Anyons on networks

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Baylor - 2/26/20

Supported by Simons Foundation collaboration grant 354583.
Quantum statistics

Single particle space $X$.

**Two particle statistics - alternative approaches:**

- Quantize $X \times 2$ and restrict Hilbert space to the symmetric or anti-symmetric subspace.

$$\psi(x_1, x_2) = \pm \psi(x_2, x_1)$$

Bose-Einstein/Fermi-Dirac statistics.
Quantum statistics

Single particle space $X$.

**Two particle statistics - alternative approaches:**

- Quantize $X^2$ and restrict Hilbert space to the symmetric or anti-symmetric subspace.
  
  $$\psi(x_1, x_2) = \pm \psi(x_2, x_1)$$

  Bose-Einstein/Fermi-Dirac statistics.

- (Leinaas and Myrheim ‘77)
  Treat particles as indistinguishable, $\psi(x_1, x_2) \equiv \psi(x_2, x_1)$.
  Quantize two particle configuration space.
Aharonov-Bohm effect
Aharonov-Bohm effect

Turn on magnetic field $\mathbf{B}$ in region inaccessible to particle.
Aharonov-Bohm effect

Path integral formulation.

\[ \mathbf{B} = \nabla \times \mathbf{A} \]

Contribution from path enclosing \( \mathbf{B} \) acquires a phase \( e^{i \theta} \) where \( \theta = \oint \mathbf{A} \cdot d\mathbf{s} \), as \( \mathbf{A} \) cannot be zero everywhere on path enclosing \( \mathbf{B} \).
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Bose-Einstein and Fermi-Dirac statistics

Two indistinguishable particles in $\mathbb{R}^3$. At constant separation relative coordinate lies on projective plane.

Exchanging particles corresponds to rotating relative coordinate around closed loop $p$.

$p$ is not contractible but $p^2$ is contractible.

A phase factor $e^{i\theta}$ associated to $p$ requires $(e^{i\theta})^2 = 1$.

Quantizing configuration space with $\theta = \pi$ corresponds to *Fermi-Dirac statistics* and $\theta = 0$ to *Bose-Einstein statistics*. 
Anyon statistics

Pair of indistinguishable particles in $\mathbb{R}^2$.

- Particles not coincident.
- Relative position coordinate in $\mathbb{R}^2 \setminus \mathbf{0}$.
- Exchange paths are closed loops about $\mathbf{0}$ in relative coordinate.
- As in the Aharonov-Bohm effect any phase factor $e^{i\theta}$ can be associated with a primitive path enclosing $\mathbf{0}$.
Braid group

For \( n \) indistinguishable particles on \( \mathbb{R}^2 \), \( \sigma_j \) exchanges adjacent particles \( j = 1, \ldots, n - 1 \).

Relations \( \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \) for \( j = 1, \ldots, n - 2 \).

Generates \( B_n \) braid group on \( n \) strands.
A potted history of anyons

(77) Leinaas and Myrheim - quantum mechanics on configuration spaces.

(82) Wilczek - anyons on surfaces.

(82) Tsui and Strömer - fractional quantum Hall effect.

(83) Laughlin wavefunction.

(05) Sarma, Freedman and Nayek - topologically protected qubits.

(08) Kitaev - network models of topological quantum computation.
**Definition**

Configuration space of $n$ indistinguishable particles in $X$,

$$C_n(X) = (X^n - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \ldots, x_n|x_i = x_j \text{ for some } i \neq j\}$. 
Definition

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$$C_n(X) = (X\times^n - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \ldots, x_n|x_i = x_j$ for some $i \neq j\}$.

1st homology groups of $C_n(\mathbb{R}^d)$:

- $H_1(C_n(\mathbb{R}^d)) = \mathbb{Z}_2$ for $d \geq 3$.
  2 abelian irreps. corresponding to Bose-Einstein & Fermi-Dirac statistics.
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  Abelian irreps. generated by $e^{i\theta}$ – anyon statistics.
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Configuration space of $n$ indistinguishable particles in $X$,

$$C_n(X) = (X^\times n - \Delta_n)/S_n$$

where $\Delta_n = \{x_1, \ldots, x_n|x_i = x_j \text{ for some } i \neq j\}$.

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  Abelian irreps. generated by $e^{i\theta}$ – anyon statistics.

- $H_1(C_n(\mathbb{R})) = 1$
  particles cannot be exchanged.
What happens on a network where the underlying space has arbitrarily complex topology?
Quantum graphs model phenomena associated with complex quantum systems.

- Free electrons in organic molecules
- Superconducting networks
- Photonic crystals
- Nanotechnology
- Quantum chaos
- Anderson localization
Exchanging indistinguishable particles on a $Y$
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Exchanging indistinguishable particles on a $Y$. 

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Graph connectivity

- Given a connected graph $\Gamma$ a \textit{k-cut} is a set of $k$ vertices whose removal makes $\Gamma$ disconnected.
- $\Gamma$ is \textit{k-connected} if the minimal cut is size $k$.
- \textbf{Theorem} (Menger) For a $k$-connected graph there exist at least $k$ independent paths between every pair of vertices.

\textbf{Example:}

![Graph example image]
Graph connectivity

- Given a connected graph $\Gamma$ a **$k$-cut** is a set of $k$ vertices whose removal makes $\Gamma$ disconnected.
- $\Gamma$ is **$k$-connected** if the minimal cut is size $k$.
- **Theorem** (Menger) For a $k$-connected graph there exist at least $k$ independent paths between every pair of vertices.

**Example:**

![Two cut](image)
Graph connectivity

Given a connected graph $\Gamma$ a $k$-cut is a set of $k$ vertices whose removal makes $\Gamma$ disconnected.

$\Gamma$ is $k$-connected if the minimal cut is size $k$.

Theorem (Menger) For a $k$-connected graph there exist at least $k$ independent paths between every pair of vertices.

Example:

Two independent paths joining $u$ and $v$. 
Features of anyon statistics on networks

**3-connected graphs:** statistics only depend on whether the graph is planar (Anyons) or non-planar (Bosons/Fermions).
Features of anyon statistics on networks

3-connected graphs: statistics only depend on whether the graph is planar (Anyons) or non-planar (Bosons/Fermions).

A planar lattice with a small section that is non-planar is locally planar but has Bose/Fermi statistics.
Features of anyon statistics on networks

**2-connected graphs:** statistics complex but independent of the number of particles.
Features of anyon statistics on networks

2-connected graphs: statistics complex but independent of the number of particles.

For example, one could construct a chain of 3-connected non-planar components where particles behave with alternating Bose/Fermi statistics.
Features of anyon statistics on networks

1-connected graphs: statistics depend on no. of particles $n$. 

Example, star with $E$ edges.

Number of anyon phases:

$$(n + E - 2E - 1)(E - 2) - (n + E - 2E - 2) + 1.$$
Features of anyon statistics on networks

1-connected graphs: statistics depend on no. of particles $n$. Example, star with $E$ edges.

![Diagram of a star with 6 edges](image)

No. of anyon phases

$$\left( \binom{n + E - 2}{E - 1} (E - 2) - \binom{n + E - 2}{E - 2} \right) + 1.$$
Basic cases

Exchange of 2 particles around loop $c$; one free phase $\phi_{c2}$.

Exchange of 2 particles at Y-junction; one free phase $\phi_Y$. 
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$$\Gamma$$

1

2

3

(12)

(23)

$$C_2(\Gamma)$$

(13)

(23)

1

2

(12)

(23)

3

4

(13)

(23)

(12)

(34)

(14)

(24)
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\[ \begin{align*} 
\Gamma & \quad 3 \quad (12) \quad (23) \\
C_2(\Gamma) & \quad (13) \quad (23) \\
\end{align*} \]
Basic cases

Exchange of 2 particles around loop $c$; one free phase $\phi_{c2}$.

Exchange of 2 particles at Y-junction; one free phase $\phi_Y$. 

1. $\Gamma$
   - 1
   - 2
   - (12)
   - (23)

2. $C_2(\Gamma)$
   - 1
   - 2
   - (13)
   - (23)

3. $\Gamma$
   - 1
   - 2
   - (12)
   - (13)
   - (23)
   - (14)
   - (24)
   - (34)
Basic cases

Exchange of 2 particles around loop $c$; one free phase $\phi_{c2}$.

Exchange of 2 particles at Y-junction; one free phase $\phi_Y$. 

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Identify three 2-particle cycles:

(i) Rotate both particles around loop $c$; phase $\phi_{c,2}$.

(ii) Exchange particles on Y-subgraph; phase $\phi_Y$.

(iii) Rotate one particle around loop $c$ other particle at vertex 1; $(1, 2) \rightarrow (1, 3) \rightarrow (1, 4) \rightarrow (1, 2)$, phase $\phi^1_{c,1}$.

Relation from contactable 2-cell $\phi_{c,2} = \phi^1_{c,1} + \phi_Y$. 
Identify three 2-particle cycles:

(i) Rotate both particles around loop \(c\); phase \(\phi_{c,2}\).

(ii) Exchange particles on \(Y\)-subgraph; phase \(\phi_Y\).

(iii) Rotate one particle around loop \(c\) other particle at vertex 1;
\( (1, 2) \rightarrow (1, 3) \rightarrow (1, 4) \rightarrow (1, 2) \), phase \(\phi_{c,1}^1\).

Relation from contactable 2-cell \(\phi_{c,2} = \phi_{c,1}^1 + \phi_Y\).
Let $c$ be a loop. What is the relation between $\phi_{c,1}^u$ and $\phi_{c,1}^v$?

(a) $u$ and $v$ joined by path disjoint from $c$.
\[
\phi_{c,1}^u = \phi_{c,1}^v \text{ as exchange cycles homotopy equivalent.}
\]

(b) $u$ and $v$ only joined by paths through $c$.
Two lasso graphs so $\phi_{c,2} = \phi_{c,1}^u + \phi_{Y_1}$ & $\phi_{c,2} = \phi_{c,1}^v + \phi_{Y_2}$.
Hence $\phi_{c,1}^u - \phi_{c,1}^v = \phi_{Y_2} - \phi_{Y_1}$.
Let $c$ be a loop. What is the relation between $\phi_{c,1}^u$ and $\phi_{c,1}^v$?

\[ (a) \quad Y_1 \quad c \quad Y_2 \]
\[ u \quad v \]

\[ (b) \quad Y_1 \quad c \quad Y_2 \]
\[ u \quad v \]

(a) $u$ and $v$ joined by path disjoint from $c$.
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Hence $\phi_{c,1}^u - \phi_{c,1}^v = \phi_{Y_2} - \phi_{Y_1}$.

- Relations between phases involving $c$ encoded in phases $\phi_Y$.
$H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where $A$ determined by $Y$-cycles.
- In (a) we have a $B$ subgraph & using (b) also $\phi_{Y_1} = \phi_{Y_2}$. 

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Anyons on networks
3-connected graphs

The prototypical 3-connected graph is a *wheel* $W^k$.

$W^5$

**Theorem (Wheel theorem)**

Let $\Gamma$ be a simple 3-connected graph different from a wheel. Then for some edge $e \in \Gamma$ either $\Gamma \setminus e$ or $\Gamma / e$ is simple and 3-connected.

- $\Gamma \setminus e$ is $\Gamma$ with the edge $e$ removed.
- $\Gamma / e$ is $\Gamma$ with $e$ contracted to identify its vertices.
Lemma

For 3-connected simple graphs all phases $\phi_Y$ are equal up to a sign.

Sketch proof. The lemma holds on $K_4$ (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases $\phi_Y$ are the same as an original phase.

Adding an edge: $\Gamma \cup e$
Lemma

For 3-connected simple graphs all phases $\phi_{\mathcal{Y}}$ are equal up to a sign.

Sketch proof. The lemma holds on $K_4$ (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases $\phi_{\mathcal{Y}}$ are the same as an original phase.

Adding an edge: $\Gamma \cup e$

Using 3-connectedness identify independent paths in $\Gamma$ to make $\mathcal{B}$. Then $\phi_{\mathcal{Y}} = \phi_{\mathcal{Y}}$. 
**Lemma**

For 3-connected simple graphs all phases $\phi_Y$ are equal up to a sign.

**Sketch proof.** The lemma holds on $K_4$ (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases $\phi_Y$ are the same as an original phase.

**Vertex expansion:** Split vertex of degree $\geq 3$ into two vertices $u$ and $v$ joined by a new edge $e$. 

![Diagram](image)
Lemma

For 3-connected simple graphs all phases $\phi_Y$ are equal up to a sign.

Sketch proof. The lemma holds on $K_4$ (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases $\phi_Y$ are the same as an original phase.

Vertex expansion: Split vertex of degree $> 3$ into two vertices $u$ and $v$ joined by a new edge $e$.

Using 3-connectedness identify independent paths in $\Gamma$ to make $B$. Then $\phi_Y = \phi_Y$. 
Theorem

For a 3-connected simple graph, $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where $A = \mathbb{Z}_2$ for non-planar graphs and $A = \mathbb{Z}$ for planar graphs.
Theorem

For a 3-connected simple graph, $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$, where $A = \mathbb{Z}_2$ for non-planar graphs and $A = \mathbb{Z}$ for planar graphs.

Proof.

- For $K_5$ and $K_{3,3}$ every phase $\phi_Y = 0$ or $\pi$. By Kuratowski’s theorem a non-planar graph contains a subgraph which is isomorphic to $K_5$ or $K_{3,3}$. 
Theorem

For a 3-connected simple graph, $H_1(C_2(Γ)) = \mathbb{Z}^{β_1}(Γ) \oplus A$, where $A = \mathbb{Z}_2$ for non-planar graphs and $A = \mathbb{Z}$ for planar graphs.

Proof.

- For $K_5$ and $K_{3,3}$ every phase $ϕ_Y = 0$ or $π$. By Kuratowski’s theorem a non-planar graph contains a subgraph which is isomorphic to $K_5$ or $K_{3,3}$.

- For planar graphs the anyon phase can be introduced by drawing the graph in the plane and integrating the anyon vector potential $\frac{α}{2π} \hat{Z} \times \frac{r_1 - r_2}{|r_1 - r_2|^2}$ along the edges of the two-particle graph.
Examples

$K_5$: 6 A-B phases, 1 discrete phase of 0 or $\pi$.

$K_{3,3}$: 4 A-B phases, 1 discrete phase of 0 or $\pi$.

$K_4$: 3 A-B phases, 1 anyon phase.
Figure: Configuration space graph $C_2(K_{3,3})$, edges shown as solid lines are in a spanning subtree with root (1, 2). Open edges are joined left to right and top to bottom.
Classification of graph statistics

*Ko & Park (2011)*

\[ H_1(C_n(\Gamma)) = \mathbb{Z}^{N_1(\Gamma)+N_2(\Gamma)+N_3(\Gamma)+\beta_1(\Gamma)} \oplus \mathbb{Z}_2^{N'_3(\Gamma)} \]

- \( N_1(\Gamma) \) sum over one cuts \( j \) of \( N(n, \Gamma, j) \).
  \[ N(n, \Gamma, j) = \binom{n + \mu_j - 2}{n-1}(\mu(j) - 2) - \binom{n + \mu_j - 2}{n} - (v_j - \mu_j - 1) \]
  \( \mu_j \) \# components of \( \Gamma \setminus j \).

- \( N_2(\Gamma) \) sum over two connected components of \( \Gamma \).

- \( N_3(\Gamma) \) \# 3-connected planar components of \( \Gamma \).

- \( N'_3(\Gamma) \) \# 3-connected non-planar components of \( \Gamma \).

- \( \beta_1(\Gamma) \) \# of loops of \( \Gamma \).
Summary

- Classification of abelian quantum statistics on graphs via graph theoretic argument.
- Physical insight into dependence of statistics on graph connectivity.
- Identified new features of anyon statistics.
- Are there phenomena associated with new forms of anyon behavior - e.g. fractional quantum Hall experiment on network?
