Topology, homology and quantum mechanics

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Baylor – 9/27/12
Outline

1. Topology in QM
2. Quantum statistics
3. Statistics on networks

Wills building Bristol University
Outline

1. Topology in QM
2. Quantum statistics
3. Statistics on networks
Quantum mechanics

Two slit experiment – wave-particle duality

Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \psi(r, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right) \psi(r, t) \]

\[ |\psi|^2 \] probability density for particle.
Quantum mechanics

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Feynman path integral

**Action** $S(p) = \int_0^T \frac{mx^2}{2} - V(x) \, dt$ – the integral of the classical Lagrangian along path $p$.

Contribution of $p$ to probability amplitude for transition between two states is $e^{i \hbar S(p)}$. Total probability amplitude obtained by summing over all possible paths connecting $A$ & $B$. 
Feynman path integral

**Action** $S(p) = \int_0^T \frac{m\dot{x}^2}{2} - V(x) \, dt$ – the integral of the classical Lagrangian along path $p$.

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Formal solution to Schrödinger eqn at time $T$ is $\exp(-\frac{i}{\hbar}HT)\psi_0$

where $H = -\frac{\hbar^2}{2m} \nabla^2 + V(x)$. Transition amplitude from $\psi_0$ to final state $\psi_f$ is $\int \psi_f^* \exp(-\frac{i}{\hbar}HT)\psi_0$. Breaking $T$ into infinitesimal time steps leads to the path integral.
Aharonov-Bohm effect
Aharonov-Bohm effect

Turn on magnetic field $\mathbf{B}$ in region inaccessible to particle.
Aharonov-Bohm effect

Path integral formulation.

\[ B = \nabla \times A \]

Contribution from path enclosing \( B \) acquires a phase \( e^{i\theta} \) where \( \theta = \oint A \, ds \), as \( A \) cannot be zero everywhere on path enclosing \( B \).
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Wavefunction $\psi(x)$ for $x \in \mathbb{R}^n$.
In general we are free to change $\psi$ by a phase factor.

- $\psi(x) \rightarrow e^{i\phi(x)}\psi(x)$
- $p \rightarrow -i\hbar \nabla + \hbar \nabla \phi$

Let $\phi(x) = \int_0^x A \cdot ds$ where $\nabla \times A = 0$
Quantum statistics

Two particles in space $X$.

**Alternative approaches:**

- Quantize $X^2$ and restrict Hilbert space to symmetric and anti-symmetric subspaces.

\[ \psi(x_1, x_2) = \pm \psi(x_2, x_1) \]  

Bose-Einstein/Fermi-Dirac statistics.
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- Quantize $X^2$ and restrict Hilbert space to symmetric and anti-symmetric subspaces.

$$\psi(x_1, x_2) = \pm \psi(x_2, x_1) \quad (1)$$

Bose-Einstein/Fermi-Dirac statistics.

- Treat particles as indistinguishable, $\psi(x_1, x_2) \equiv \psi(x_2, x_1)$.

Quantize configuration space,

$$C_2(X) = (X^2 - \Delta) / S_2 \quad (2)$$
Bose-Einstein and Fermi-Dirac statistics

In $\mathbb{R}^3$ using the relative coordinate, at a constant separation the configuration space $C_2(\mathbb{R}^3)$ is the projective plane.

Exchanging particles corresponds to traveling around a closed loop $p$ in $C_2$. On the projective plane $p$ is not contractible but $p^2$ is contractible. To associate a phase $e^{i\theta}$ to $p$ requires $(e^{i\theta})^2 = 1$ or $e^{i\theta} = \pm 1$. Quantizing $C_2$ with a phase $-1$ associated to exchange paths corresponds to Fermi-Dirac statistics while a phase $+1$ corresponds to Bose-Einstein statistics.
Fermi-Dirac statistics are only consistent with multi-particle states constructed from *different* single-particle states. The electron has Fermi-Dirac statistics ⇒
Fermi-Dirac statistics are only consistent with multi-particle states constructed from *different* single-particle states. The electron has Fermi-Dirac statistics $\Rightarrow$ chemistry!
Anyon statistics

Pair of indistinguishable particles in $\mathbb{R}^2$.

Any phase $e^{i\theta}$ can be associated with a primitive exchange path.
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- Relative position coordinate $x \in \mathbb{R}^2 \setminus 0$.
- Topology of configuration space the same as puncturing space with a magnetic field in A-B experiment.
- Exchange paths; closed loops about $0$.
- A phase change around loops can be encoded in a vector potential.

Any phase $e^{i\theta}$ can be associated with a primitive exchange path.
Fractional quantum Hall effect

Hall effect (1879)
Fractional quantum Hall effect

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Quantum Hall effect (1975)
Low temp & large $B$.

Conductance quantized $E_n = \frac{\hbar e B}{m} (n + 1/2)$. 
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**Hall effect** (1879)

![Hall effect diagram]

**Quantum Hall effect** (1975)
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**Fractional quantum Hall effect** (Tsui and Strömer 1982)
Gallium arsenide.
Integers can be replaced with rational numbers $p/q$.
Explained using Laughlin wavefn, *composite particles with anyon statistics.*
Configuration space of $n$ particles

$X$ one particle configuration space.

**Definition**

Configuration space of $n$ indistinguishable particles in $X$,

$$C_n(X) = (X^n - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \ldots, x_n | x_i = x_j \text{ for some } i \neq j\}$. 

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- $\pi_1(C_n(\mathbb{R}^d)) = S_n$ for $d \geq 3$ abelianization $H_1(C_n(\mathbb{R}^d)) = \mathbb{Z}/2$
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- \( \pi_1(C_n(\mathbb{R}^2)) = B_n \) braid group of \( n \) strands, \( H_1(C_n(\mathbb{R}^2)) = \mathbb{Z} \).
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- $\pi_1(C_n(\mathbb{R}^2)) = B_n$ braid group of $n$ strands, $H_1(C_n(\mathbb{R}^2)) = \mathbb{Z}$.
- $\pi_1(C_n(\mathbb{R})) = 1$
What happens on a graph where the underlying space has arbitrarily complex topology?
Quantum graph

Graph $G$
Vertices $\{1, \ldots, v\}$
Edges $\mathcal{E} = \{(i, j)\}$

- Adjacency matrix $A$, $A_{jk} = 1$ if $j \sim k$, $A_{jk} = 0$ otherwise.
- $v_j$ valency of vertex $j$.
- For many particles on a metric graph boundary conditions are hard to incorporate as particles become coincident.
|ψ⟩ = \sum_{j=1}^{\nu} \psi_j |j⟩ \text{ in Hilbert space } \mathbb{C}^\nu.

Hamiltonian \( H \), \( \nu \times \nu \) Hermitian matrix.

e.g. discrete Laplacian \( H = A - D \), where \( D = \text{diag}\{v_1, \ldots, v_\nu\} \).

Transitions possible between adjacent vertices.
Single particle QM on combinatorial graph

- $|\psi\rangle = \sum_{j=1}^{V} \psi_{j} |j\rangle$ in Hilbert space $\mathbb{C}^{V}$.
- Hamiltonian $H$, $V \times V$ Hermitian matrix.
  
  e.g. discrete Laplacian $H = A - D$, where $D = \text{diag}\{v_1, \ldots, v_V\}$.

Transitions possible between adjacent vertices.

Gauge potential

$\Omega$ is a $V \times V$ real antisymmetric matrix, $\Omega_{ij} = 0$ if $(ij) \notin E$.

Incorporate gauge potential in Hamiltonian, $H \rightarrow H^{\Omega}$ where

$H^{\Omega}_{ij} = e^{i\Omega_{ij}} H_{ij}$. 
Gauge transformation

Let $U = \text{diag}\{e^{i\theta_1}, \ldots, e^{i\theta_v}\}$,

$$|\psi\rangle \rightarrow U|\psi\rangle$$

$$H \rightarrow UHU^*$$

A gauge potential $\Omega$ is \textit{trivial} if $H^\Omega = UHU^*$ for some $U$, i.e. $H^\Omega$ is generated by a gauge transformation. For a trivial gauge potential

$$\Omega_{jk} = \begin{cases} 
\theta_k - \theta_j + 2\pi M_{jk}, & j \sim k, \\
0, & \text{otherwise},
\end{cases} \quad (3)$$

where $M$ is an antisymmetric integer matrix.
Let $T$ be a *spanning tree* of $G$, a connected subgraph whose cycles are all self-retracing.
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Index every edge not in $T$ with $\phi \in \{1, \ldots, f\}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tree_diagram.png}
\caption{Example of a spanning tree $T$ of a graph $G$.
\end{figure}
Let $T$ be a spanning tree of $G$, a connected subgraph whose cycles are all self-retracing.

Index every edge not in $T$ with $\phi \in \{1, \ldots, f\}$. $c_\phi(\star)$ denotes fundamental cycle using edge $e_\phi$ based at $\star$. $\pi_1(G)$ generated by $\{c_\phi(\star)\}$. $H_1(G) = \pi_1(G)/(c_ic_j \sim c_jc_i)$ abelianized homotopy group.
$C_2(G) = (\mathcal{N}^2 - \Delta)/S_2$ can be regarded as graph $G_2$. An edge in $G_2$ corresponds to keeping one particle fixed while the other is moved along an edge of $G$. $A_2$ is adjacency matrix of $G_2$.

**Example:** $K_3$

$G = K_3$

![Diagram of graphs $G$ and $G_2$](image)
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![Graphs](image)
Two-particle graph

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\[ G = K_3 \]

![Diagram of two graphs: G and G_2.](image)
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\[ G = K_3 \]

\( \begin{align*}
G & \quad \quad \quad \quad G_2 \\
1 & \quad \quad \quad \quad (1,2)
\end{align*} \]

\[ \begin{align*}
2 & \quad \quad \quad \quad (1,3)
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Definition

*Contactable cycles* are cycles on $G_2$ that are not self-retracing but are metrically contractible for two particles on $\Gamma$.

Pairs of disjoint edges in $G$ generate contractible cycles.
Contractible cycles

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Pairs of disjoint edges in $G$ generate contractible cycles. Let $\mathcal{K}(G_2) \subset \mathcal{C}(G_2)$ be set of contractible cycles mod self-retracing.

\[
\mathcal{C}(G_2)/\mathcal{K}(G_2) \cong \pi_1(C_2(\Gamma))
\]

\[
H_1(C_2(\Gamma)) = \pi_1(C_2(\Gamma))/(c_i c_j \sim c_j c_i)
\]
Example: Lasso graph

\[ H_1(G_2) \sim \mathbb{Z}^2 \]
Example: Lasso graph

$H_1(G_2) \sim \mathbb{Z}^2$
Topological gauge potentials

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$\Omega_2$ is a *topological gauge potential* for $G_2$ if $\Omega_2(c) = 0 \mod 2\pi$ for every contractible loop $c$ on $G_2$. 
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Topology and quantum mechanics
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- $\omega := (\omega_1, \ldots, \omega_{f_2})$ where $\omega_{\phi_2}$ is flux of $\Omega_2$ through $c_{\phi_2}$. 
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- $\omega := (\omega_1, \ldots, \omega_{f_2})$ where $\omega_{\phi_2}$ is flux of $\Omega_2$ through $c_{\phi_2}$.
- Constraints from contractible loops $R\omega = 0 \mod 2\pi$, where $R$ is integer matrix.
Aharonov-Bohm phases and two-body phases

Aharonov-Bohm phase

A free statistics phase that corresponds to taking one particle around a cycle in $G$ with the other fixed.

An Aharonov-Bohm phase can be produced by threading a cycle of $G$ with a magnetic flux.
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Free phases not parameterized by Aharov-Bohm phases and discrete phases are *two body phases* which characterize Abelian graph statistics.
1 A-B phase.

1 free 2-body phase.

\[
\frac{(|E| - 1)(|E| - 2)}{2}
\]
free 2-body phases.

3 A-B & 1 free 2-body phase.
Non-planar graphs

Theorem (Kuratowski)

Every nonplanar graph – a graph which cannot be drawn in the plane without edges crossing – contains $K_5$ or $K_{3,3}$ as a subgraph or contains a subgraph that is homeomorphic to $K_5$ or $K_{3,3}$.
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6 A-B phases & 1 discrete phase of 0 or $\pi$.

4 A-B phases & 1 discrete phase of 0 or $\pi$.

12 A-B, 6 2-body & 2 phases of 0 or $\pi$. 
Figure: Configuration space graph $G_2$ of $K_{3,3}$, edges shown as solid lines are in the spanning subtree with root $(1, 2)$. Open edges are joined left to right and top to bottom.
Counting phases

Ko & Park (2011)

\[ H_1(C_n(G)) = \mathbb{Z}^{N_1(G)+N_2(G)+N_3(G)+\beta_1(G)} \oplus \mathbb{Z}_2^{N'_3(G)} \]  \hspace{1cm} (4)

- **\( N_1(G) \)** sum over one cuts \( j \) of \( N(n, G, j) \).

\[ N(n, G, j) = \binom{n + \mu_j - 2}{n-1}(\mu(j) - 2) - \binom{n + \mu_j - 2}{n} - (v_j - \mu_j - 1) \]

\( \mu_j \) \# components of \( G \setminus j \).

- **\( N_2(G) \)** sum over two connected components of \( G \).
- **\( N_3(G) \)** \# 3-connected planar components of \( G \).
- **\( N'_3(G) \)** \# 3-connected non-planar components of \( G \).
- **\( \beta_1(G) \)** \# of loops of \( G \).
3-connected components

Lemma

A 3-connected graph has a single 2-body exchange phase.

Sketch proof:
We already know lemma holds for $K_4$ the simplest 3-connected graph.
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![Diagram of a 3-connected graph and a 2-body exchange phase](image)
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Conclusions

- Formulated multi-particle QM on combinatorial graphs.
- Vector field used to incorporate statistics in Hamiltonian.
- Demonstrated new forms of quantum statistics for graphs.
- Graph statistics allow multiple anyon phases and discrete phases.
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Outlook

- Metric graphs
- Non-abelian statistics
- Many body properties: transport, analogue of fractional quantum Hall effect, Hartree-Fock.
- Physical mechanism for exotic graph statistics.
References


Thank You

What you were missing in Bristol