) in \( 2^{m-1-i} \) ways.

Thus
\[
\wp(\alpha_i + \cdots + \alpha_{n+1}) = 1 + 1 + 2 + \cdots + 2^{r-1-i} + 2^{r-i} = 1 + \sum_{i=0}^{r-1-i} 2^i = 1 + \frac{1 - 2^{r+1-i}}{1 - 2} = 1 - 1 + 2^{r+1-i} = 2^{r+1-i}.
\]

\[ \square \]

Proposition 3.1.8. If \( r \geq 1 \) and \( \sigma \in A(\tilde{\alpha}, 0) \), then \( \wp(\sigma(\tilde{\alpha} + \rho) - \rho) = 2^{r-1-2\ell(\sigma)} \).
Proof. Assume that \( r \geq 1 \) and \( \sigma \in A(\tilde{\alpha},0) \), with \( k = \ell(\sigma) \). We proceed by induction on \( k \). If \( \sigma \in A(\tilde{\alpha},0) \), with \( k = 0 \), then \( \sigma = 1 \). Hence, by Proposition 3.1.6, \( \varphi(\sigma(\tilde{\alpha}+\rho)-\rho) = \varphi(\tilde{\alpha}) = 2^{r-1} \). Now if \( \sigma \in A(\tilde{\alpha},0) \), with \( k = 1 \), then \( \sigma = s_i \) for some \( 2 \leq i \leq r-1 \). Hence \( \sigma(\tilde{\alpha}+\rho) = \alpha_i = \alpha_1 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_r \). Notice that by Proposition 3.1.7, \( \varphi(\alpha_1 + \cdots + \alpha_{i-1}) = 2^{i-2} \), and \( \varphi(\alpha_{i+1} + \cdots + \alpha_r) = 2^{r-i-1} \).

Since the subset of the positive roots used to write \( \alpha_1 + \cdots + \alpha_{i-1} \), is disjoint from the subset of the positive roots used to write \( \alpha_{i+1} + \cdots + \alpha_r \), we have that \( \varphi(\alpha_1 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_r) = 2^{i-2} \cdot 2^{r-i-1} = 2^{r-3} = 2^{r-1-2(1)} \).

Assume that for any \( \sigma \in A(\tilde{\alpha},0) \) with \( \ell(\sigma) \leq k \), \( \varphi(\sigma(\tilde{\alpha}+\rho)-\rho) = 2^{r-1-2\ell(\sigma)} \).

If \( \sigma \in A(\tilde{\alpha},0) \) with \( \ell(\sigma) = k + 1 \), then \( \sigma = s_{i_1}s_{i_2} \cdots s_{i_k}s_{i_{k+1}} \) for some nonconsecutive integers \( i_1, \ldots, i_{k+1} \) satisfying \( 2 \leq i_1 < i_2 < \cdots < i_k < i_{k+1} \leq r-1 \). By Proposition 3.1.2,

\[
\sigma(\tilde{\alpha}+\rho)-\rho = \tilde{\alpha} - \sum_{j=1}^{k+1} \alpha_{i_j}
\]

\[
= \sum_{j=1}^{i_1-1} \alpha_j + \sum_{j=i_1+1}^{i_2-1} \alpha_j + \sum_{j=i_2+1}^{i_3-1} \alpha_j + \cdots + \sum_{j=i_{k+1}+1}^{r} \alpha_j.
\]

Now we want to know the number of ways to write

\[
\sum_{j=1}^{i_1-1} \alpha_j + \sum_{j=i_1+1}^{i_2-1} \alpha_j + \sum_{j=i_2+1}^{i_3-1} \alpha_j + \cdots + \sum_{j=i_{k+1}+1}^{r} \alpha_j.
\]
as a nonnegative integral sum of positive roots. By Proposition 3.1.7, we have that
\[\wp\left(\sum_{j=1}^{i_1-1} \alpha_j\right) = 2^{i_1-2},\]
\[\wp\left(\sum_{j=i_1+1}^{i_2-1} \alpha_j\right) = 2^{(i_2-1)-(i_1+1)} = 2^{i_2-i_1-2},\]
\[\wp\left(\sum_{j=i_2+1}^{i_3-1} \alpha_j\right) = 2^{(i_3-1)-(i_2+1)} = 2^{i_3-i_2-2},\]
\[\vdots\]
\[\wp\left(\sum_{j=i_k+1}^{i_{k+1}-1} \alpha_j\right) = 2^{(i_{k+1}-1)-(i_k+1)} = 2^{i_{k+1}-i_k-2},\]
\[\wp\left(\sum_{j=i_{k+1}+1}^{r} \alpha_j\right) = 2^{r-(i_{k+1}+1)} = 2^{r-i_k-1}.\]
Since the subsets of the positive roots of \(\mathfrak{sl}_{r+1}\) used to write each of the above sums are pairwise disjoint we have that
\[\wp(\sigma(\tilde{\alpha} + \rho) - \rho) = \wp(\sum_{j=1}^{i_1-1} \alpha_j + \sum_{j=i_1+1}^{i_2-1} \alpha_j + \sum_{j=i_2+1}^{i_3-1} \alpha_j + \cdots + \sum_{j=i_k+1}^{i_{k+1}-1} \alpha_j + \sum_{j=i_{k+1}+1}^{r} \alpha_j)\]
\[= \wp(\sum_{j=1}^{i_1-1} \alpha_j)\wp(\sum_{j=i_1+1}^{i_2-1} \alpha_j)\wp(\sum_{j=i_2+1}^{i_3-1} \alpha_j)\cdots\wp(\sum_{j=i_k+1}^{i_{k+1}-1} \alpha_j)\wp(\sum_{j=i_{k+1}+1}^{r} \alpha_j)\]
\[= 2^{i_1-2} \cdot 2^{i_2-i_1-2} \cdot 2^{i_3-i_2-2} \cdots 2^{i_{k+1}-i_k-2} \cdot 2^{r-i_k-1}\]
\[= 2^{r-1-2(k+1)}.\]

The following proposition will be used in the proof of Theorem 3.1.3.

**Proposition 3.1.9.** For \(r \geq 1\),
\[\sum_{k=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k} (1+q)^{r-1-2k} = \sum_{i=1}^{r} q^i.\]

**Proof.** Equation (4.3.7) in [14] shows that for integers \(k\) and \(n \geq 0,\)
\[\sum_{k \leq \frac{n}{2}} (-1)^k \binom{n-k}{k} q^k (1+q)^{n-2k} = \frac{1-q^{n+1}}{1-q}.\]
Let \( r \geq 1 \). By substituting \( n = r - 1 \geq 0 \) into (3.3) we have that

\[
\sum_{k=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k} = q \left( \frac{1-q^r}{1-q} \right).
\]

Now observe that \( \sum_{i=1}^{r} q^i = q \sum_{i=0}^{r-1} q^i = q \left( \frac{1-q^r}{1-q} \right) \). Therefore

\[
\sum_{k=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k} = \sum_{i=1}^{r} q^i. \quad \Box
\]

The following proof of Proposition 3.1.9 does not use generating functions.

Alternate proof of Proposition 3.1.9. We proceed by induction on \( r \). If \( r = 1 \), then \((-1)^0 \binom{0}{0} q^1 (1+q)^0 = q \). Now assume that for any \( m \leq r \),

\[
\sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (-1)^k \binom{m-1-k}{k} q^{1+k}(1+q)^{m-1-2k} = \sum_{i=1}^{m} q^i.
\]

Let \( m = r + 1 \). Using the identity \( \binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b} \) for any integers \( a, b \geq 0 \) we get

\[
\sum_{k=0}^{\left\lfloor \frac{r}{2} \right\rfloor} (-1)^k \binom{r-k}{k} q^{1+k}(1+q)^{r-2k} = S_1 + S_2, \quad \text{where} \quad (3.4)
\]

\[
S_1 := \sum_{k=0}^{\left\lfloor \frac{r}{2} \right\rfloor} (-1)^k \binom{r-1-k}{k-1} q^{1+k}(1+q)^{r-2k}, \quad \text{and} \quad (3.5)
\]

\[
S_2 := (1+q) \sum_{k=0}^{\left\lfloor \frac{r}{2} \right\rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k}. \quad (3.6)
\]
By using \( \binom{r-1}{k} = 0 \), and the shift \( k \to k + 1 \) we can write (3.5) as

\[
S_1 = - \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r - 2 - k}{k} q^{2+k} (1 + q)^{r-2-2k} \\
= -q \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{(r - 1) - 1 - k}{k} q^{1+k} (1 + q)^{(r-1)-1-2k} \\
= -q(q + q^3 + \cdots + q^{r-1}) , \text{ by induction hypothesis} \\
= -(q^2 + q^3 + \cdots + q^{r}), \quad (3.7)
\]

Now we compute \( S_2 \) in two cases.

**Case 1:** If \( r \) is odd, then \( \lfloor \frac{r}{2} \rfloor = \lfloor \frac{r-1}{2} \rfloor \). Hence (3.6) can be written as

\[
S_2 = (1 + q) \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r - 1 - k}{k} q^{1+k} (1 + q)^{r-1-2k} \\
= (1 + q)(q + q^3 + \cdots + q^{r}), \text{ by induction hypothesis.}
\]

In which case (3.4) is given by

\[
S_1 + S_2 = -(q^2 + q^3 + \cdots + q^{r}) + (1 + q)(q + q^2 + \cdots + q^{r}) \\
= q + q^2 + q^3 + \cdots + q^{r+1}, \text{ thus completing the induction step.}
\]

**Case 2:** If \( r \) is even, then \( \lfloor \frac{r}{2} \rfloor = \lfloor \frac{r+1}{2} \rfloor = \lfloor \frac{r-1}{2} \rfloor + 1 \). Hence (3.6) can be written as

\[
S_2 = (1 + q) \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor + 1} (-1)^k \binom{r - 1 - k}{k} q^{1+k} (1 + q)^{r-1-2k} \\
= (1 + q) \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r - 1 - k}{k} q^{1+k} (1 + q)^{r-1-2k} + \\
+ (1 + q)(-1)^{\lfloor \frac{r-1}{2} \rfloor + 1} \binom{r - 2 - \lfloor \frac{r-1}{2} \rfloor}{\lfloor \frac{r-1}{2} \rfloor + 1} q^{2+\lfloor \frac{r-1}{2} \rfloor} (1 + q)^{r-3-2\lfloor \frac{r-1}{2} \rfloor} \\
= (1 + q)(q + \cdots + q^{r}) + (-1)^{\lfloor \frac{r-1}{2} \rfloor + 1} \binom{r - 2 - \lfloor \frac{r-1}{2} \rfloor}{\lfloor \frac{r-1}{2} \rfloor + 1} q^{2+\lfloor \frac{r-1}{2} \rfloor} (1 + q)^{r-2-2\lfloor \frac{r-1}{2} \rfloor},
\]
by induction hypothesis.

Since \((\binom{a}{b}) = 0\) when \(b > a\) and \(\lfloor \frac{r}{2} \rfloor = \lfloor \frac{r-1}{2} \rfloor + 1\), it is easy to show that
\[(r - 2 - \lfloor \frac{r}{2} \rfloor) = 0.\]
Hence \((-1)^{\lfloor \frac{r}{2} \rfloor + 1} (r - 2 - \lfloor \frac{r}{2} \rfloor) q^{2 + \lfloor \frac{r-1}{2} \rfloor + 1} (1 + q)^{r-2-2\lfloor \frac{r}{2} \rfloor} = 0.\]
Thus (3.4) is given by
\[S_1 + S_2 = -(q^2 + q^3 + \cdots + q^r) + (1 + q)(q + q^2 + \cdots + q^r)\]
\[= q + q^2 + q^3 + \cdots + q^{r+1},\]
which completes the proof.

\[\square\]

**Remark 3.1.3.** Suppose \(r \geq 1\). If we define \(F_r(t) = \sum_{k=0}^{\infty} \binom{r-1-k}{k} t^k\), then \(F_r(1)\) is the \(r^{th}\) Fibonacci number. So \(F_r(t)\) is a \(t\)-analog of the Fibonacci numbers. Also notice if \(t = \frac{-q}{(1+q)^2}\), then \(q(1+q)^{r-1} F_r(t)\) is the sum we encountered in Proposition 3.1.9.

**Proof of Theorem 3.1.3.** By Lemma 3.1.2 and Propositions 3.1.5 and 3.1.9, if \(k = \ell(\sigma)\), then
\[m_q(\tilde{\alpha}, 0) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \varphi_q(\sigma(\tilde{\alpha} + \rho) - \rho)\]
\[= \sum_{\sigma \in A(\tilde{\alpha}, 0)} (-1)^{\ell(\sigma)} \varphi_q(\sigma(\tilde{\alpha} + \rho) - \rho)\]
\[= \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k}\]
\[= q + q^2 + q^3 + \cdots + q^r.\]

\[\square\]

### 3.1.3 Nonzero weight spaces of \(\mathfrak{sl}_{r+1}(\mathbb{C})\)

It is fundamental in Lie theory that the zero weight space is a Cartan subalgebra. If \(\tilde{\alpha}\) is the highest root of \(\mathfrak{g}\), then the nonzero weights of \(L(\tilde{\alpha})\), the adjoint representation of \(\mathfrak{g}\), are the roots and have multiplicity 1. We visit this picture from our point of view in the case when \(\mathfrak{g} = \mathfrak{sl}_{r+1}\). Let \(r \geq 1\) and \(n = r + 1\).
Theorem 3.1.4. If $\mu \in P_+(\mathfrak{sl}_n)$ and $\mu \neq 0$, then $A(\tilde{\alpha}, \mu) = \begin{cases} \{1\} & \text{if } \mu = \tilde{\alpha} \\ \emptyset & \text{otherwise.} \end{cases}$

We begin by proving the following propositions.

Proposition 3.1.10. If $\tilde{\alpha}$ is the highest root of $\mathfrak{sl}_n$, then $A(\tilde{\alpha}, \tilde{\alpha}) = \{1\}$.

Proof. It suffices to show that $\wp(\sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) > 0$ if and only if $\sigma = 1$.

$(\Rightarrow)$ Assume that $\sigma \in S_n$ such that $\wp(\sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) > 0$. Hence

$$(\varpi_i, (\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) \in \mathbb{N}, \text{ for all } 1 \leq i \leq r.$$

By (3.2) we have that

$$\sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha} = (a_{\sigma^{-1}(1)} - n, a_{\sigma^{-1}(2)} - n + 2, \ldots, a_{\sigma^{-1}(n-1)} - 1, a_{\sigma^{-1}(n)} + 1),$$

where $a_i$ is given by (3.1).

Let $M = \{i \mid \sigma^{-1}(i) \neq i\}$. Suppose $M \neq \emptyset$ and let $j = \text{min}(M)$. Hence $\sigma^{-1}(j) = k$, for some integer $j < k \leq n$. By definition of $a_i$, since $\sigma^{-1}(i) = i$, for all $1 \leq i \leq j - 1$, we have that $(\varpi_i, (\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) = 0$, for all $1 \leq i \leq j - 1$.

Thus

$$(\varpi_j, (\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) = a_{\sigma^{-1}(j)} - n + j = \begin{cases} j - n - 1 & \text{if } k = n \\ j - k & \text{if } j < k < n. \end{cases}$$

In either case $j < k \leq n$ implies that $(\varpi_j, (\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) < 0$, a contradiction. Thus $M = \emptyset$ and $\sigma^{-1}(i) = i$, for all $1 \leq i \leq n$. Therefore $\sigma = 1$.

$(\Leftarrow)$ If $\sigma = 1$, then $\wp(\sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) = \wp(0) = 1 > 0$. \qed

Proposition 3.1.11. Let $\mu \in P_+(\mathfrak{sl}_n) \setminus \{0\}$. Then there exists $\sigma \in S_n$ such that $\wp(\sigma(\tilde{\alpha} + \rho) - \rho - \mu) > 0$ if and only if $\mu = \tilde{\alpha}$.

Proof. $(\Rightarrow)$ If $\mu \in P_+(\mathfrak{sl}_n)$, then $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, for some $\mu_1, \ldots, \mu_n \in \mathbb{Z}$, satisfying $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\sum_{i=1}^n \mu_i = 0$. Assume $\mu \neq 0$, hence $(\mu_1, \ldots, \mu_n) \neq (0, \ldots, 0)$. If $\mu_1 < 0$, then $\mu_i < 0$, for all $2 \leq i \leq n$ and $\sum_{i=1}^n \mu_i \neq 0$, a contradiction. So we may assume $\mu_1 \geq 0$. 

Now suppose there exists $\sigma \in S_n$ such that $\phi(\sigma(\tilde{\alpha} + \rho) - \rho - \mu) > 0$. Hence $(\varpi_i, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) \in \mathbb{N}$, for all $1 \leq i \leq n - 1$. In particular, 

$$(\varpi_1, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) = a_{\sigma^{-1}(1)} - n + 1 - \mu_1 \geq 0$$ if and only if $\mu_1 \leq a_{\sigma^{-1}(1)} - n + 1$.

Observe that

$$a_{\sigma^{-1}(1)} - n + 1 = \begin{cases} 1 & \text{if } \sigma^{-1}(1) = 1 \\ -n & \text{if } \sigma^{-1}(1) = n \\ 1 - \sigma^{-1}(1) & \text{if } 1 < \sigma^{-1}(1) < n. \end{cases}$$

Then $(\varpi_1, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) > 0$ if and only if $\sigma^{-1}(1) = 1$ and $\mu_1 \leq 1$. Now because $\mu_1 \in \mathbb{Z}$ and $0 \leq \mu_1 \leq 1$, we have that $\mu_1 = 0$ or $1$. If $\mu_1 = 0$, then $\sum_{i=1}^{n} \mu_i = 0$ if and only if $\mu = 0$, a contradiction. Therefore $\mu_1 = 1$.

Now observe that $(\varpi_{n-1}, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) = \sum_{i=1}^{n-1} (a_{\sigma^{-1}(i)} - n + i - \mu_i) = \mu_n - a_{\sigma^{-1}(n)}$. Hence $(\varpi_{n-1}, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) \geq 0$ if and only if $\mu_n \geq a_{\sigma^{-1}(n)}$.

If $1 < \sigma^{-1}(n) < n$, then $a_{\sigma^{-1}(n)} = n - \sigma^{-1}(n) \geq 1$ and hence $\mu_n \geq 1$. Then $1 = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 1$ implies that $\mu_i = 1$, for all $1 \leq i \leq n$ and so $\sum_{i=1}^{n} \mu_i \neq 0$, a contradiction. Therefore $\sigma^{-1}(n) = n$ and $\mu_n = -1$.

Observe that since $\mu_1 = 1$, $\mu_n = -1$, and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, we have that $\mu_i \in \{1, 0, -1\}$, for all $2 \leq i \leq n - 1$. If $\mu_2 = -1$, then $\mu = (1, -1, \ldots, -1)$ and $\sum_{i=1}^{n} \mu_i \neq 0$, a contradiction. Suppose $\mu_2 = 1$. Since $\sigma^{-1}(1) = 1$ and $\sigma^{-1}(n) = n$, we have that $1 < \sigma^{-1}(2) < n$. Hence $(\varpi_2, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) = a_{\sigma^{-1}(2)} - n + 1 = 1 - \sigma^{-1}(2) < 0$, a contradiction. Thus $\mu_2 = 0$.

Notice that if $\mu_j = -1$, for some $2 < j \leq n - 1$, then $\mu_i = -1$, for all $j < i \leq n - 1$. In which case $\sum_{i=1}^{n} \mu_i \neq 0$, giving rise to a contradiction. So $\mu_i = 0$, for all $2 \leq i \leq n - 1$, and thus $\mu = (1, 0, \ldots, 0, -1) = \tilde{\alpha}$.

$(\Leftarrow)$ Follows from Proposition 3.1.10. 

\textbf{Proof of Theorem 3.1.4.} Follows from Proposition 3.1.10 and the contrapositive of Proposition 3.1.11. 

The following corollary is fundamental in Lie theory. We give an alternate proof of this well known result by using Weyl alternation sets.
Corollary 3.1.4. If $\mu \in P(\mathfrak{sl}_n)$, then $m(\tilde{\alpha}, \mu) = \begin{cases} r & \text{if } \mu = 0 \\ 1 & \text{if } \mu \in \Phi \\ 0 & \text{otherwise.} \end{cases}$

Proof. By Proposition 3.1.20 in [5], if $\mu \in P(\mathfrak{sl}_n)$, then there exists $w \in W$ and $\xi \in P_+(\mathfrak{sl}_n)$ such that $w(\xi) = \mu$. Also by Proposition 3.2.27 in [5] we know that weight multiplicities are invariant under $W$. Thus it suffices to compute $m(\tilde{\alpha}, \mu)$ for $\mu \in P_+(\mathfrak{sl}_n)$. By Corollary 3.1.2, $m(\tilde{\alpha}, 0) = r$. By Theorem 3.1.4 we know $A(\tilde{\alpha}, \tilde{\alpha}) = \{1\}$ and $A(\tilde{\alpha}, \mu) = \emptyset$, whenever $\mu \in P_+(\mathfrak{sl}_{n+1}) \setminus \{0, \tilde{\alpha}\}$. This implies that $m(\tilde{\alpha}, \tilde{\alpha}) = \varphi(1(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) = \varphi(0) = 1$, and that $m(\tilde{\alpha}, \mu) = 0$, whenever $\mu \in P_+(\mathfrak{sl}_{n+1}) \setminus \{0, \tilde{\alpha}\}$. □

3.2 Lie algebra $\mathfrak{so}_{2r+1}(\mathbb{C})$

Let $r \geq 2$ and let $G = SO_{2r+1}(\mathbb{C}) = \{g \in SL_{2r+1}(\mathbb{C}) : g^t = g^{-1}\}$ be the special orthogonal group of $(2r+1) \times (2r+1)$ matrices over $\mathbb{C}$. Let $\mathfrak{g} = \mathfrak{so}_{2r+1}(\mathbb{C}) = \{X \in M_{2r+1}(\mathbb{C}) : X^t = -X\}$ and $\mathfrak{h} = \{\text{diag}[a_1, \ldots, a_r, 0, -a_r, \ldots, -a_1] | a_1, \ldots, a_r \in \mathbb{C}\}$ be a fixed choice of Cartan subalgebra. For $1 \leq i \leq r$, define the linear functionals $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(H) = a_i$, for any $H = \text{diag}[a_1, \ldots, a_r, 0, -a_r, \ldots, -a_1] \in \mathfrak{h}$.

For each $1 \leq i \leq r-1$, let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and let $\alpha_r = \varepsilon_r$. Then the set of simple and positive roots are $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ and $\Phi^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j : 1 \leq i < j \leq n\} \cup \{\varepsilon_i : 1 \leq i \leq r\}$, respectively. The fundamental weights are defined by

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i, \text{ for } 1 \leq i \leq r-1, \text{ and}$$

$$\varpi_r = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r).$$

Then observe that

$$\varpi_1 = \varepsilon_1 = \alpha_1 + \cdots + \alpha_r \text{ and}$$

$$\rho = \varpi_1 + \varpi_2 + \cdots + \varpi_r$$

$$= (r - \frac{1}{2})\varepsilon_1 + (r - \frac{3}{2})\varepsilon_2 + \cdots + \frac{3}{2}\varepsilon_{r-1} + \frac{1}{2}\varepsilon_r.$$
For \(1 \leq i \leq r - 1\), \(s_{\alpha_i}(\varepsilon_k) = \varepsilon_{\sigma(k)}\), where \(\sigma\) is the transposition \((i\ i + 1) \in S_r\). If \(k \neq r\), then \(s_{\alpha_r}(\varepsilon_k) = \varepsilon_k\) and \(s_{\alpha_r}(\varepsilon_r) = -\varepsilon_r\). For any \(1 \leq i \leq r\), let \(s_i := s_{\alpha_i}\). Then the Weyl group, \(W\), acts on \(\mathfrak{h}^*\) by signed permutations of \(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\) and is generated by the simple root reflections \(s_1, \ldots, s_r\).

### 3.2.1 The zero weight space of \(\mathfrak{so}_{2r+1}(\mathbb{C})\)

In this section we present results regarding the Weyl alternation set corresponding to the fundamental weight \(\varpi_1\) and the zero weight of \(\mathfrak{so}_{2r+1}\).

**Proposition 3.2.1.** Let \(\sigma = s_{i_1}s_{i_2}\cdots s_{i_k} \in W\), for some nonconsecutive integers \(i_1, i_2, \ldots, i_k\) between 2 and \(r\). Then \(\sigma(\varpi_1 + \rho) - \rho = \varpi_1 - \sum_{j=1}^{k} \alpha_{i_j}\) is a nonnegative integral combination of positive roots.

**Proof.** Recall that for any \(1 \leq i, j \leq r\), \(s_i(\varpi_j) = \varpi_j - \delta_{ij}\alpha_j\). Also, by definition, \(s_i(\alpha_j) = \alpha_j - \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i\), for any \(1 \leq i, j \leq r\). Observe that \((\alpha_i, \alpha_j) = 0\), whenever \(i\) and \(j\) are nonconsecutive integers between \(1\) and \(r\). Thus, given \(\sigma = s_{i_1}s_{i_2}\cdots s_{i_k} \in W\), with \(i_1, i_2, \ldots, i_k\) nonconsecutive integers between 2 and \(r\), we have that

\[
\sigma(\varpi_1 + \rho) - \rho = s_{i_1}s_{i_2}\cdots s_{i_k}(\varpi_1 + (\varpi_1 + \cdots + \varpi_r)) - (\varpi_1 + \cdots + \varpi_r)
= 2\varpi_1 + \varpi_2 + \cdots + \varpi_r - (\alpha_{i_1} + \cdots + \alpha_{i_k}) - (\varpi_1 + \cdots + \varpi_r)
= \varpi_1 - \sum_{j=1}^{k} \alpha_{i_j}.
\]

Now since \(\varpi_1 = \alpha_1 + \cdots + \alpha_r\), notice \(\varpi_1 - \sum_{j=1}^{k} \alpha_{i_j}\) is a nonnegative integral combination of positive roots. \(\square\)

**Theorem 3.2.1.** Let \(\sigma \in W\). Then \(\sigma \in A(\varpi_1, 0)\) if and only if \(\sigma = 1\) or \(\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}\), for some nonconsecutive integers \(i_1, \ldots, i_k\) between 2 and \(r\).

**Proof.** (\(\Rightarrow\)) Let \(\sigma \in A(\varpi_1, 0)\) and proceed by induction on \(\ell(\sigma)\). If \(\ell(\sigma) = 0\), then \(\sigma = 1\) and we are done. If \(\sigma = s_i\), for some \(1 \leq i \leq r\), then

\[
s_i(\varpi_1 + \rho) - \rho = s_i(2\varpi_1 + \varpi_2 + \cdots + \varpi_r) - (\varpi_1 + \cdots + \varpi_r)
= \begin{cases} 
\alpha_1 + \alpha_2 + \cdots + \alpha_r & \text{if } i = 1 \\
\alpha_1 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_r & \text{otherwise}.
\end{cases}
\]

(Continues on the next page.)
If \( i = 1 \), then \( \sigma \notin A(\varpi, 0) \), a contradiction. Thus \( \sigma = s_i \), where \( 2 \leq i \leq r \). If \( \ell(\sigma) = 2 \), then \( \sigma = s_i s_j \) for some integers \( 1 \leq i, j \leq r \). If \( i = j \), then \( \sigma = s_i s_i = s_i^2 = 1 \) and so \( \ell(\sigma) \neq 2 \), gives a contradiction. Hence \( i \neq j \). Now notice

\[
\begin{align*}
  s_i s_j (\varpi + \rho) - \rho &= \varpi + \rho - \alpha_i - \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \alpha_i - \rho \\
  &= \varpi - \alpha_i - \alpha_j + \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \alpha_i.
\end{align*}
\]

If \( i \) and \( j \) are consecutive integers between 2 and \( r \), then \( j = i \pm 1 \) and, in either case, \((\alpha_i, \alpha_j) = -1\). Hence \( \sigma \notin A(\varpi, 0) \), a contradiction. Thus \( i \) and \( j \) are nonconsecutive integers between 2 and \( r \).

Now assume that for any \( \sigma \in A(\varpi, 0) \) with length \( n \leq k \), there exist nonconsecutive integers \( i_1, \ldots, i_n \) between 2 and \( r \) such that \( \sigma = s_{i_1} s_{i_2} \cdots s_{i_n} \). Now let \( \sigma \in A(\varpi, 0) \), such that \( \ell(\sigma) = k + 1 \). Hence there exist distinct integers \( i_1, i_2, \ldots, i_{k+1} \) between 1 and \( r \) such that \( \sigma = s_{i_1} s_{i_2} \cdots s_{i_{k+1}} \). Let \( \sigma = s_{i_1} \pi \), where \( \pi = s_{i_2} s_{i_3} \cdots s_{i_{k+1}} \) and \( \ell(\pi) = k \). Since \( \sigma \in A(\varpi, 0) \), we have that \( \pi \in A(\varpi, 0) \), and by induction hypothesis, \( i_2, \ldots, i_{k+1} \) are nonconsecutive integers between 2 and \( r \).

By Proposition 3.2.1, \( \pi(\varpi + \rho) = \varpi + \rho - \sum_{j=2}^{j=k+1} \alpha_{i_j} \).

Hence

\[
\begin{align*}
  \sigma(\varpi + \rho) - \rho &= s_{i_1} \pi (\varpi + \rho) - \rho \\
  &= s_{i_1} (\varpi + \rho - \sum_{j=2}^{j=k+1} \alpha_{i_j}) - \rho \\
  &= \varpi + \rho - \alpha_{i_1} - \sum_{j=2}^{j=k+1} (\alpha_{i_j} - \frac{2(\alpha_{i_j}, \alpha_{i_1})}{(\alpha_{i_1}, \alpha_{i_1})} \alpha_{i_1}) - \rho \\
  &= \varpi - \sum_{j=1}^{j=k+1} \alpha_{i_j} + 2 \sum_{j=2}^{j=k+1} \left(\frac{\alpha_{i_j}, \alpha_{i_1}}{(\alpha_{i_1}, \alpha_{i_1})}\alpha_{i_1}\right).
\end{align*}
\]

Observe that if \( i_1 = 1 \), then \( \varphi(\sigma(\varpi + \rho) - \rho) = 0 \) and hence \( \sigma \notin A(\varpi, 0) \), a contradiction. Now suppose there exists an integer, \( 2 \leq j \leq r \), such that \( i_j \) and \( i_1 \) are consecutive integers. Then \((\alpha_{i_j}, \alpha_{i_1}) = -1\) and hence the coefficient of \( \alpha_{i_1} \) in \( \sigma(\varpi + \rho) - \rho \) is negative. Thus \( \sigma \notin A(\varpi, 0) \), again a contradiction. Therefore \( i_1, i_2, \ldots, i_{k+1} \) must be nonconsecutive integers between 2 and \( r \).
By Proposition 3.2.1, if \( \sigma = s_{i_1} s_{i_2} \cdots s_{i_k} \in W \), for some nonconsecutive integers \( i_1, \ldots, i_k \) between 2 and \( r \), then \( \sigma(\varpi_1 + \rho) - \rho \) is a nonnegative integral combination of positive roots. Thus \( \sigma \in A(\varpi_1, 0) \).

**Lemma 3.2.1.** If \( m \geq 1 \), then the total number of sequences of nonconsecutive integers between 1 and \( m \) is given by \( F_{m+2} \).

**Proof.** Suppose \( m = 1 \), then the number of sequences of nonconsecutive integers between 1 and 1, is \( 2 = F_3 \), namely choose none or choose the 1. If \( m = 2 \), then we can choose none, or choose the 1 or the 2. Hence there are \( 3 = F_4 \) number of sequences of nonconsecutive integers between 1 and 2. Assume that for any \( m \leq k \), the number of sequences of nonconsecutive integers between 1 and \( m \) is \( F_{m+2} \).

Now to find the number of sequences of nonconsecutive integers between 1 and \( k+1 \) we note that this number depends on whether \( k+1 \) is chosen or not. If \( k+1 \) is not chosen then the number of number of sequences of nonconsecutive integers between 1 and \( k \), by our induction hypothesis, is \( F_{k+2} \). If \( k+1 \) is chosen then we cannot choose \( k \), so number of sequences of nonconsecutive integers between 1 and \( k-1 \) is \( F_{k+1} \). Therefore the number of sequences of nonconsecutive integers between 1 and \( k+1 \) is given by \( F_{k+2} + F_{k+1} = F_{k+3} \).

We can now give a proof of the main result of this section:

**Theorem 3.2.2.** If \( r \geq 2 \) and \( \varpi_1 = \sum_{\alpha \in \Delta} \alpha \) is a fundamental weight of \( \mathfrak{so}_{2r+1} \), then \( |A(\varpi_1, 0)| = F_{r+1} \).

**Proof.** By Theorem 3.2.1, we have that \( A(\varpi_1, 0) = \{ \sigma \in W | \sigma = s_{i_1} \cdots s_{i_k}, \) for some nonconsecutive integers \( 2 \leq i_1, \ldots, i_k \leq r \}. \) Therefore, by Lemma 3.2.1, \( |A(\varpi_1, 0)| = F_{r+1} \).

### 3.2.2 \( \mathfrak{so}_{2r+1} \) and a \( q \)-analog

It is known that the multiplicity of the zero weight of \( \mathfrak{so}_{2r+1} \) in the finite-dimensional representation \( L(\varpi_1) \) is equal to 1, see [2]. In this section, we give a combinatorial proof of this fact, by proving:
Theorem 3.2.3. Let \( r \geq 2 \) and let \( \varpi_1 = \sum_{\alpha \in \Delta} \alpha \) be a fundamental weight of \( \mathfrak{so}_{2r+1} \). Then \( m_q(\varpi_1, 0) = q^r \).

Lemma 3.2.2. Let \( \varpi_1 = \sum_{\alpha \in \Delta} \alpha \) be a fundamental weight of \( \mathfrak{so}_{2r+1} \). Then
\[
|\{\sigma \in A(\varpi_1, 0) : \ell(\sigma) = k \text{ and } \sigma \text{ contains no } s_r \text{ factor}\}| = \binom{r-1-k}{k},
\]
\[
|\{\sigma \in A(\varpi_1, 0) : \ell(\sigma) = k \text{ and } \sigma \text{ contains an } s_r \text{ factor}\}| = \binom{r-2-k}{k},
\]
max\(\{\ell(\sigma) : \sigma \in A(\varpi_1, 0) \text{ and } \sigma \text{ contains no } s_r \text{ factor}\}\) = \(\lfloor \frac{r-1}{2} \rfloor\), and
max\(\{\ell(\sigma) : \sigma \in A(\varpi_1, 0) \text{ and } \sigma \text{ contains an } s_r \text{ factor}\}\) = \(\lfloor \frac{r-2}{2} \rfloor\).

Proposition 3.2.2. Let \( \sigma \in A(\varpi_1, 0) \). Then
\[
\varphi_q(\sigma(\varpi_1 + \rho) - \rho) = \begin{cases} 
q^{1+\ell(\sigma)}(1+q)^{r-1-2\ell(\sigma)} & \text{if } \sigma \text{ contains no } s_r \text{ factor} \\
q^{1+\ell(\sigma)}(1+q)^{r-2-2\ell(\sigma)} & \text{if } \sigma \text{ contains an } s_r \text{ factor}.
\end{cases}
\]

Observe that for any \( \sigma \in A(\varpi_1, 0) \) the subset of positive roots of \( \mathfrak{so}_{2r+1} \) used to write \( \sigma(\varpi_1 + \rho) - \rho \), consists of sums of simple roots, just as in the \( \mathfrak{sl}_{r+1} \) case. Therefore, Lemma 3.2.2 and Proposition 3.2.2 follow from analogous arguments as those used in Lemma 3.1.2 and Proposition 3.1.5, respectively.

Now can now prove the closed formula for the \( q \)-multiplicity of the zero weight in \( L(\varpi_1) \).

**Proof of Theorem 3.2.3.** Observe that
\[
m_q(\varpi_1, 0) = \sum_{\sigma \in A(\varpi_1, 0) \text{ with no } s_r \text{ factor}} (-1)^{\ell(\sigma)} \varphi_q(\sigma(\varpi_1 + \rho) - \rho) + \sum_{\sigma \in A(\varpi_1, 0) \text{ with an } s_r \text{ factor}} (-1)^{\ell(\sigma)} \varphi_q(\sigma(\varpi_1 + \rho) - \rho).
\]

By Lemma 3.2.2, Proposition 3.2.2 and Proposition 3.1.9 it follows that
\[
\sum_{\sigma \in A(\varpi_1, 0) \text{ with no } s_r \text{ factor}} (-1)^{\ell(\sigma)} \varphi_q(\sigma(\varpi_1 + \rho) - \rho) = \sum_{k=0}^{r-1} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k} = \sum_{i=1}^{r} q^i.
\]
Similarly, 
\[
\sum_{\sigma \in A(\varpi_1,0)} (-1)^{\ell(\sigma)} q_{\varpi}(\sigma(\varpi_1 + \rho) - \rho) = \sum_{k=0}^{\left\lfloor \frac{r-2}{2} \right\rfloor} (-1)^{1+k} \binom{r-2-k}{k} q^{1+k}(1+q)^{r-2-2k} = -\sum_{i=1}^{r-1} q^i.
\]

Therefore \(m_q(\varpi_1,0) = (q + q^2 + \cdots + q^{r-1} + q^r) - (q + q^2 + \cdots + q^{r-1}) = q^r\). \(\square\)

**Corollary 3.2.1.** Let \(r \geq 2\) and let \(\varpi_1 = \sum_{\alpha \in \Delta} \alpha\) be a fundamental weight of \(\mathfrak{so}_{2r+1}\). Then \(m(\varpi_1,0) = 1\).

**Proof.** Follows directly from Theorem 3.2.3, since \(m(\varpi_1,0) = m_q(\varpi_1,0)|_{q=1} = 1\). \(\square\)

### 3.2.3 Nonzero weight spaces of \(\mathfrak{so}_{2r+1}(\mathbb{C})\)

We now consider the nonzero dominant weights, \(\mu\), of \(\mathfrak{so}_{2r+1}\) and compute the Weyl alternation sets \(A(\varpi_1,\mu)\). Throughout this section we let \(r \geq 2\).

**Theorem 3.2.4.** If \(\mu \in P_+(\mathfrak{so}_{2r+1}) \setminus \{0\}\), then \(A(\varpi_1,\mu) = \begin{cases} \{1\} & \text{if } \mu = \varpi_1 \\ \emptyset & \text{otherwise} \end{cases}\)

The following Propositions will be used in the proof of Theorem 3.2.4.

**Proposition 3.2.3.** If \(\varpi_1 = \sum_{\alpha \in \Delta} \alpha\) is a fundamental weight of \(\mathfrak{so}_{2r+1}\), then \(A(\varpi_1,\varpi_1) = \{1\}\).

**Proof.** Since \(\varpi_1 = \alpha_1 + \cdots + \alpha_r\), notice \(\sigma(\varpi_1 + \rho) - \rho - \varpi_1\) is a nonnegative integral sum of positive roots only if \(\sigma(\varpi_1 + \rho) - \rho\) is. By Theorem 3.2.1 we know \(\sigma(\varpi_1 + \rho) - \rho\) is a nonnegative integral sum of positive roots if and only if \(\sigma = s_{i_1}s_{i_2} \cdots s_{i_k}\), for some nonconsecutive integers \(i_1, \ldots, i_k\) between \(2\) and \(r\). Hence \(A(\varpi_1,\varpi_1) \subset A(\varpi_1,0)\).

Suppose that \(\sigma \in A(\varpi_1,\varpi_1)\) with \(\ell(\sigma) = k \geq 1\), then there exist nonconsecutive integers \(i_1, \ldots, i_k\) between \(2\) and \(r\) such that \(\sigma = s_{i_1}s_{i_2} \cdots s_{i_k}\). By Proposition 3.2.1 we have that \(\sigma(\varpi_1 + \rho) - \rho = \varpi_1 - \sum_{j=1}^{k} \alpha_{i_j}\). Then notice \(\sigma(\varpi + \rho) - \rho - \varpi_1 = -\sum_{j=1}^{k} \alpha_{i_j}\) cannot be expressed as a nonnegative integral sum of positive roots, a contradiction. Thus \(\ell(\sigma) = 0\) and \(\sigma = 1\). \(\square\)
Proposition 3.2.4. Let $\mu \in P_{+}(\mathfrak{so}_{2r+1}) \setminus \{0\}$. Then there exists $\sigma \in W$ such that $\varphi(\sigma(\varpi_1 + \rho) - \rho - \mu) > 0$ if and only if $\mu = \varpi_1$.

Proof. ($\Rightarrow$) Let $\mu \in P_{+}(\mathfrak{so}_{2r+1}) \setminus \{0\}$ and assume $\sigma \in W$ such that $\varphi(\sigma(\varpi_1 + \rho) - \rho - \mu) > 0$. By Proposition 3.1.20 in [5], we know that $P_{+}(\mathfrak{so}_{2r+1})$ consists of all weights $\mu = k_1 \varepsilon_1 + k_2 \varepsilon_2 + \cdots + k_r \varepsilon_r$, with $k_1 \geq k_2 \geq \cdots \geq k_r \geq 0$. Here $2k_i$ and $k_i - k_j$ are integers for all $1 \leq i, j \leq r$.

Now observe that $\sigma(\varpi_1 + \rho) - \rho - \mu = \sigma((r + \frac{1}{2}) \varepsilon_1 + (r - \frac{3}{2}) \varepsilon_2 + (r - \frac{5}{2}) \varepsilon_3 + \cdots + \frac{1}{2} \varepsilon_{r-1} + \varepsilon_r) - ((r - \frac{1}{2}) \varepsilon_1 + (r - \frac{3}{2}) \varepsilon_2 + \cdots + \frac{1}{2} \varepsilon_r) - (k_1 \varepsilon_1 + \cdots + k_r \varepsilon_r)$. Let $a_i$ denote the coefficient of $\alpha_i$ in $\sigma(\varpi_1 + \rho) - \rho - \mu$. Then

$$a_1 = \begin{cases} -i + 1 - k_1 & \text{if } \sigma(\varepsilon_1) = \varepsilon_i \text{ for } 2 \leq i \leq r, \\ -2r + i - k_1 & \text{if } \sigma(\varepsilon_1) = -\varepsilon_i \text{ for } 2 \leq i \leq r, \\ 1 - k_1 & \text{if } \sigma(\varepsilon_1) = \varepsilon_1, \\ -2r - k_1 & \text{if } \sigma(\varepsilon_1) = -\varepsilon_1. \end{cases}$$

Since $r \geq 2$ and $a_1 \in \mathbb{N}$, we have that $\sigma(\varepsilon_1) = \varepsilon_1$ and $a_1 = 1 - k_1$. If $k_1 = 0$, then $k_i = 0$ for all $1 \leq i \leq r$. So $\mu = 0$, a contradiction. Hence $k_1 = 1$. Since $k_i - k_j \in \mathbb{Z}$, for all $i$ and $j$, and since $1 = k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_r \geq 0$, we have that $k_i = 0$ or $1$, for all $2 \leq i \leq r$. We want to show that $k_i = 0$ for all $2 \leq i \leq r$. It suffices to show $k_2 = 0$. Simple computations shows that

$$a_2 = \begin{cases} -i + 2 - k_2 & \text{if } \sigma(\varepsilon_2) = \varepsilon_i \text{ for } 3 \leq i \leq r, \\ -2r + i + 1 - k_2 & \text{if } \sigma(\varepsilon_2) = -\varepsilon_i \text{ for } 3 \leq i \leq r, \\ -k_2 & \text{if } \sigma(\varepsilon_2) = \varepsilon_2, \\ -2r + 3 - k_2 & \text{if } \sigma(\varepsilon_2) = -\varepsilon_2. \end{cases}$$

Since $r \geq 2$ and $a_2 \in \mathbb{N}$, we have that $\sigma(\varepsilon_2) = \varepsilon_2$ and hence $k_2 = 0$. Thus $\mu = \varepsilon_1 = \varpi_1$.

($\Leftarrow$) By Proposition 3.2.3, we know if $\mu = \varpi_1$, then $\varphi(\sigma(\varpi_1 + \rho) - \rho - \varpi_1) > 0$ when $\sigma = 1$. \hfill $\Box$

Theorem 3.2.5. If $\mu \in P(\mathfrak{so}_{2r+1})$, then $m(\varpi_1, \mu) =$ \begin{cases} 1 & \text{if } \mu = 0 \text{ or } \mu \in W \cdot \varpi_1, \\ 0 & \text{otherwise}. \end{cases}
Proof. Recall that given \( \mu \in P(\mathfrak{so}_{2r+1}) \), there exists \( w \in W \) and \( \xi \in P_+(\mathfrak{so}_{2r+1}) \) such that \( w(\xi) = \mu \) and also recall that weight multiplicities are invariant under \( W \) (Propositions 3.1.20, 3.2.27 in [5]). Thus it suffices to consider \( \mu \in P_+(\mathfrak{so}_{2r+1}) \). Corollary 3.2.1 gives \( m(\bar{\alpha}, 0) = 1 \), while Theorem 3.2.4 implies \( m(\varpi_1, \varpi_1) = 1 \) and \( m(\varpi_1, \mu) = 0 \), whenever \( \mu \in P_+(\mathfrak{so}_{2r+1}) \setminus \{0, \varpi_1\} \). \( \square \)

3.3 The classical Lie algebras \( \mathfrak{sp}_{2r}(\mathbb{C}) \) and \( \mathfrak{so}_{2r}(\mathbb{C}) \)

In this section we consider the classical Lie algebras \( \mathfrak{sp}_{2r}(\mathbb{C}) \) and \( \mathfrak{so}_{2r}(\mathbb{C}) \) and prove:

**Theorem 3.3.1.** If \( \mathfrak{g} \) is the classical Lie algebra \( \mathfrak{sp}_{2r}(\mathbb{C}) \) (with \( r \geq 3 \)) or \( \mathfrak{so}_{2r}(\mathbb{C}) \) (with \( r \geq 4 \)) and \( \Delta \) denotes a set of simple roots of \( \mathfrak{g} \), then the weight \( \sum_{\alpha \in \Delta} \alpha \) is not a dominant integral weight of \( \mathfrak{g} \).

Proof. We follow the notation in [5]. In the case of \( \mathfrak{sp}_{2r} = \mathfrak{sp}_{2r}(\mathbb{C}) \), with \( r \geq 3 \), for each \( 1 \leq i \leq r-1 \), let \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \) and let \( \alpha_r = 2\varepsilon_r \). Then the set of simple roots is \( \Delta = \{\alpha_1, \ldots, \alpha_r\} \). The fundamental weights are defined by \( \varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \), for \( 1 \leq i \leq r \). Now notice \( \alpha_1 + \cdots + \alpha_r = \varepsilon_1 + \varepsilon_r = \varpi_1 - \varpi_{r-1} + \varpi_r \). Thus, the weight defined by the sum of the simple roots is not a dominant weight of \( \mathfrak{sp}_{2r} \), for \( r \geq 3 \).

In the case \( \mathfrak{so}_{2r} = \mathfrak{so}_{2r}(\mathbb{C}) \), with \( r \geq 4 \), for each \( 1 \leq i \leq r-1 \), let \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \) and let \( \alpha_r = \varepsilon_{r-1} + \varepsilon_r \). Then the set of simple roots is \( \Delta = \{\alpha_1, \ldots, \alpha_r\} \). The fundamental weights are defined by \( \varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \), if \( 1 \leq i \leq r-1 \), and \( \varpi_{r-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r) \) and \( \varpi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} + \varepsilon_r) \). Now notice \( \alpha_1 + \cdots + \alpha_r = \varepsilon_1 + \varepsilon_{r-1} = \varpi_1 - \varpi_{r-2} + \varpi_{r-1} + \varpi_r \). Thus, the weight defined by the sum of the simple roots is not a dominant weight of \( \mathfrak{so}_{2r} \), for \( r \geq 4 \). \( \square \)

3.4 The exceptional Lie algebras

We conclude this chapter by considering the exceptional simple Lie algebras over \( \mathbb{C} \) and proving:

**Theorem 3.4.1.** If \( \mathfrak{g} \) is an exceptional simple Lie algebra of type \( G_2, F_4, E_6, E_7, \) or \( E_8 \) and \( \Delta \) denotes a set of simple roots of \( \mathfrak{g} \), then the weight \( \sum_{\alpha \in \Delta} \alpha \) is not a
dominant integral weight of \( g \).

**Proof.** In each case we will describe, as in [11], the underlying vector space \( V \) and the root system \( \Phi \) as a subset of \( V \). In each case the root system will be a subspace of some \( \mathbb{R}^k = \{ \sum_{i=1}^k a_i e_i \} \), where \( \{ e_i : 1 \leq i \leq k \} \) is the standard orthonormal basis and the \( a_i \)'s are real numbers.

The underlying vector space of the exceptional Lie algebra \( G_2 \) is \( V = \{ v \in \mathbb{R}^3 | (v, e_1 + e_2 + e_3) = 0 \} \) and the root system is given by \( \Phi = \{ \pm (e_1 - e_2), \pm (e_2 - e_3), \pm (e_1 - e_3) \} \cup \{ \pm (2e_1 - e_2 - e_3), \pm (2e_2 - e_1 - e_3), \pm (2e_3 - e_1 - e_2) \} \}. The set of simple roots is \( \Delta = \{ \alpha_1, \alpha_2 \} \), where

\[
\alpha_1 = e_1 - e_2 \quad \text{and} \quad \alpha_2 = -2e_1 + e_2 + e_3.
\]

The fundamental weights, in terms of the simple roots, are

\[
\varpi_1 = 2\alpha_1 + \alpha_2 \quad \text{and} \quad \varpi_2 = 3\alpha_1 + 2\alpha_2.
\]

Observe that \( \alpha_1 + \alpha_2 = -\varpi_1 + \varpi_2 \), which is not a dominant integral weight of \( G_2 \).

The underlying vector space of the exceptional Lie algebra \( F_4 \) is \( V = \mathbb{R}^4 \) and the root system is given by \( \Phi = \{ \pm e_i \pm e_j | i < j \} \cup \{ \pm e_i \} \cup \{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \} \). The set of simple roots is \( \Delta = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \), where

\[
\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4), \\
\alpha_2 = e_4, \\
\alpha_3 = e_3 - e_4, \quad \text{and} \\
\alpha_4 = e_2 - e_3.
\]
The fundamental weights, in terms of the simple roots, are

\[
\begin{align*}
\varpi_1 &= 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 \\
\varpi_2 &= 3\alpha_1 + 6\alpha_2 + 4\alpha_3 + 2\alpha_4 \\
\varpi_3 &= 4\alpha_1 + 8\alpha_2 + 6\alpha_3 + 3\alpha_4 \\
\varpi_4 &= 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4
\end{align*}
\]

Observe that \(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \varpi_1 - \varpi_2 + \varpi_4\), which is not a dominant integral weight of \(F_4\).

The underlying vector space of the exceptional Lie algebra \(E_6\) is \(V = \{v \in \mathbb{R}^8 | (v, e_6 - e_7) = (v, e_7 + e_8) = 0\}\) and the root system is given by \(\Phi = \{\pm e_i \pm e_j | i < j \leq 5\} \cup \{\frac{1}{2} \sum_{i=1}^{8} (-1)^{n(i)} e_i \in V | \sum_{i=1}^{8} n(i) \text{ even}\}\). The set of simple roots is \(\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}\), where

\[
\begin{align*}
\alpha_1 &= \frac{1}{2} (e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1), \\
\alpha_2 &= e_2 + e_1, \\
\alpha_3 &= e_2 - e_1, \\
\alpha_4 &= e_3 - e_2, \\
\alpha_5 &= e_4 - e_3, \text{ and} \\
\alpha_6 &= e_5 - e_4.
\end{align*}
\]

The fundamental weights, in terms of the simple roots, are

\[
\begin{align*}
\varpi_1 &= \frac{1}{3} (4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) \\
\varpi_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\
\varpi_3 &= \frac{1}{3} (5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6) \\
\varpi_4 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 \\
\varpi_5 &= \frac{1}{3} (4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6) \\
\varpi_6 &= \frac{1}{3} (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6)
\end{align*}
\]
Observe that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = \varpi_1 + \varpi_2 - \varpi_4 + \varpi_6$, which is not a dominant integral weight of $E_6$.

The underlying vector space of the exceptional Lie algebra $E_7$ is $V = \{v \in \mathbb{R}^8|(v, e_7 + e_8) = 0\}$, and the root system is given by $\Phi = \{\pm e_i \pm e_j|i < j \leq 6\} \cup \{\pm(e_7 - e_8)\} \cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)}e_i \in V|\sum_{i=1}^{8} n(i) \text{ even}\}$. The set of simple roots is $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, where

$$
\begin{align*}
\alpha_1 &= \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1), \\
\alpha_2 &= e_2 + e_1, \\
\alpha_3 &= e_2 - e_1, \\
\alpha_4 &= e_3 - e_2, \\
\alpha_5 &= e_4 - e_3, \\
\alpha_6 &= e_5 - e_4, \text{ and} \\
\alpha_7 &= e_6 - e_5.
\end{align*}
$$

The fundamental weights, in terms of the simple roots, are

$$
\begin{align*}
\varpi_1 &= 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \\
\varpi_2 &= \frac{1}{2}(4\alpha_1 + 7\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7), \\
\varpi_3 &= 3\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7, \\
\varpi_4 &= 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7, \\
\varpi_5 &= \frac{1}{2}(6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7), \\
\varpi_6 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7, \\
\varpi_7 &= \frac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7).
\end{align*}
$$

Observe that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = \varpi_1 + \varpi_2 - \varpi_4 + \varpi_6$, which is not a dominant integral weight of $E_7$.

The underlying vector space of the exceptional Lie algebra $E_8$ is $V = \mathbb{R}^8$ and the root system is given by $\Phi = \{\pm e_i \pm e_j|i < j \leq 6\} \cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)}e_i \in V|\sum_{i=1}^{8} n(i) \text{ even}\}$. The set of simple roots is $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$, where
The fundamental weights, in terms of the simple roots, are

\[
\begin{align*}
\varpi_1 &= 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8 \\
\varpi_2 &= 5\alpha_1 + 8\alpha_2 + 10\alpha_3 + 15\alpha_4 + 12\alpha_5 + 9\alpha_6 + 6\alpha_7 + 3\alpha_8 \\
\varpi_3 &= 7\alpha_1 + 10\alpha_2 + 14\alpha_3 + 20\alpha_4 + 16\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8 \\
\varpi_4 &= 10\alpha_1 + 15\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8 \\
\varpi_5 &= 8\alpha_1 + 12\alpha_2 + 16\alpha_3 + 24\alpha_4 + 20\alpha_5 + 15\alpha_6 + 10\alpha_7 + 5\alpha_8 \\
\varpi_6 &= 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8 \\
\varpi_7 &= 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 8\alpha_6 + 6\alpha_7 + 3\alpha_8 \\
\varpi_8 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8
\end{align*}
\]

Observe that \(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = \varpi_1 + \varpi_2 - \varpi_4 + \varpi_8\), which is not a dominant integral weight of \(E_8\). \(\Box\)
BIBLIOGRAPHY


PAMELA E. HARRIS

CURRICULUM VITAE

PERSONAL DATA

Place of Birth: Guadalajara, Jalisco Mexico

Nationality: US Citizen

Language: Native Spanish speaker, Fluent in English

RESEARCH INTEREST

Representation theory of Lie algebras.

EDUCATION

2012 PhD in Mathematics from the University of Wisconsin - Milwaukee.

Thesis: Combinatorial Problems Related to Kostant’s Weight Multiplicity Formula

Advisor: Jeb F. Willenbring

2008 MS in Mathematics from the University of Wisconsin - Milwaukee.

2005 BS in Mathematics from Marquette University in Milwaukee.

2003 AS & AA from Milwaukee Area Technical College in Milwaukee.

EMPLOYMENT

July 2012 Assistant Professor, Department of Mathematics at United States Military Academy-West Point.

2011-2012 Faculty in the Department of Mathematics, Statistics, and Computer Science at Marquette University.

2011 Math Instructor in the Educational Opportunity Program, at Marquette University.

2006-2011 Graduate Teaching Assistant in the Mathematical Sciences Department at UW-Milwaukee.
Publications and Preprints


Selected Talks

Apr. 2012 Underrepresented Students in Topology and Algebra Symposium, University of Iowa–Conference speaker.


Jan. 2012 Bridgewater State University, Bridgewater, MA – Mathematics Presentation.


Nov. 2011 University of Wisconsin-Milwaukee – Algebra Seminar.


Apr. 2011 University of Wisconsin-Milwaukee – Algebra Seminar.

May 2010 University of Wisconsin-Milwaukee – Algebra Seminar.

Conferences and Symposiums

2012 Underrepresented Students in Topology and Algebra Research Symposium (USTARS)
University of Iowa, Iowa City, Iowa April 2012.

2012 Infinite Possibilities Conference
University of Maryland, Baltimore County, March 30-31, 2012.

2012 Joint Mathematics Meetings
Hynes Convention Center, Boston, MA, January 4-7, 2012.

2011 40 Years and Counting: AWM’s Celebration of Women in Mathematics

2011 Southeastern Lie Theory Workshop: Finite and Algebraic Groups
University of Virginia, Charlottesville, Virginia, June 1-4, 2011.

2011 Underrepresented Students in Topology and Algebra Research Symposium (USTARS)
University of Iowa, Iowa City, Iowa April 1-3, 2011.

2011 Lie Theory and Its Applications-Conference in Honor of Nolan Wallach
University of California, San Diego, California March 18-21, 2011.

2011 Women in Mathematics Symposium
Institute for Pure and Applied Mathematics Los Angeles, California, February 24 - 26, 2011.

Honors and Distinctions
2011-2012 Recipient of *Arnold L. Mitchem Dissertation Fellowship* at Marquette University. This fellowship is intended to increase the presence of underrepresented ethnic groups by supporting doctoral candidates in completing their final academic requirement, the dissertation.

2009-2012 Recipient of *Advanced Opportunity Program Fellowship*. This fellowship is awarded to graduate students who are members of groups underrepresented in graduate study.

2011 Recipient of *Hispanic Professionals of Greater Milwaukee (HPGM) graduate student scholarship*. HPGM is one of the few organizations in Wisconsin aimed at offering scholarship to Hispanic Professionals strictly for graduate degrees. Award recipients were chosen based on academic record, plans and career goals, and community service.

2006-2011 Recipient of *Graduate Assistantship in Areas of National Need Fellowship*. This fellowship is awarded to students with excellent records who demonstrate financial need and plan to pursue the highest degree available in their course of study in a field designated as an area of national need.

2011, 2010 Recipient of *Mark Lawrence Teply Award* to support the purchase of mathematical research books.

2009 Recipient of *Ernst Schwandt Teaching Award* for outstanding teaching performance.

**Teaching Experience**

2011-2012 Faculty, Marquette University, Mathematics, Statistics, and Computer Science Department. Full responsibility for math course:

**Spring 2012:** Theory of Numbers (U/G)-Integers, unique factorization theorems, arithmetic functions, theory of congruences, quadratic residues, partition theory.
2011 Mathematics instructor, Student Support Services in the Educational Opportunity Program at Marquette University. Student Support Services provides an opportunity for students from minority groups and low-income families to attend Marquette University. Incoming freshmen attend a 5 week intensive summer program in preparation for fall semester courses. I had full responsibility for the math curriculum, which covered courses in Probability and Statistics, Functions and Graphs, and Calculus.

2011 Mathematics instructor, Upward Bound Math & Science Pre-College Division of the Educational Opportunity Program at Marquette University. Upward Bound Math & Science prepares high school students who are low-income and first-generation college students to enter and successfully complete college. I had full responsibility for the high school seniors Pre-calculus course in their 5 week intensive summer program. Since this program is designed for high school students with a serious interest in math and science careers, a component of the course involved a career project. In this project students researched college requirements in pursuing desired STEM careers.

2006-2011 Teaching Assistant, UWM Mathematical Sciences Department.

Had full responsibility for a variety of college level courses:

**Fall 2010:** Calculus and Analytic Geometry II- Applications of integration, techniques of integration; infinite sequences and series; parametric equations, conic sections, and polar coordinates.

**Fall 2009:** Calculus and Analytic Geometry I- Limits, derivatives, and graphs of algebraic, trigonometric, exponential, and logarithmic functions; antiderivatives, the definite integral, and the fundamental theorem of calculus, with applications.

**Fall 2008:** Intermediate Algebra- Algebraic techniques with polynomials, rational expressions, equations and inequalities, exponen-
tial and logarithmic functions, rational exponents, conic sections, systems of linear equations.

Assisted with the following courses:

**Fall 2007:** Discussion leader for Survey in Analytic Calculus-A one-semester survey with applications to business administration, economics, and non-physical sciences. Topics include coordinate systems, equations of curves, limits, differentiation, integration, applications.

**Spring 2007:** Algebraic Structures for Elementary Education Majors- Topics for K-8 teachers. Basic patterns and rules that govern number systems, geometric transformations, and manipulation of algebraic expressions.


**Service**

**2011 Poster presenter,** for the University of Wisconsin-Milwaukee’s Open House, October 30th, 2011.

**2011 Volunteer Tutor,** Office of Student Educational Services at Marquette University. This involves a four hour per week commitment, which includes drop-by tutoring and scheduled tutoring sessions for Calculus.

**2009, 2010 Graduate Student Recruitment,** UW-Milwaukee. Participated in lunch outings with prospective graduate students, as well as provided tours of the campus.
Memberships

American Mathematical Society
Association for Women in Mathematics
Mathematical Association of America