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Symplectic-mixed finite element approximation of linear acoustic wave equations

Robert C. Kirby · Thinh Tri Kieu

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Abstract We apply mixed finite element approximations to the first-order form of the acoustic wave equation. The semidiscrete method exactly conserves the system energy. A fully discrete method employing the symplectic Euler time method in time exactly conserves a positive-definite pertubed energy functional that is equivalent to the actual energy under a CFL condition. In addition to proving optimal-order $L^\infty(L^2)$ estimates, we also develop a bootstrap technique that allows us to derive stability and error bounds for the time derivatives and divergence of the vector variable beyond the standard under some additional regularity assumptions.

Keywords Mixed finite element, acoustic wave equation, symplectic integration

Baylor University

1 Introduction

We consider the linear acoustic wave equation

$$\begin{aligned} \varrho p_t + \nabla \cdot u &= f, \\ \kappa^{-1} u_t + \nabla p &= g, \end{aligned} \tag{1}$$

posed on some domain $\Omega \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}$ with $d = 2,3$. We assume that T is finite and, for simplicity, that Ω is polyhedral so that it may be tesselated exactly into simplices. We pose initial conditions $p(x,0) = p_0(x)$ and $u(x,0) = u_0(x)$ and the boundary condition $u \cdot \nu = 0$ on $\partial\Omega$, where ν is the unit outward normal to Ω. Our analysis also covers homogeneous Dirichlet boundary conditions on part or all of the boundary, and extensions to inhomogeneous Dirichlet condition requires simply modifying the right-hand side of the weak form. We assume that the material density, ϱ, is some measurable function bounded below and above by positive ϱ_* and ϱ^*. The parameter κ is the bulk modulus of compressibility, assumed bounded between positive κ_* and κ^*. These equations are of essential interest in, among many other areas of application, seismic imaging.

We are interested in discretization of these equations using mixed finite element spaces, where p is discretized in some L^2 space and u in some $H(\text{div})$ space. Geveci [11] first considered such a discretization for the constant coefficient case, proving existence and uniqueness and optimal *a priori* error estimates in $L^\infty(L^2)$ for the semidiscrete formulation considered here. He also formulated but did not analyze a backward Euler time-stepping scheme. Glowinski and Rossi [12] utilize without analysis the first-order mixed formulation with a forward Euler time discretization in a control problem. Other analysis of mixed methods seems to focus on a slightly different formulation in which two time derivatives appear on the vector variable and none on the other equation. This more clearly represents the acoustic wave equation as a limiting result of elastodynamics, but less explicitly reveals the conservation-law framework of the equations. Cowsar, Dupont and Wheeler [10] prove *a priori* error estimates for this formulation and stability results for a family of time discretizations. This analysis was extended by Jenkins, Rivière, and Wheeler [17], and Jenkins provided numerical experimentation related to these results [16]. Chung and Engquist [9] also address the first-order formulation used by Geveci, but use a provably optimal discontinuous Galerkin method with specially chosen local spaces. Their numerical results use the (symplectic) leapfrog scheme, but they do not address the interaction of this time discretization with energy conservation or prove fully discrete estimates.

Our present work returns to the mixed formulation of Geveci, and strengthens the existing results in two major ways. First, we are able to control the temporal derivatives of both variables and the divergence of u_h, and such estimates are absent in related publications. Second, we study energy conservation of fully discrete schemes in the first-order system. In fact, the mixed form gives a Hamiltonian ODE with an appropriately defined Poisson bracket. Rather than utilizing the general theory of geometric numerical integration, however, we use a direct approach to calculate the approximately conserved Hamiltonian. Because a discretized PDE gives a family of ordinary differential equations, it is important to know how any implied constants scale under mesh refinement. In fact, such behavior gives rise to our CFL condition.

In a more abstract framework for spatial operators, Boffi, Buffa, and Gastaldi [5] study semidiscrete hyperbolic equations where the underlying spaces have commuting properties. Their techniques capture the current first-order form using

Raviart-Thomas spaces as well as Maxwell's equations and others. In this paper, we restrict to the particular case of acoustice wave equations, but we develop stronger estimates that include the error in $\nabla \cdot u_h$ and time derivatives instead of just $L^\infty(L^2)$ norms. We also explicitly analyze fully discrete methods.

The interaction of our spatial and temporal discretizations represent the intersection of two important trends in modern numerical analysis. On one hand, research on the finite element exterior calculus [2] and mimetic methods [3] demonstrate that the effectiveness of Raviart-Thomas-Nédeléc and related spaces derives from the discrete preservation of the de Rham complex. On the other hand, the theory of geometric integration has provided algorithms that reproduces essential qualitative structures such as energy conservation, with practical implications for long-term dynamics. In this way, our current discretization can be seen to preserve the essential structure in both the spatial and temporal aspects of the wave equation. This combination has been formulated for electromagnetics [23, 24], but ours seems to be the first theoretical analysis combining symplectic time integration with some form of mixed finite element space.

1.1 Mathematical Preliminaries

Let $\{\mathcal{T}_h\}_h$ be a family of quasiuniform triangulations of Ω [6]. We let $W = L^2(\Omega)$ and V the subspace of $H(\text{div})$ with vanishing normal trace. We let V_h be the Raviart-Thomas-Nédélec space [21, 22] of order $r \geq 0$ over each triangulation $\mathcal{T}_h$ and W_h the space of discontinuous piecewise polynomials of degree r over $\mathcal{T}_h$. It is possible to extend these mixed spaces to domains with a single curved facet [18], although we do not dwell on this. Our techniques apply equally well to rectangular meshes, but the poor approximation capabilities for quadrilateral meshes discussed in [1] suggest that the techniques of Bochev and Ridzal [4] may be required to extend our analysis to that case. The results also hold for the simplicial Brezzi-Douglas-Fortin-Marini spaces [7], but we suspect that the typical extra order of convergence for the velocity variable in L^2 would not be obtained for the Brezzi-Douglas-Marini elements.

Throughout, $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ or $\left(L^2(\Omega)\right)^d$ inner product, as needed. We also make use of the standard Sobolev spaces $H^m(\Omega)$ and $(H^m(\Omega))^d$ with norms denoted by $\|\cdot\|_m$ and seminorms by $|\cdot|_m$ in our error estimates, where m is some nonnegative integer. We also use the spaces $L^p(0,T;X)$ consisting of functions taking values in a normed space X at each time such that $\int_0^T \|f(\cdot,s)\|_X^p ds < \infty$, with the usual modification for $p=\infty$.

Our estimates make use of coefficient-weighted norms. For some strictly positive, bounded function ω, we define the weighted L^2 norm $\|f\|_\omega$ by

$$\|f\|_\omega^2 \equiv \int_\Omega \omega\,|f|^2\,dx, \tag{2}$$

and if $0 < \omega_* \le \omega(x) \le \omega^*$ throughout Ω, then we have the equivalence

$$\sqrt{\omega_*}\,\|f\| \le \|f\|_\omega \le \sqrt{\omega^*}\,\|f\|\,. \tag{3}$$

These norms admit weighted versions of Cauchy-Schwarz. With such a weight function ω, we can bound a standard inner product as

$$(f,g) = \left(\omega^{\frac{1}{2}} f, \omega^{-\frac{1}{2}} g\right) \le \|f\|_\omega\,\|g\|_{\omega^{-1}}\,. \tag{4}$$

We make the standard inverse assumption about our spaces, namely that there exists a positive constant C_0 such that

$$\|\nabla\cdot v_h\| \le \frac{C_0}{h}\,\|v_h\|\,. \tag{5}$$

for all $v_h \in V_h$.

We also use the standard projection operators $\pi : W \to W_h$ and $\Pi : V \to V_h$ for the mixed spaces.

For any $q \in W$, πq is the L^2 projection satisfying

$$(\pi q, w_h) = (q, w_h) \tag{6}$$

for all $w_h \in W_h$, and for $v \in V$, $\Pi v \in V_h$ is defined by

$$(\nabla \cdot \Pi v, w_h) = (\nabla \cdot v, w_h) \tag{7}$$

for all $w_h \in W_h$.

These projections have well-known approximation properties [7]. In particular, there exists a positive constant C_1 such that

$$\|\pi q - q\| \leq C_1 h^m \left|q\right|_m \tag{8}$$

whenever $q \in H^m(\Omega)$ and $1 \leq m \leq r+1$.

There also exists a positive C_2 such that

$$\|\Pi v - v\| \leq C_2 h^m |v|_m \tag{9}$$

for any $v \in \left(H^m\left(\Omega\right)\right)^d$ and for $1 \leq m \leq r+1$.

Because of the commuting relation between π, Π and the divergence (i.e., that $\nabla \cdot \Pi u = \pi\left(\nabla \cdot u\right)$, we also have the bound

$$\|\nabla \cdot \left(\Pi v - v\right)\| \leq C_1 h^m |\nabla \cdot v|_m, \tag{10}$$

provided $\nabla \cdot v \in \left(H^m\right)^d$ for $1 \leq m \leq r+1$.

Similar facts hold for $H(\text{curl})$ elements, and so we could easily adapt our techniques for curl-type equations.

We introduce the energy functional $\mathcal{E}$: $W \times V \to \mathbb{R}$ by

$$\mathcal{E}(w,v) = \frac{1}{2}\|w\|_{\varrho}^2 + \frac{1}{2}\|v\|_{\kappa^{-1}}^2 . \tag{11}$$

The square root of this quantity defines a norm on the product space $W \times V$, that is

$$\|(w,v)\|_{\mathcal{E}} \equiv \sqrt{\mathcal{E}(w,v)}. \tag{12}$$

1.2 Weak form and discretization

The solution of (1) satisfies the weak form of finding $u : [0,T] \to V \equiv H_0(\mathrm{div})$ and $p : [0,T] \to W \equiv L^2$ such that

$$\begin{aligned} (\varrho p_t, w) + (\nabla \cdot u, w) &= (f, w)\,, \\ \left(\kappa^{-1} u_t, v\right) - (p, \nabla \cdot v) &= (g, v)\,, \end{aligned} \tag{13}$$

together with initial conditions

$$\begin{aligned} p(x,0) &= p_0(x), \\ u(x,0) &= u_0(x). \end{aligned} \tag{14}$$

Here $H_0(\mathrm{div})$ is the subspace of $H(\mathrm{div})$ with functions of normal component vanishing on the boundary of the domain.

The semidiscrete mixed formulation of (13) is to find $u_h : [0,T] \to V_h$ and $p_h : [0,T] \to W_h$ such that

$$\begin{aligned} \left(\varrho p_{h,t}, w_h\right) + (\nabla \cdot u_h, w_h) &= (f, w_h)\,, \\ \left(\kappa^{-1} u_{h,t}, v_h\right) - (p_h, \nabla \cdot v_h) &= (g, v_h) \end{aligned} \tag{15}$$

for all $w_h \in W_h$ and $v_h \in V_h$. We take as initial conditions for the semidiscrete problem

$$\begin{aligned} p_h(\cdot,0) &= \pi p_0, \\ u_h(\cdot,0) &= \Pi u_0, \end{aligned} \tag{16}$$

where we have chosen the divergence projection rather than the L^2 projection of the initial condition for u_h.

The mixed formulation can be rendered as a Hamiltonian system as follows. Let F be a continuous mappings from $H(\mathrm{div}) \times L^2$ into the reals. We define $\frac{\delta F}{\delta u}$ and $\frac{\delta F}{\delta p}$ as the typical functional derivatives, acting on $v \in H(\mathrm{div})$ and $w \in L^2$ by

$$\left(\frac{\delta F}{\delta u}, v\right) + \left(\frac{\delta F}{\delta p}, w\right) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(F(u + \epsilon v, p + \epsilon w) - F(u, p)\right).$$

The Riesz Representation Theorem shows that $\frac{\delta F}{\delta u} \in H(\mathrm{div})$ and $\frac{\delta F}{\delta p} \in L^2$.

We can define the Poisson bracket on a pair of such functionals F and G by

$$\{F, G\} = \left(\nabla \cdot \left(\frac{\delta F}{\delta u}\right), \frac{\delta G}{\delta p}\right) - \left(\nabla \cdot \left(\frac{\delta G}{\delta u}\right), \frac{\delta F}{\delta p}\right)$$

This bracket is antisymmetric and can be shown to satisfy the Jacobi identity $\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0$. With this definition, the Hamiltoninan is

$$H(u, p) = \left(\kappa^{-1} u, u\right) + (\varrho p, p).$$

This structure is inherited by the mixed spatial discretization, so that we obtain a Hamiltonian family of ODE on any mesh.

Let $\{\phi_i\}_{i=1}^{|W_h|}$ and $\{\psi_i\}_{i=1}^{|V_h|}$ be bases for W_h and V_h, respectively. We define the weighted mass matrices

$$\begin{aligned} \mathrm{M}_{ij}^{\varrho} &= (\varrho \phi_j, \phi_i) \\ \widetilde{\mathrm{M}}_{ij}^{\kappa^{-1}} &= \left(\kappa^{-1} \psi_j, \psi_i\right), \end{aligned} \tag{17}$$

and we omit the superscript when $\varrho \equiv 1$ or $\kappa \equiv 1$.

We also require the weak divergence operator $\mathrm{D}_{ij} = (\phi_i, \nabla \cdot \psi_j)$. We represent the discrete solutions $W_h \ni p_h(\cdot, t) \equiv \sum_{i=1}^{|W_h|} \mathrm{p}_i(t)\phi_i$ and $V_h \ni u_h(\cdot, t) \equiv \sum_{i=1}^{|V_h|} \mathrm{u}_i(t)\psi_i$. We similarly let $\mathrm{f}_i(t)$ and $\mathrm{g}_i(t)$ denote the expansion coefficients of the L^2 projections of the forcing terms f and g. That is, $\pi f(\cdot,t) = \sum_{i=1}^{|W_h|} \mathrm{f}_i(t)\phi_i$ and $\pi g(\cdot,t) = \sum_{i=1}^{|V_h|} \mathrm{g}_i(t)\psi_i$. Finally, with p^0 and u^0 the vectors of expansion coefficients of the initial conditions (16), the semidiscrete form (15) satisfies the system of ordinary differential equations

$$\begin{aligned} \mathrm{M}^{\varrho}\mathrm{p}_t + \mathrm{D}\mathrm{u} &= \mathrm{M}\mathrm{f}, \\ \widetilde{\mathrm{M}}^{\kappa^{-1}}\mathrm{u}_t - \mathrm{D}^T\mathrm{p} &= \widetilde{\mathrm{M}}\mathrm{g}, \end{aligned} \tag{18}$$

with initial conditions

$$\begin{aligned} \mathrm{p}(0) &= \mathrm{p}^0, \\ \mathrm{u}(0) &= \mathrm{u}^0. \end{aligned} \tag{19}$$

An essential feature of the wave equation is its energy conservation, and it is important to understand how the energy fares under time-stepping schemes. It is well-known that the standard forward Euler method, such as used in [12], provides nondecreasing energy at each time step, while the backward Euler method, proposed by Geveci [11], generates nonincreasing energy. Here, we will consider the symplectic Euler method [13, 25], which we will show exactly conserves a perturbed version of the system energy. Generally, symplectic methods nearly preserve a nearby energy functional over long time periods, but for this linear problem, we are able to exactly construct a conserved quantity.

Now, we partition the time interval $[0,T]$ into equispaced time steps $0 \equiv t_0 < t_1 < t_2 < \cdots < t_N$, where $t_i = i\Delta t$. Then, we approximate the solution to the semidiscrete method (15) with $p_h(t_n) \approx p_h^n \in W_h$ and $u_h(t_n) \approx u_h^n \in V_h$ by the rule

$$\begin{aligned} \left(\varrho\frac{\Delta p_h^n}{\Delta t}, w_h\right) + \left(\nabla\cdot u_h^n, w_h\right) &= \left(f^n, w_h\right), \\ \left(\kappa^{-1}\frac{\Delta u_h^n}{\Delta t}, v_h\right) - \left(p_h^{n+1}, \nabla\cdot v_h\right) &= \left(g^{n+1}, v_h\right), \end{aligned} \tag{20}$$

where $f^n = f(t_n)$, $g^n = g(t_n)$, and $\Delta p_h^n = p_h^{n+1} - p_h^n$ is the standard forward difference operator. This is sometimes called the semiimplicit Euler method, since the first equation uses data at time level t_n and the second at t_{n+1}. In practice, we use the first equation to solve for p_h^{n+1} only by inverting a mass matrix and then use that value of p_h^{n+1} to solve for u_h^{n+1} in the second equation, again only inverting a mass matrix.

2 Stability of semidiscrete method

The existence and uniqueness of the solution of the semidiscrete system (15) with constant coefficients, as well as basic $L^\infty(L^2)$ stability results, are proven by Geveci [11]. We will prove similar stability results with different techniques that will more readily translate to the fully discrete methods later.

Before doing this, we prove a lemma that we will use in lieu of Young's inequality on several occasions.

Lemma 1 *Suppose that a real number x satisfies the quadratic inequality*

$$x^2 \leq \gamma^2 + \beta x$$

for $\beta, \gamma \geq 0$ but $\beta^2 + \gamma^2 > 0$. Then

$$x \leq \beta + \gamma.$$

Proof Rewrite the inequality as $x^2 - \beta x - \gamma^2 \leq 0$. The left-hand side is a quadratic function with positive leading term. So, for the inequality to hold, x must be to the left of the larger of the two roots of $x^2 - \beta x - \gamma^2$, which is

$$\frac{\beta + \sqrt{\beta^2 + 4\gamma^2}}{2}.$$

This is readily bounded by $\beta + \gamma$. Young's inequality instead would only give $x \leq \sqrt{2}\gamma + \beta$.

We define the energy of the semidiscrete solution by

$$a(t) = \left\| \left(p_h\left(\cdot, t\right), u_h\left(\cdot, t\right) \right) \right\|_{\mathcal{E}}. \tag{21}$$

Theorem 1 *Let (p_h, u_h) be the solution to the semidiscrete problem (15). Provided that $f \in L^1\left(0, T; L^2\left(\Omega\right)\right)$ and $g \in L^1\left(0, T; \left(L^2\left(\Omega\right)\right)^d\right)$, we have the stability bound*

$$\sup_{0 \leq s \leq T} a(s) \leq a(0) + \sqrt{2} \int_0^T \left\| f\left(\cdot, s\right) \right\|_{\varrho^{-1}} + \left\| g\left(\cdot, s\right) \right\|_{\kappa} ds. \tag{22}$$

Proof By selecting $w_h = p_h$ and $v_h = u_h$ at each time level in (15) and adding the two equations, we find that

$$\left(\varrho p_{h,t}, p_h \right) + \left(\kappa^{-1} u_{h,t}, u_h \right) = \left(f, p_h \right) + \left(g, u_h \right)$$

at each time $0 \leq t \leq T$. We rewrite the left-hand side as

$$\left(\varrho p_{h,t} p_h \right) + \left(\kappa^{-1} u_{h,t}, u_h \right) = \frac{1}{2} \frac{d}{dt} \left\| p_h\left(\cdot, t\right) \right\|_{\varrho}^2 + \frac{1}{2} \frac{d}{dt} \left\| u_h\left(\cdot, t\right) \right\|_{\kappa^{-1}}^2 = \frac{d}{dt} a^2(t).$$

Now, we pick any $0 \leq \tilde{t} \leq T$ and integrate from 0 to $\tilde{t}$ to obtain

$$a^2(\tilde{t}) = a^2(0) + \int_0^{\tilde{t}} \left(f, p_h \right) + \left(g, u_h \right) dt.$$

In the absence of forcing terms, this demonstrates that energy is conserved in the semidiscrete system.

To proceed, we use the weighted Cauchy-Schwarz estimate and extend the domain of integration to make the bound

$$\begin{aligned} a^2\left(\tilde{t}\right) &\leq a^2\left(0\right) + \int_0^{\tilde{t}} \left\| f\left(\cdot, s\right) \right\|_{\varrho^{-1}} \left\| p_h\left(\cdot, s\right) \right\|_{\varrho} ds + \int_0^{\tilde{t}} \left\| g\left(\cdot, s\right) \right\|_{\kappa} \left\| u_h\left(\cdot, s\right) \right\|_{\kappa^{-1}} ds \\ &\leq a^2\left(0\right) + \int_0^T \left\| f\left(\cdot, s\right) \right\|_{\varrho^{-1}} \left\| p_h\left(\cdot, s\right) \right\|_{\varrho} + \left\| g\left(\cdot, s\right) \right\|_{\kappa} \left\| u_h\left(\cdot, s\right) \right\|_{\kappa^{-1}} ds. \end{aligned}$$

Using the discrete Cauchy-Schwartz inequality under the integral sign, we have

$$\begin{aligned} a^2\left(\tilde{t}\right) &\leq a^2\left(0\right) + \int_0^T \sqrt{2}a(s)\sqrt{\left\|f\left(\cdot,s\right)\right\|_{\varrho^{-1}}^2 + \left\|g\left(\cdot,s\right)\right\|_{\kappa}^2} ds \\ &\leq a^2\left(0\right) + \sqrt{2}\left(\sup_{0\leq s\leq T} a(s)\right)\int_0^T \left\|f\left(\cdot,s\right)\right\|_{\varrho^{-1}} + \left\|g\left(\cdot,s\right)\right\|_{\kappa} ds. \end{aligned}$$

Now, the right-hand side is independent of t and $f(t) = t^2$ is monotonic, so we have the bound

$$\sup_{0\leq s\leq T} a^2\left(s\right) \leq a^2\left(0\right) + \sqrt{2}\left(\sup_{0\leq s\leq T} a(s)\right)\int_0^T \left\|f\left(\cdot,s\right)\right\|_{\varrho^{-1}} + \left\|g\left(\cdot,s\right)\right\|_{\kappa} ds.$$

This has the form

$$x^2 \leq \gamma^2 + \beta x,$$

where

$$\begin{aligned} x &= \sup_{0\leq t\leq T} a(t), \\ \beta &= \sqrt{2}\int_0^T \left\|f\left(\cdot,s\right)\right\|_{\varrho^{-1}} + \left\|g\left(\cdot,s\right)\right\|_{\kappa} ds > 0, \\ \gamma &= a(0) \geq 0. \end{aligned}$$

Applying Lemma 1 finishes the proof.

Now, we will develop a bootstrap technique that will allow us to prove, assuming greater regularity on the data, estimates in stronger norms. The system (18) is a linear system of ODE, so the solution will be as differentiable as we like assuming sufficient differentiability of the forcing terms. So, assuming that f_t and g_t exist in a reasonable space (made precise in the following theorem), we may differentiate the system to obtain

$$\begin{aligned} \mathrm{M}^{\varrho}\mathrm{p}_{tt} + \mathrm{D}\mathrm{u}_t &= \mathrm{M}\mathrm{f}_t, \\ \widetilde{\mathrm{M}}^{\kappa^{-1}}\mathrm{u}_{tt} - \mathrm{D}^T\mathrm{p}_t &= \widetilde{\mathrm{M}}\mathrm{g}_t, \end{aligned}$$

where M and $\widetilde{\mathrm{M}}$ are the mass matrices defined earlier in (17).

We define $\mathrm{q}(t) = \mathrm{p}_t$ and $\mathrm{r}(t) = \mathrm{u}_t$, and we rewrite the system above as

$$\begin{aligned} \mathrm{M}^{\varrho}\mathrm{q}_t + \mathrm{D}\mathrm{r} &= \mathrm{M}\mathrm{f}_t, \\ \widetilde{\mathrm{M}}^{\kappa^{-1}}\mathrm{r}_t - \mathrm{D}^T\mathrm{q} &= \widetilde{\mathrm{M}}\mathrm{g}_t. \end{aligned} \tag{23}$$

Given the equivalence between the system of ODE (18) and the variational problem (15), we can take $W_h \ni q_h(\cdot,t) = \sum_{i=1}^{|W_h|} \mathrm{q}_i(t)\phi_i$ and $V_h \ni r_h(\cdot,t) = \sum_{i=1}^{|V_h|} \mathrm{r}_i(t)\psi_i$. This shows that the time derivatives $p_{h,t} = q_h$ and $u_{h,t} = r_h$ satisfy the variational problem

$$\begin{aligned} \left(\varrho q_{h,t}, w_h\right) + \left(\nabla \cdot r_h, w_h\right) &= \left(f_t, w_h\right), \\ \left(\kappa^{-1} r_{h,t}, v_h\right) - \left(q_h, \nabla \cdot v_h\right) &= \left(g_t, v_h\right). \end{aligned} \tag{24}$$

The initial conditions for this system are specified by evaluating the system (15) at time 0:

$$\begin{aligned} \left(\varrho q_h(\cdot,0), w_h\right) + \left(\nabla \cdot u_h(\cdot,0), w_h\right) &= \left(f(\cdot,0), w_h\right), \\ \left(\kappa^{-1} r_h(\cdot,0), v_h\right) - \left(p_h(\cdot,0), \nabla \cdot v_h\right) &= \left(g(\cdot,0), v_h\right), \end{aligned}$$

so that the initial derivatives are weighted projections of the initial values of the forcing terms and derivatives of the initial conditions. With $u_h(\cdot,0) \in V_h$, $p_{h,t}(\cdot,0)$, we pick $w_h = q_h(\cdot,0)$ to give

$$\left\|q_h\left(\cdot,0\right)\right\|_{\varrho}^2 = \left(f(\cdot,0), q_h(\cdot,0)\right) - \left(\nabla \cdot u_h\left(\cdot,0\right), q_h(\cdot,0)\right),$$

from which follows the bound

$$\left\|q_h\left(\cdot,0\right)\right\|_{\varrho} \le \left\|f\left(\cdot,0\right)\right\|_{\varrho^{-1}} + \left\|\nabla \cdot u_h\left(\cdot,0\right)\right\|_{\varrho^{-1}}. \tag{25}$$

We may also bound $r_h(\cdot,0)$. Taking $v_h = r_h(\cdot,0)$ in the second equation in (15) gives

$$\left\|r_h\left(\cdot,0\right)\right\|_{\kappa^{-1}}^2 = \left(g(\cdot,0), r_h(\cdot,0)\right) + \left(p_h(\cdot,0), \nabla \cdot r_h(\cdot,0)\right).$$

Using Cauchy-Schwarz and the inverse assumption (5) to bound $\|\nabla \cdot r_h(\cdot,0)\|$ by the L^2 norm gives the estimate

$$\|r_h(\cdot,0)\|_{\kappa^{-1}} \leq \|g(\cdot,0)\|_{\kappa} + \frac{C_0\sqrt{\kappa^*}}{h}\|p_h(\cdot,0)\|. \tag{26}$$

This is only an $\mathcal{O}(h^{-1})$ bound, degrading as the mesh is refined. However, if the initial condition $p^0 \in H^1$, the use of the inverse assumption and hence the h^{-1} factor may be avoided. Since $\nabla \cdot r_h(\cdot,0) \in W_h$ and $p_h(\cdot,0)$ is the L^2 projection of p^0, we have

$$(p_h(\cdot,0), \nabla \cdot r_h(\cdot,0)) = \left(\pi p^0, \nabla \cdot r_h(\cdot,0)\right) = \left(p^0, \nabla \cdot r_h(\cdot,0)\right) = -\left(\nabla p^0, r_h(\cdot,0)\right),$$

giving the bound

$$\|r_h(\cdot,0)\|_{\kappa^{-1}} \leq \|g(\cdot,0)\|_{\kappa} + \left\|\nabla p^0\right\|_{\kappa}. \tag{27}$$

If $p^0 \notin H^1$, but is not merely L^2 (say, it is piecewise smooth with jump discontinuities), it may be possible to get more refined estimates using elementwise integration by parts. However, as we will typically apply this stability estimate to error estimates with zero initial pressure, we do not pursue this further here.

A straightforward application of Theorem 1 to (24) gives a bound on the time derivatives. We define

$$b(t) \equiv \left\|\left(p_{h,t}(\cdot,t), u_{h,t}(\cdot,t)\right)\right\|_{\mathcal{E}}. \tag{28}$$

Theorem 2 *Let p_h and u_h be the solutions to (15). If $f_t \in L^1(0,T;L^2(\Omega))$ and $g_t \in L^1(0,T;\left(L^2(\Omega)\right)^d)$, then*

$$\sup_{0\leq s\leq T} b(s) \leq b(0) + \sqrt{2}\int_0^T \|f_t(\cdot,s)\|_{\varrho^{-1}} + \|g_t(\cdot,s)\|_{\kappa}\, ds. \tag{29}$$

Bounds on the time derivatives lead to bounds on the divergence of the discrete solution at each time. Selecting $w_h = \nabla \cdot u_h$ in the first equation of (15) gives

$$\left(\varrho p_{h,t}, \nabla \cdot u_h\right) + \|\nabla \cdot u_h\|^2 = (f, \nabla \cdot u_h), \tag{30}$$

so that

$$\|\nabla \cdot u_h\| \le \|f\| + \sqrt{\varrho^*}\left\|p_{h,t}\right\|_\varrho, \tag{31}$$

and in light of the previous theorem, we have

Theorem 3 *Under the assumptions of Theorem 2, and assuming also that $f \in L^\infty(0,T;L^2(\Omega))$, we have that*

$$\begin{aligned}\sup_{0\le s\le T}\|\nabla \cdot u_h(\cdot,s)\| \le \sup_{0\le s\le T}\|f(\cdot,s)\| + \sqrt{\varrho^*}b(0)\\ + \sqrt{2\varrho^*}\int_0^T \|f_t(\cdot,s)\|_{\varrho^{-1}} + \|g_t(\cdot,s)\|_\kappa \, ds.\end{aligned} \tag{32}$$

3 Semidiscrete error estimates

We will bound the error in the semidiscrete method in various norms by comparing the computed solution to the projections of the true solutions. To do this, we restrict the test functions in (13) to the finite-dimensional spaces. Then, using the properties of the projections π and Π, we have that

$$\begin{aligned}(\varrho\pi p_t, w_h) + (\nabla\cdot\Pi u, w_h) = (f, w_h) + (\varrho(\pi p_t - p_t), w_h),\\ \left(\kappa^{-1}\Pi u_t, v_h\right) - (\pi p, \nabla\cdot v_h) = (g, v_h) + \left(\kappa^{-1}(\Pi u_t - u_t), v_h\right).\end{aligned} \tag{33}$$

We may subtract the semidiscrete form (15) from these to obtain the error equations

$$\begin{aligned}\left(\varrho\left(\pi p_t - p_{h,t}\right), w_h\right) + (\nabla\cdot(\Pi u - u_h), w_h) = (\varrho(\pi p_t - p_t), w_h),\\ \left(\kappa^{-1}\left(\Pi u_t - u_{h,t}\right), v_h\right) - (\pi p - p_h, \nabla\cdot v_h) = \left(\kappa^{-1}(\Pi u_t - u_t), v_h\right).\end{aligned} \tag{34}$$

We define $W_h \ni \theta_h(\cdot,t) \equiv \pi p(\cdot,t) - p_h(\cdot,t)$ and $V_h \ni \chi_h(\cdot,t) \equiv \Pi u(\cdot,t) - u_h(\cdot,t)$ to be the differences between projections and computed solutions. We also let $W \ni \xi(\cdot,t) \equiv \pi p(\cdot,t) - p(\cdot,t)$ and $V \ni \eta(\cdot,t) \equiv \Pi u(\cdot,t) - u(\cdot,t)$ be the differences between the exact solutions and their projections.

Because of the definitions of ξ and η and the properties of the projections, these differences satisfy the semidiscrete problem.

$$\begin{aligned} \left(\varrho\theta_{h,t}, w_h\right) + \left(\nabla\cdot\chi_h, w_h\right) &= \left(\varrho\xi_t, w_h\right), \\ \left(\kappa^{-1}\chi_{h,t}, v_h\right) - \left(\theta_h, \nabla\cdot v_h\right) &= \left(\kappa^{-1}\eta_t, v_h\right). \end{aligned} \tag{35}$$

Because of the choice of initial conditions for p_h and u_h, the initial values $\theta_h(\cdot,0)$ and $\chi_h(\cdot,0)$ are both zero.

A direct application of Theorems 1, 2 and 3 allows us to bound θ_h and χ_h in various norms in terms of the projection errors. We define

$$\varepsilon(t) = \left\|\left(\theta_h(\cdot,t), \chi_h(\cdot,t)\right)\right\|_{\mathcal{E}}. \tag{36}$$

Lemma 2 *Suppose that the true solution (p,u) has time derivatives $p_t \in L^1\left(0,T;H^{r+1}(\Omega)\right)$ and $u_t \in L^1\left(0,T;\left(H^{r+1}(\Omega)\right)^d\right)$. Then, for $1 \leq m \leq r+1$, we have the bound*

$$\varepsilon(t) \leq C_1 h^m \sqrt{2\varrho^*}\int_0^T \left|p_t\left(\cdot,s\right)\right|_m ds + C_2 h^m \sqrt{\frac{2}{\kappa_*}}\int_0^T \left|u_t\left(\cdot,s\right)\right|_m ds. \tag{37}$$

Proof Applying Theorem 1 to (35) and using the fact that $\varepsilon^2(0) = 0$, we have that

$$\sup_{0\leq s\leq T}\varepsilon(s) \leq \sqrt{2}\int_0^T \left\|\varrho\xi_t\left(\cdot,s\right)\right\|_{\varrho^{-1}} + \left\|\kappa^{-1}\eta_t\left(\cdot,s\right)\right\|_{\kappa} ds.$$

We rewrite the weighted norms

$$\left\|\varrho\xi_t\right\|^2_{\varrho^{-1}} = \int_\Omega \varrho^{-1}\left(\varrho\xi_t\right)^2 dx = \int_\Omega \varrho\left(\xi_t\right)^2 dx = \left\|\xi_t\right\|^2_\varrho,$$

with an analagous calculation showing $\left\|\kappa^{-1}\eta_t\right\|_\kappa = \left\|\eta_t\right\|_{\kappa^{-1}}$. This gives

$$\sup_{0\leq s\leq T}\varepsilon(s) \leq \sqrt{2}\int_0^T \left\|\xi_t\left(\cdot,s\right)\right\|_\varrho + \left\|\eta_t\left(\cdot,s\right)\right\|_{\kappa^{-1}} ds,$$

and then we use norm equivalence and the approximation theoretic bounds (8) and (9) to finish the proof.

We define the error quantity

$$\epsilon(t) = \|(p(\cdot,t) - p_h(\cdot,t), u(\cdot,t) - u_h(\cdot,t))\|_{\mathcal{E}}. \tag{38}$$

Theorem 4 *Suppose that the solutions have time derivatives $p_t \in L^1\left(0,T;H^{r+1}(\Omega)\right)$ and $u_t \in L^1\left(0,T;\left(H^{r+1}(\Omega)\right)^d\right)$ and additionally that $p \in L^\infty\left(0,T;H^{r+1}(\Omega)\right)$ and $u \in L^\infty\left(0,T;\left(H^{r+1}(\Omega)\right)^d\right)$. Then, for each $1 \le m \le r+1$, we have*

$$\begin{aligned}\sup_{0\le s\le T} \epsilon(s) \le & C_1 h^m \sqrt{\varrho^*}\left(\frac{1}{\sqrt{2}}\sup_{0\le s\le T}|p(\cdot,s)|_m + \sqrt{2}\int_0^T |p_t(\cdot,s)|_m\, ds\right) \\ & + \frac{C_2 h^m}{\sqrt{\kappa_*}}\left(\frac{1}{\sqrt{2}}\sup_{0\le s\le T}|u(\cdot,s)|_m + \sqrt{2}\int_0^T |u_t(\cdot,s)|_m\, ds\right).\end{aligned} \tag{39}$$

Proof The triangle inequality gives

$$\sup_{0\le s\le T}\epsilon(s) \le \sup_{0\le s\le T}\|(\xi(\cdot,s),\eta(\cdot,s))\|_{\mathcal{E}} + \sup_{0\le s\le T}\|(\theta(\cdot,s),\chi(\cdot,s))\|_{\mathcal{E}}. \tag{40}$$

We use the approximation-theoretic bounds (8) and (9) on the first term:

$$\begin{aligned}\|(\xi(\cdot,s),\eta(\cdot,s))\|_{\mathcal{E}} &\le \frac{1}{\sqrt{2}}\|\xi(\cdot,s)\|_{\varrho} + \frac{1}{\sqrt{2}}\|\eta(\cdot,s)\|_{\kappa^{-1}} \\ &\le \sqrt{\frac{\varrho^*}{2}}\|\xi(\cdot,s)\| + \frac{1}{\sqrt{2\kappa_*}}\|\eta(\cdot,s)\| \\ &\le C_1 h^m\sqrt{\frac{\varrho^*}{2}}|p(\cdot,s)|_m + \frac{C_2 h^m}{\sqrt{2\kappa_*}}|u(\cdot,s)|_m.\end{aligned}$$

The result follows by applying the previous theorem to the second term in (40) and combining terms.

Now, we turn to the error in the time derivatives. We define

$$\beta(t) = \left\|\left(\pi p_t(\cdot,t) - p_{h,t}(\cdot,t), \Pi u_t(\cdot,t) - u_{h,t}(\cdot,t)\right)\right\|_{\mathcal{E}} = \left\|\left(\theta_{h,t}(\cdot,t),\chi_{h,t}(\cdot,t)\right)\right\|_{\mathcal{E}}. \tag{41}$$

Lemma 3 *Suppose that* $p_{tt} \in L^1(0,T;H^m(\Omega))$, $u_{tt} \in L^1\left(0,T;(H^m(\Omega))^d\right)$ *and also that* $p_t(\cdot,0) \in H^m(\Omega)$ *and* $u_t(\cdot,0) \in (H^m(\Omega))^d$. *Then*

$$\begin{aligned}\sup_{0\le s\le T}\beta(s) \le & C_1\sqrt{\varrho^*}h^m\left(\frac{1}{\sqrt{2}}\left|p_t(\cdot,0)\right|_m + \sqrt{2}\int_0^T \left|p_{tt}(\cdot,s)\right| ds\right) \\ & + \frac{C_2h^m}{\sqrt{\kappa_*}}\left(\frac{1}{\sqrt{2}}\left|u_t(\cdot,0)\right|_m + \sqrt{2}\int_0^T \left|u_{tt}(\cdot,s)\right| ds\right).\end{aligned} \tag{42}$$

Proof If we apply the stability result of Theorem 2 to the error equations (35), we find that

$$\sup_{0\le s\le T}\beta(s) \le \beta(0) + \sqrt{2}\int_0^T \left\|\varrho\xi_{tt}(\cdot,s)\right\|_{\varrho^{-1}} + \left\|\kappa^{-1}\eta_{tt}(\cdot,s)\right\|_{\kappa} ds.$$

We can bound $\beta(0)$ using the error equations (35) together with bounds on the initial derivatives (25) and (26):

$$\begin{aligned}\beta(0) &\le \frac{1}{\sqrt{2}}\left\|\theta_{h,t}(\cdot,0)\right\|_{\varrho} + \frac{1}{\sqrt{2}}\left\|\chi_{h,t}(\cdot,0)\right\|_{\kappa^{-1}} \\ &\le \frac{1}{\sqrt{2}}\left\|\xi_t(\cdot,0)\right\|_{\varrho} + \frac{1}{\sqrt{2}}\left\|\eta_t(\cdot,0)\right\|_{\kappa^{-1}}.\end{aligned}$$

The approximation results (8) and (9) give then that

$$\beta(0) \le C_1h^m\sqrt{\frac{\varrho^*}{2}}\left|p_t(\cdot,0)\right|_m + \frac{C_2h^m}{\sqrt{2\kappa_*}}\left|u_t(\cdot,0)\right|_m.$$

For the remaining terms, rewriting the weighted norms and applying the approximation estimates finishes the proof.

Now, we let

$$\gamma(t) = \left\|\left(p_t(\cdot,t) - p_{h,t}(\cdot,t), u_t(\cdot,t) - u_{h,t}(\cdot,t)\right)\right\|_{\mathcal{E}}, \tag{43}$$

and we have an error estimate. Lemma 3 plus the approximation results give us a bound on $\gamma(t)$.

Theorem 5 *Under the assumptions of Lemma 3, we have that*

$$\sup_{0\leq s\leq T}\gamma(s) \leq C_1 h^m \sqrt{\varrho^*}\left(\frac{1}{\sqrt{2}}\left|p_t(\cdot,0)\right|_m + \frac{1}{\sqrt{2}}\sup_{0\leq s\leq T}\left|p_t(\cdot,s)\right|_m + \sqrt{2}\int_0^T \left|p_{tt}(\cdot,s)\right|_m ds\right) + \frac{C_2 h^m}{\sqrt{\kappa_*}}\left(\frac{1}{\sqrt{2}}\left|u_t(\cdot,0)\right|_m + \frac{1}{\sqrt{2}}\sup_{0\leq s\leq T}\left|u_t(\cdot,s)\right|_m + \sqrt{2}\int_0^T \left|u_{tt}(\cdot,s)\right|_m ds\right). \tag{44}$$

Proof First, we use the triangle inequality to write that

$$\sup_{0\leq s\leq T}\gamma(s) \leq \sup_{0\leq s\leq T}\beta(s) + \sup_{0\leq s\leq T}\left\|(\xi_t(\cdot,s),\eta_t(\cdot,s))\right\|_{\mathcal{E}}.$$

The first term is bounded by the previous lemma. The second term satisfies the bound

$$\begin{aligned}\sup_{0\leq s\leq T}\left\|(\xi_t(\cdot,s),\eta_t(\cdot,s))\right\|_{\mathcal{E}} &\leq \sup_{0\leq s\leq T}\sqrt{\frac{\varrho^*}{2}}\left\|\xi_t(\cdot,s)\right\| + \sup_{0\leq t\leq T}\frac{1}{\sqrt{2\kappa_*}}\left\|\eta_t(\cdot,s)\right\| \\ &\leq \frac{C_1 h^m\sqrt{\varrho^*}}{\sqrt{2}}\sup_{0\leq s\leq T}\left|p_t(\cdot,s)\right|_m + \frac{C_2 h^m}{\sqrt{2\kappa_*}}\sup_{0\leq s\leq T}\left|u_t(\cdot,s)\right|_m,\end{aligned}$$

and the final result follows by combining these bounds with Lemma 3.

Finally, we consider the error in divergence of the vector variable.

Theorem 6 *If the assumptions of Lemma 3 hold, plus that* $\nabla\cdot u \in L^\infty(0,T;H^m(\Omega))$, *we have*

$$\begin{aligned}\sup_{0\leq s\leq T}\left\|\nabla\cdot(u-u_h)(\cdot,s)\right\| \leq{}& C_1 h^m \varrho^*\Bigg(\left|p_t(\cdot,0)\right|_m + \sup_{0\leq s\leq T}\left|p_t(\cdot,s)\right|_m \\ &+ \sup_{0\leq s\leq T}\left|\nabla\cdot u(\cdot,s)\right|_m + 2\int_0^T \left|p_{tt}(\cdot,s)\right|_m ds\Bigg) \\ &+ C_2 h^m\sqrt{\frac{\varrho^*}{\kappa_*}}\left(\left|u_t(\cdot,0)\right|_m + 2\int_0^T \left|u_{tt}(\cdot,s)\right|_m ds\right).\end{aligned} \tag{45}$$

Proof First, we bound the divergence error at each time by the triangle inequality.

$$\left\|\nabla\cdot(u-u_h)(\cdot,s)\right\| \leq \left\|\nabla\cdot\chi_h(\cdot,s)\right\| + \left\|\nabla\cdot\eta(\cdot,s)\right\|. \tag{46}$$

The latter term is bounded by (10). For the former, we note that an estimate like (31) for the error equations (35) gives

$$\sup_{0\le s\le T} \|\nabla\cdot\chi_h(\cdot,s)\| \le \sup_{0\le s\le T}\|\varrho\xi_t(\cdot,s)\| + \sqrt{\varrho^*}\sup_{0\le s\le T}\left\|\theta_{h,t}(\cdot,s)\right\|_{\varrho}.$$

The first term is bounded by

$$\sup_{0\le s\le T}\|\varrho\xi_t(\cdot,s)\| \le C_1\varrho^* h^m \sup_{0\le s\le T}|p_t(\cdot,s)|_m .$$

For the second term, $\left\|\theta_{h,t}(\cdot,s)\right\|_{\varrho}\le\sqrt{2}\beta(s)$, so applying Lemma 3 gives

$$\begin{aligned}
\sqrt{\varrho^*}\sup_{0\le s\le T}\left\|\theta_{h,t}(\cdot,s)\right\|_{\varrho} \le&\sqrt{2\varrho^*}\sup_{0\le s\le T}\beta(s)\\
\le&\sqrt{2\varrho^*}\left[C_1\sqrt{\varrho^*}h^m\left(\frac{1}{\sqrt{2}}|p_t(\cdot,0)|_m+\sqrt{2}\int_0^T|p_{tt}(\cdot,s)|_m\,ds\right)\right]\\
&+\sqrt{2\varrho^*}\left[\frac{C_2h^m}{\sqrt{\kappa_*}}\left(\frac{1}{\sqrt{2}}|u_t(\cdot,0)|_m+\sqrt{2}\int_0^T|u_{tt}(\cdot,s)|_m\,ds\right)\right]\\
=&C_1h^m\varrho^*\left(|p_t(\cdot,0)|_m+2\int_0^T|p_{tt}(\cdot,s)|_m\,ds\right)\\
&+C_2\sqrt{\frac{\varrho^*}{\kappa_*}}h^m\left(|u_t(\cdot,0)|_m+2\int_0^T|u_{tt}(\cdot,s)|_m\,ds\right).
\end{aligned}$$

4 Fully discrete method

Energy conservation drives the semidiscrete estimates above, both for the stability and also for the error estimates. Such a property for a time stepping scheme would not only honor the physics of the system, but would also admit a similar technique for the fully discrete analysis. Except for special time discretizations (e.g. Crank-Nicholson and certain higher-order implicit methods), few methods exactly conserve the system energy. Rather than dealing with implicit methods, we will show that the symplectic Euler method applied to our problem preserves a quantity that approximates the system energy. This is similar to the backward analysis known for Hamiltonian systems [13].

Energy techniques based on conservation are quite standard for semidiscrete PDE, but finding and using a conserved quantity for the fully discrete case is a new wrinkle in this paper. It makes techniques for the discrete-time case closely parallel the semidiscrete. However, as our analysis relies on an explicit construction rather than the existence of the conserved energy functional, we leave it as an open question whether similar backward techniques can be adapted to nonlinear problems.

4.1 Discrete energy and conservation

Given that we seek a quantity that is a perturbation of the exact system energy, we will find it with some basic manipulations of the discrete equations. The true system energy at each time level n is $a_n^2 = \mathcal{E}(p_h^n, u_h^n)$, but we will also show that that functional

$$\widetilde{\mathcal{E}}_{\Delta t}(w_h, v_h) \equiv \mathcal{E}(w_h, v_h) - \frac{\Delta t}{2} (\nabla \cdot v_h, w_h) \tag{47}$$

on $W_h \times V_h$ appears and gives rise to our conserved approximate energy, with

$$\widetilde{a}_n^2 \equiv \widetilde{\mathcal{E}}_{\Delta t}(p_h^n, u_h^n). \tag{48}$$

Before seeing how this quantity emerges from the analysis, we can see that under a CFL-like restriction on Δt it is in fact equivalent to the actual energy on the finite element spaces

Lemma 4 *Let*

$$\alpha \equiv \frac{C_0 \sqrt{\kappa^*}}{2h\sqrt{\varrho_*}}, \tag{49}$$

and suppose that

$$\alpha \Delta t < 1. \tag{50}$$

Then $\widetilde{\mathcal{E}}_{\Delta t}$ is positive-definite and satisfies

$$(1-\alpha\Delta t)\,\mathcal{E}\left(w_h,v_h\right) \leq \widetilde{\mathcal{E}}_{\Delta t}\left(w_h,v_h\right) \leq 2\mathcal{E}\left(w_h,v_h\right) \tag{51}$$

for all $w_h \in W_h$ and $v_h \in V_h$.

Proof Let $w_h \in W_h$ and $v_h \in V_h$. Then using weighted Cauchy-Schwarz, the equivalence of weighted norms, the inverse assumption (5) and Young's inequality, we have

$$\begin{aligned}
\left|\left(\nabla\cdot v_h, w_h\right)\right| &\leq \left\|\nabla\cdot v_h\right\|_{\varrho^{-1}}\left\|w_h\right\|_{\varrho} \\
&\leq \frac{1}{\sqrt{\varrho_*}}\left\|\nabla\cdot v_h\right\|\left\|w_h\right\|_{\varrho} \\
&\leq \frac{C_0}{h\sqrt{\varrho_*}}\left\|v_h\right\|\left\|w_h\right\|_{\varrho} \\
&\leq \frac{C_0\sqrt{\kappa^*}}{h\sqrt{\varrho_*}}\left\|v_h\right\|_{\kappa^{-1}}\left\|w_h\right\|_{\varrho} \\
&\leq \frac{C_0\sqrt{\kappa^*}}{h\sqrt{\varrho_*}}\mathcal{E}\left(w_h,v_h\right).
\end{aligned}$$

With this estimate, we have that

$$\begin{aligned}
\widetilde{\mathcal{E}}_{\Delta t}\left(w_h,v_h\right) &= \mathcal{E}\left(w_h,v_h\right) - \frac{\Delta t}{2}\left(\nabla\cdot v_h, w_h\right) \\
&\geq \mathcal{E}\left(w_h,v_h\right) - \frac{C_0\Delta t\sqrt{\kappa^*}}{2h\sqrt{\varrho_*}}\mathcal{E}\left(w_h,v_h\right) \qquad (52) \\
&= (1-\alpha\Delta t)\,\mathcal{E}\left(w_h,v_h\right),
\end{aligned}$$

the CFL condition and the positive-definiteness of $\mathcal{E}$ rendering the quantity positive-definite. The upper bound follows in a similar fashion.

The perturbed energy is certainly well-defined on all of $W \times V$, but our proof of equivalence requires the inverse estimate, which only holds on the finite element spaces.

Now, we can show that the quantity $\tilde{a}_n^2$ is in fact conserved.

Lemma 5 *If the forcing functions f, g both vanish, then for each $0 \leq n < N$, we have that*

$$\tilde{a}_{n+1}^2 = \tilde{a}_n^2. \tag{53}$$

Proof Select $w_h = \frac{p_h^{n+1}+p_h^n}{2}$ and $v_h = \frac{u_h^{n+1}+u_h^n}{2}$ in (20) to find that

$$\frac{1}{2\Delta t}\left\|p_h^{n+1}\right\|_\varrho^2 - \frac{1}{2\Delta t}\left\|p_h^n\right\|_\varrho^2 + \left(\nabla\cdot u_h^n, \frac{p_h^{n+1}+p_h^n}{2}\right) = 0,$$

$$\frac{1}{2\Delta t}\left\|u_h^{n+1}\right\|_{\kappa^{-1}}^2 - \frac{1}{2\Delta t}\left\|u_h^n\right\|_{\kappa^{-1}}^2 - \left(p_h^{n+1}, \nabla\cdot \frac{u_h^{n+1}+u_h^n}{2}\right) = 0.$$

Adding the equations together, multiplying by Δt, and some straightforward manipulations give the desired result.

4.2 Stability

Theorem 7 *Supposing that the sum $\sum_{n=0}^{N-1}\left(\|f^n\|_{\varrho^{-1}} + \left\|g^{n+1}\right\|_\kappa\right)\Delta t$ is bounded and the CFL condition (50) holds, the energy a_n of the solution to (20) satisfies the bound*

$$\max_{0\leq n\leq N} a_n \leq \sqrt{\frac{1}{1-\alpha\Delta t}}\tilde{a}_0 + \frac{\sqrt{2}}{1-\alpha\Delta t}\sum_{n=0}^{N-1}\left(\|f^n\|_{\varrho^{-1}} + \left\|g^{n+1}\right\|_\kappa\right)\Delta t \tag{54}$$

Proof Selecting the test functions as in the previous theorem, the same manipulations give us

$$\tilde{a}_{n+1}^2 - \tilde{a}_n^2 = \Delta t\left(f^n, \frac{p_h^{n+1}+p_h^n}{2}\right) + \Delta t\left(g^{n+1}, \frac{u_h^{n+1}+u_h^n}{2}\right).$$

If we fix some $0 \leq M \leq N$ and sum this equation from $n = 0$ to $M - 1$, we obtain

$$\tilde{a}_M^2 = \tilde{a}_0^2 + \sum_{n=0}^{M-1}\Delta t\left(f^n, \frac{p_h^{n+1}+p_h^n}{2}\right) + \sum_{n=0}^{M-1}\Delta t\left(g^{n+1}, \frac{u_h^{n+1}+u_h^n}{2}\right).$$

Using Lemma 4 on the left-hand side and weighted Cauchy-Schwarz on the right and extending the interval of summation give

$$\begin{aligned}(1-\alpha\Delta t)\, a_M^2 &\leq \tilde{a}_0^2 + \sum_{n=0}^{M-1}\frac{\Delta t}{2}\left(\|f^n\|_{\varrho^{-1}}\left\|p_h^{n+1}+p_h^n\right\|_\varrho + \left\|g^{n+1}\right\|_\kappa\left\|u_h^{n+1}+u_h^n\right\|_{\kappa^{-1}}\right)\\ &\leq \tilde{a}_0^2 + \sum_{n=0}^{N-1}\frac{\Delta t}{2}\left(\|f^n\|_{\varrho^{-1}}\left\|p_h^{n+1}+p_h^n\right\|_\varrho + \left\|g^{n+1}\right\|_\kappa\left\|u_h^{n+1}+u_h^n\right\|_{\kappa^{-1}}\right).\end{aligned}$$

Now, we apply the triangle inequality and the fact that $\|p_h^n\|_\varrho$ and $\|u_h^n\|_{\kappa^{-1}}$ are each bounded by $\sqrt{2}a_n$ to find that

$$\max_{0\le n\le N} a_n^2 \le \frac{1}{1-\alpha\Delta t}\tilde{a}_0^2 + \frac{\sqrt{2}}{1-\alpha\Delta t}\left(\max_{0\le n\le N} a_n\right)\sum_{n=0}^{N-1}\left(\|f^n\|_{\varrho^{-1}} + \left\|g^{n+1}\right\|_\kappa\right)\Delta t$$

To complete the proof, we use Lemma 1 with $x = \max_{0\le n\le N} a_n$,

$\beta = \left(\frac{\sqrt{2}}{1-\alpha\Delta t}\right)\sum_{n=0}^{N-1}\left(\|f^n\|_{\varrho^{-1}} + \left\|g^{n+1}\right\|_\kappa\right)\Delta t$, and $\gamma = \frac{1}{\sqrt{1-\alpha\Delta t}}\tilde{a}_0$.

As with the semidiscrete case, we can apply a boostrapping argument to obtain bounds in stronger norms. However, we will only deal with the case of discrete initial conditions $p_h^0 = 0$ and $u_h^0 = 0$. This is all that is required for subsequent error estimates. Using linearity, these results may also be adapted to nonzero initial conditions by converting them to forcing functions.

We define the difference quotients.

$$\begin{aligned} q_h^n &= \frac{\Delta p_h^n}{\Delta t}, \\ r_h^n &= \frac{\Delta u_h^n}{\Delta t}. \end{aligned} \tag{55}$$

Then, time-differencing (20) gives us the new system of equations

$$\begin{aligned} \left(\varrho\frac{\Delta q_h^n}{\Delta t}, w_h\right) + (\nabla\cdot r_h^n, w_h) &= \left(\frac{\Delta f^n}{\Delta t}, w_h\right), \\ \left(\kappa^{-1}\frac{\Delta r_h^n}{\Delta t}, v_h\right) - \left(q_h^{n+1}, \nabla\cdot v_h\right) &= \left(\frac{\Delta g^{n+1}}{\Delta t}, v_h\right), \end{aligned} \tag{56}$$

which holds for $0 \le n \le N-1$ rather than N. The initial conditions for q_h^0 and r_h^0 are defined by evaluating (20) at $n = 0$

$$\begin{aligned} \left(\varrho q_h^0, w_h\right) + \left(\nabla\cdot u_h^0, w_h\right) &= \left(f^0, w_h\right), \\ \left(\kappa^{-1} r_h^0, v_h\right) - \left(p_h^1, v_h\right) &= \left(g^1, v_h\right). \end{aligned} \tag{57}$$

Selecting $w_h = q_h^0$ in the first equation, applying the fact that $u_h^0 = 0$ and using Cauchy Schwarz immediately gives that

$$\left\|q_h^0\right\|_\varrho \le \left\|f^0\right\|_{\varrho^{-1}},$$

but the presence of p_h^1 rather than p_h^0 in the second complicates matters somewhat. We use the first equation, with $u_h^0 = 0$, in (20) to bound p_h^1:

$$\left(\varrho p_h^1, w_h\right) = \Delta t \left(f^0, w_h\right),$$

so that

$$\left\|p_h^1\right\|_\varrho \leq \Delta t \left\|f^0\right\|_{\varrho^{-1}}.$$

Then, the second equation in (20) gives that

$$\left(\kappa^{-1} r_h^0, v_h\right) - \left(p_h^1, \nabla \cdot v_h\right) = \left(g^1, v_h\right).$$

Picking $v_h = r_h^0$ and using weighted Cauchy-Schwarz gives

$$\left\|r_h^0\right\|_{\kappa^{-1}}^2 \leq \left\|p_h^1\right\|_\varrho \left\|\nabla \cdot r_h^0\right\|_{\varrho^{-1}} + \left\|g^1\right\|_\kappa \left\|r_h^0\right\|_{\kappa^{-1}}.$$

Now, we use our bound on $\left\|p_h^1\right\|_\varrho$, equivalence of various norms, and the inverse assumption to find that

$$\begin{aligned}
\left\|p_h^1\right\|_\varrho \left\|\nabla \cdot r_h^0\right\|_{\varrho^{-1}} &\leq \frac{\Delta t}{\sqrt{\varrho_*}} \left\|f^0\right\|_{\varrho^{-1}} \left\|\nabla \cdot r_h^0\right\| \\
&\leq \frac{C_0 \Delta t}{h\sqrt{\varrho_*}} \left\|f^0\right\|_{\varrho^{-1}} \left\|r_h^0\right\| \\
&\leq \frac{C_0 \sqrt{\kappa^*} \Delta t}{h\sqrt{\varrho_*}} \left\|f^0\right\|_{\varrho^{-1}} \left\|r_h^0\right\|_{\kappa^{-1}}.
\end{aligned}$$

Hence, we have that

$$\left\|r_h^0\right\|_{\kappa^{-1}} \leq \frac{C_0 \sqrt{\kappa^*} \Delta t}{h\sqrt{\varrho_*}} \left\|f^0\right\|_{\varrho^{-1}} + \left\|g^1\right\|_\kappa. \tag{58}$$

The CFL condition requires $\Delta t = \mathcal{O}(h)$, so this bound does not degrade under mesh refinement.

Now, we define the sequence b_n to be the energy functional applied to q_h^n, r_h^n:

$$b_n^2 = \left\|(q_h^n, r_h^n)\right\|_{\mathcal{E}}^2 = \left\|\left(\frac{\Delta p_h^n}{\Delta t}, \frac{\Delta u_h^n}{\Delta t}\right)\right\|_{\mathcal{E}}^2 = \frac{1}{2}\left\|\frac{\Delta p_h^n}{\Delta t}\right\|_\varrho^2 + \frac{1}{2}\left\|\frac{\Delta u_h^n}{\Delta t}\right\|_{\kappa^{-1}}^2. \tag{59}$$

We can apply our previous stability theorem to to the equations (56), at least up to the penultimate time step to obtain

Theorem 8 *If the initial conditions p_h^0 and u_h^0 vanish and the CFL condition (50) holds, the time differences $\frac{\Delta p_h^n}{\Delta t}$ and $\frac{\Delta u_h^n}{\Delta t}$ satisfy the stability bound*

$$\begin{aligned}\max_{0\leq n\leq N-1} b_n \leq & \frac{1}{\sqrt{1-\alpha\Delta t}}\left(\left(1+\frac{C_0\Delta t}{h}\sqrt{\frac{\kappa^*}{\varrho_*}}\right)\left\|f^0\right\|_{\varrho^{-1}}+\left\|g^1\right\|_{\kappa}\right)\\ &+\frac{\sqrt{2}}{1-\alpha\Delta t}\sum_{n=0}^{N-2}\left(\left\|\frac{\Delta f^n}{\Delta t}\right\|_{\varrho^{-1}}+\left\|\frac{\Delta g^{n+1}}{\Delta t}\right\|_{\kappa}\right)\Delta t,\end{aligned} \tag{60}$$

provided that the quantities on the right-hand side are bounded.

Proof We apply the stability result of Theorem 7 to the equations (56) to find

$$\max_{0\leq n\leq N-1} b_n \leq \frac{1}{\sqrt{1-\alpha\Delta t}}\tilde{b}_0+\frac{\sqrt{2}}{1-\alpha\Delta t}\sum_{n=0}^{N-1}\left(\left\|\frac{\Delta f^n}{\Delta t}\right\|_{\varrho^{-1}}+\left\|\frac{\Delta g^{n+1}}{\Delta t}\right\|_{\kappa}\right)\Delta t,$$

and only the bound of the initial term requires explanation. We have

$$\tilde{b}_0\leq\sqrt{2}b_0\leq\left\|q_h^0\right\|_{\varrho}+\left\|r_h^0\right\|_{\kappa^{-1}}\leq\left\|f^0\right\|_{\varrho^{-1}}+\frac{C_0\Delta t}{h}\sqrt{\frac{\kappa^*}{\varrho_*}}\left\|f^0\right\|_{\varrho^{-1}}+\left\|g^1\right\|_{\kappa},$$

and collecting terms finishes the proof.

Using (20), we can also bound the divergence at each time level:

$$\left\|\nabla\cdot u_h^n\right\|\leq\left\|f^n\right\|+\sqrt{\varrho^*}\left\|\frac{\Delta p_h^n}{\Delta t}\right\|_{\varrho}\leq\left\|f^n\right\|+\sqrt{2\varrho^*}b^n, \tag{61}$$

which gives the following theorem

Theorem 9 *Under the hypotheses of Theorem 8, we have the bound*

$$\begin{aligned}\max_{0\leq n\leq N-1}\left\|\nabla\cdot u_h^n\right\|\leq & \max_{0\leq n\leq N-1}\left\|f^n\right\|\\ &+\sqrt{\frac{2\varrho^*}{1-\alpha\Delta t}}\left(\left(1+\frac{C_0\Delta t}{h}\sqrt{\frac{\kappa^*}{\varrho_*}}\right)\left\|f^0\right\|_{\varrho^{-1}}+\left\|g^1\right\|_{\kappa}\right)\\ &+\frac{2\sqrt{\varrho^*}}{1-\alpha\Delta t}\sum_{n=0}^{N-2}\left(\left\|\frac{\Delta f^n}{\Delta t}\right\|_{\varrho^{-1}}+\left\|\frac{\Delta g^{n+1}}{\Delta t}\right\|_{\kappa}\right)\Delta t.\end{aligned} \tag{62}$$

4.3 Error estimates

Now, we turn to the question of estimating the error. Let $p^n(\cdot) = p(\cdot, t_n)$ and $u^n(\cdot) = u(\cdot, t_n)$ be the true solution evaluated at the discrete time levels. Let $\pi p^n \in W_h$ and $\Pi u^n \in V_h$ be the projections of the true solutions at the discrete time levels.

As before, let $\xi = \pi p - p$. Then $\xi_t = \pi p_t - p_t$, and define ξ^n and ξ_t^n be evaluating ξ and its time derivative at the discrete time levels. We similarly define $\eta = \Pi u - u$ and its time derivative and evaluation at each t_n. As in the semidiscrete case, we let θ_h and χ_h denote the differences between the projections and computed solutions, but now we only have these at discrete time levels: $\theta_h^n = \pi p^n - p_h^n$ and $\chi_h^n = \Pi u^n - u_h^n$.

To handle the fully discrete estimate, differences between time derivatives and difference quotients also appear. We need difference operators applied to functions of time. For some $f(\cdot, t)$, we define $\Delta f(\cdot, t) = f(\cdot, t + \Delta t) - f(\cdot, t)$. This exactly agrees with differencing at discrete time levels. We define

$$\begin{aligned} \zeta(\cdot, t) &= \frac{\Delta p(\cdot, t)}{\Delta t} - p_t(\cdot, t), \\ \psi(\cdot, t) &= \frac{\Delta u(\cdot, t)}{\Delta t} - u_t(\cdot, t), \end{aligned} \tag{63}$$

and also $\zeta^n = \zeta(\cdot, t_n)$ and $\psi^n = \psi(\cdot, t_n)$. With these definitions, standard manipulations show that the true solution satisfies the discrete equation

$$\begin{aligned} \left(\varrho \frac{\Delta \pi p^n}{\Delta t}, w_h\right) + (\nabla \cdot \Pi u^n, w_h) &= (f^n, w_h) + \left(\varrho \frac{\Delta \xi^n}{\Delta t}, w_h\right) + (\varrho \zeta^n, w_h), \\ \left(\kappa^{-1} \frac{\Delta \Pi u^n}{\Delta t}, v_h\right) - \left(\pi p^{n+1}, \nabla \cdot v_h\right) &= \left(g^{n+1}, v_h\right) + \left(\kappa^{-1} \frac{\Delta \eta^n}{\Delta t}, v_h\right) + \left(\kappa^{-1} \psi^n, v_h\right), \end{aligned} \tag{64}$$

and subtracting (20) from this gives error equations

$$\begin{aligned} \left(\varrho\frac{\Delta\theta_h^n}{\Delta t}, w_h\right) + \left(\nabla\cdot\chi_h^n, w_h\right) &= \left(\varrho\frac{\Delta\xi^n}{\Delta t}, w_h\right) + \left(\varrho\zeta^n, w_h\right), \\ \left(\kappa^{-1}\frac{\Delta\chi_h^n}{\Delta t}, v_h\right) - \left(\theta_h^{n+1}, \nabla\cdot v_h\right) &= \left(\kappa^{-1}\frac{\Delta\eta^n}{\Delta t}, v_h\right) + \left(\kappa^{-1}\psi^n, v_h\right). \end{aligned} \tag{65}$$

Because the initial conditions for the discrete method coincide with the projections of the true solution, we have that θ_h^0 and χ_h^0 both vanish.

We define

$$\varepsilon^n = \left\|\left(\theta_h^n, \chi_h^n\right)\right\|_{\mathcal{E}}, \tag{66}$$

and make the bound:

Lemma 6 *Suppose that* $p_{tt} \in L^1\left(0,T;L^2\left(\Omega\right)\right)$, $u_{tt} \in L^1\left(0,T;\left(L^2\left(\Omega\right)\right)^d\right)$, $p_t \in L^1\left(0,T;H^m\left(\Omega\right)\right)$, $u_t \in L^1\left(0,T;\left(H^m\left(\Omega\right)\right)^d\right)$, *and that (50) holds. Then we have the error estimate*

$$\begin{aligned} \max_{0\le n\le N}\varepsilon^n \le &\frac{\sqrt{2\varrho^*}}{1-\alpha\Delta t}\left(C_1h^m\int_0^T\left|p_t\left(\cdot,s\right)\right|_m ds + \Delta t\int_0^T\left\|p_{tt}\left(\cdot,s\right)\right\| ds\right) \\ &+ \frac{\sqrt{2}}{\left(1-\alpha\Delta t\right)\sqrt{\kappa_*}}\left(C_2h^m\int_0^T\left|u_t\left(\cdot,s\right)\right|_m ds + \Delta t\int_0^T\left\|u_{tt}\left(\cdot,s\right)\right\| ds\right). \end{aligned} \tag{67}$$

Proof First, we apply the stability result of Theorem 7 to (65), noting that the initial conditions vanish, to find that

$$\begin{aligned} \max_{0\le n\le N}\varepsilon^n \le &\frac{\sqrt{2}}{1-\alpha\Delta t}\sum_{n=0}^{N-1}\left(\left\|\varrho\left(\frac{\Delta\xi^n}{\Delta t}+\zeta^n\right)\right\|_{\varrho^{-1}} + \left\|\kappa^{-1}\left(\frac{\Delta\eta^n}{\Delta t}+\psi^n\right)\right\|_{\kappa}\right)\Delta t \\ = &\frac{\sqrt{2}}{1-\alpha\Delta t}\sum_{n=0}^{N-1}\left(\left\|\frac{\Delta\xi^n}{\Delta t}+\zeta^n\right\|_{\varrho} + \left\|\frac{\Delta\eta^n}{\Delta t}+\psi^n\right\|_{\kappa^{-1}}\right)\Delta t. \end{aligned}$$

Now, we bound each of these terms separately. For the first, we have that

$$\sum_{n=0}^{N-1}\left\|\frac{\Delta\xi^n}{\Delta t}+\zeta_n\right\|_{\varrho}\Delta t \le \sqrt{\varrho^*}\sum_{n=0}^{N-1}\left[\left\|\frac{\Delta\xi^n}{\Delta t}\right\| + \left\|\zeta^n\right\|\right]\Delta t. \tag{68}$$

Since

$$\begin{aligned}
\sum_{n=0}^{N-1} \|\Delta\xi^n\| &= \sum_{n=0}^{N-1} \left\| \int_{t_n}^{t_{n+1}} \xi_t(\cdot, s) ds \right\| \\
&\leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\xi_t(\cdot, s) ds\| \\
&\leq C_1 h^m \int_0^T |p_t(\cdot, s)|_m \, ds.
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{N-1} \|\zeta^n\| \Delta t &= \sum_{n=0}^{N-1} \left\| p^{n+1} - p^n - \Delta t p_t^n \right\| \\
&= \sum_{n=0}^{N-1} \left\| \int_{t_n}^{t_{n+1}} (t_{n+1} - s) \, p_{tt}(\cdot, s) \, ds \right\| \\
&\leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} (t_{n+1} - s) \|p_{tt}(\cdot, s)\| \, ds \\
&\leq \sum_{n=0}^{N-1} \Delta t \int_{t_n}^{t_{n+1}} \|p_{tt}(\cdot, s)\| \, ds \\
&\leq \Delta t \int_0^T \|p_{tt}(\cdot, s)\| \, ds,
\end{aligned}$$

the first term in (68) is bounded by

$$\frac{\sqrt{2\varrho^*}}{1 - \alpha\Delta t} \left[C_1 h^m \int_0^T |p_t(\cdot, s)|_m \, ds + \Delta t \int_0^T \|p_{tt}(\cdot, s)\| \, ds \right].$$

Similar techniques for the second term complete the proof.

This lemma allows us to bound the error

$$\epsilon^n = \left\| (p^n - p_h^n, u^n - u_h^n) \right\|_{\mathcal{E}}. \tag{69}$$

Theorem 10 *Under the assumptions of the previous Lemma, we have that*

$$\max_{0\le n\le N}\epsilon^n \le \frac{\sqrt{2\varrho^*}}{1-\alpha\Delta t}\left(\frac{C_1h^m}{2}\sup_{0\le s\le T}\left|p\left(\cdot,s\right)\right|_m + C_1h^m\int_0^T\left|p_t\left(\cdot,s\right)\right|_m ds + \Delta t\int_0^T\left\|p_{tt}\left(\cdot,s\right)\right\| ds\right) + \frac{\sqrt{2}}{\left(1-\alpha\Delta t\right)\sqrt{\kappa_*}}\left(\frac{C_2h^m}{2}\sup_{0\le s\le T}\left|u\left(\cdot,s\right)\right|_m dt + C_2h^m\int_0^T\left|u_t\left(\cdot,s\right)\right|_m ds + \Delta t\int_0^T\left\|u_{tt}\left(\cdot,s\right)\right\| ds\right). \tag{70}$$

Proof From the triangle inequality, we have

$$\epsilon^n \le \left\|\left(\xi^n,\eta^n\right)\right\|_{\mathcal{E}} + \varepsilon^n.$$

The first term satisfies the bound

$$\left\|\left(\xi^n,\eta^n\right)\right\|_{\mathcal{E}} \le \frac{1}{\sqrt{2}}\left\|\xi^n\right\|_{\varrho} + \frac{1}{\sqrt{2}}\left\|\eta^n\right\|_{\kappa^{-1}} \le \sqrt{\frac{\varrho^*}{2}}\left\|\xi^n\right\| + \frac{1}{\sqrt{2\kappa_*}}\left\|\eta^n\right\|,$$

and our approximation estimates give that

$$\max_{0\le n\le N}\left\|\left(\xi^n,\eta^n\right)\right\|_{\mathcal{E}} \le C_1h^m\sqrt{\frac{\varrho^*}{2}}\sup_{0\le s\le T}\left|p\left(\cdot,s\right)\right|_m + \frac{C_2h^m}{\sqrt{2\kappa_*}}\sup_{0\le s\le T}\left|u\left(\cdot,s\right)\right|_m.$$

Combining this bound with the previous lemma and grouping terms completes the proof.

Next, we estimate the error in time difference quotients. We define

$$\beta_n = \left\|\left(\frac{\Delta\theta_h^n}{\Delta t},\frac{\Delta\chi_h^n}{\Delta t}\right)\right\|_{\mathcal{E}}. \tag{71}$$

Lemma 7 *Suppose that* $p_{tt}\in L^1\left(0,T;H^m\left(\Omega\right)\right)$*,* $u_{tt}\in L^1\left(0,T;\left(H^m\left(\Omega\right)\right)^d\right)$*, and* $p_{ttt}\in L^1(0,T;L^2(\Omega))$*,* $u_{ttt}\in L^1(0,T;(L^2(\Omega))^d)$ *and that the CFL condition (50)* *holds. Then we have the estimate*

$$
\begin{aligned}
\max_{0\le n\le N-1}\beta^n \le h^m \Bigg[& \frac{1}{\sqrt{1-\alpha\Delta t}}\left(C_1\sqrt{\varrho^*}\left(1+\frac{C_0\Delta t}{h}\sqrt{\frac{\kappa^*}{\varrho_*}}\right)\sup_{0\le s\le\Delta t}\left|p_t(\cdot,s)\right|_m \right.\\
&\left. + \frac{C_2}{\sqrt{\kappa_*}}\sup_{0\le s\le\Delta t}\left|u_t(\cdot,s)\right|_m\right)\\
&+\frac{2\sqrt{2}}{1-\alpha\Delta t}\left(C_1\sqrt{\varrho^*}\int_0^T\left|p_{tt}(\cdot,s)\right|_m ds\right.\\
&\left.+\frac{C_2}{\sqrt{\kappa_*}}\int_0^T\left|u_{tt}(\cdot,s)\right|_m ds\right)\Bigg]\\
+\Delta t\Bigg[& \frac{1}{\sqrt{1-\alpha\Delta t}}\left(\sqrt{\varrho^*}\left(1+\frac{C_0\Delta t}{h}\sqrt{\frac{\kappa^*}{\varrho_*}}\right)\sup_{0\le s\le\Delta t}\left\|p_{tt}(\cdot,s)\right\|\right.\\
&\left.+\frac{1}{\sqrt{\kappa_*}}\sup_{0\le s\le\Delta t}\left\|u_{tt}(\cdot,s)\right\|\right)\\
&+\frac{2\sqrt{2}}{1-\alpha\Delta t}\left(\sqrt{\varrho^*}\int_0^T\left\|p_{ttt}(\cdot,s)\right\| ds\right.\\
&\left.+\frac{1}{\sqrt{\kappa_*}}\int_0^T\left\|u_{ttt}(\cdot,s)\right\| ds\right)\Bigg].
\end{aligned}
\tag{72}
$$

Proof Since the initial conditions for the error equations (65) vanish, we can apply the stability estimate in Lemma 8 to obtain

$$
\begin{aligned}
\max_{0\le n\le N-1}\beta_n \le & \frac{1}{\sqrt{1-\alpha\Delta t}}\left(\left(1+\frac{C_0\Delta t}{h}\sqrt{\frac{\kappa^*}{\varrho_*}}\right)\left\|\frac{\Delta\xi^0}{\Delta t}+\zeta^0\right\|_\varrho+\left\|\frac{\Delta\eta^0}{\Delta t}+\psi^0\right\|_{\kappa^{-1}}\right)\\
&+\frac{\sqrt{2}}{1-\alpha\Delta t}\sum_{n=0}^{N-2}\left(\left\|\frac{\Delta^2\xi^n}{(\Delta t)^2}+\frac{\Delta\zeta^n}{\Delta t}\right\|_\varrho+\left\|\frac{\Delta^2\eta^n}{(\Delta t)^2}+\frac{\Delta\psi^n}{\Delta t}\right\|_{\kappa^{-1}}\right)\Delta t.
\end{aligned}
\tag{73}
$$

We take each of the norms on the right in turn. The first two are evaluated at the initial condition. We start with norm equivalence and the triangle inequality:

$$
\left\|\frac{\Delta\xi^0}{\Delta t}+\zeta^0\right\|_\varrho \le \sqrt{\varrho^*}\left\|\frac{\Delta\xi^0}{\Delta t}\right\|+\sqrt{\varrho^*}\left\|\zeta^0\right\|.
$$

The first of these is estimated by

$$
\left\|\frac{\Delta\xi^0}{\Delta t}\right\|=\frac{1}{\Delta t}\left\|\int_0^{\Delta t}\xi_t(\cdot,s)\,ds\right\|\le C_1h^m\sup_{0\le s\le\Delta t}\left|p_t(\cdot,s)\right|_m,
$$

and the second by

$$\left\|\zeta^0\right\| = \left\|\frac{\Delta p^0}{\Delta t} - p_t^0\right\| \leq \frac{1}{\Delta t}\left\|\int_0^{\Delta t} (\Delta t - s)\, p_{tt}(\cdot, s)\, ds\right\| \leq \Delta t \sup_{0\leq s\leq \Delta t} \|p_{tt}(\cdot, s)\| .$$

In a similar fashion, we find that

$$\left\|\frac{\Delta \eta^0}{\Delta t} + \psi^0\right\|_{\kappa^{-1}} \leq \frac{C_2 h^m}{\sqrt{\kappa_*}} \sup_{0\leq s\leq \Delta t} |u_t(\cdot, s)|_m + \frac{\Delta t}{\sqrt{\kappa_*}} \sup_{0\leq s\leq \Delta t} \|u_{tt}(\cdot, s)\| .$$

Now, we turn to the sums in (73). First, we have

$$\begin{aligned}\sum_{0\leq n\leq N-2} \left\|\frac{\Delta^2 \xi^n}{(\Delta t)^2} + \frac{\Delta \zeta^n}{\Delta t}\right\|_{\varrho} \Delta t &\leq \sqrt{\varrho^*} \sum_{0\leq n\leq N-2} \left(\left\|\frac{\Delta^2 \xi^n}{\Delta t}\right\| + \|\Delta \zeta^n\|\right) \\ &\equiv \sqrt{\varrho^*}\,(I + II),\end{aligned}$$

and we handle these in turn. To bound I, we start with the calculation

$$\begin{aligned}\frac{\Delta^2 \xi^n}{\Delta t} &= \frac{1}{\Delta t}\left(\Delta \xi^{n+1} - \Delta \xi^n\right) \\ &= \frac{1}{\Delta t}\left[\left(\Delta \xi^{n+1} - \Delta t \xi_t^{n+1}\right) - \left(\Delta \xi^n - \Delta t \xi_t^{n+1}\right)\right] \\ &= \frac{1}{\Delta t}\int_{t_{n+1}}^{t_{n+2}} (t_{n+2} - s)\xi_{tt}(\cdot, s)\, ds + \int_{t_n}^{t_{n+1}} (s - t_n)\xi_{tt}(\cdot, s)\, ds.\end{aligned}$$

In both of these integrals, we make the change variables $\sigma = s - t_{n+1}$ to find that

$$\begin{aligned}\frac{\Delta^2 \xi^n}{\Delta t} &= \frac{1}{\Delta t}\int_0^{\Delta t} (\Delta t - s)\,\xi_{tt}(\cdot, \sigma + t_{n+1})\, ds + \int_{-\Delta t}^{0} (\Delta t + s)\,\xi_{tt}(\cdot, \sigma + t_{n+1})\, ds \\ &= \frac{1}{\Delta t}\int_{-\Delta t}^{\Delta t} (\Delta t - |\sigma|)\,\xi_{tt}(\cdot, \sigma + t_{n+1})\, d\sigma \\ &= \frac{1}{\Delta t}\int_{t_n}^{t_{n+2}} (\Delta t - |s - t_{n+1}|)\,\xi_{tt}(\cdot, s)\, ds.\end{aligned}$$

Using this calculation in I lets us make the bound

$$\begin{aligned}\sum_{n=0}^{N-2}\left\|\frac{\Delta^2 \xi^n}{\Delta t}\right\| &= \sum_{n=0}^{N-2}\left\|\frac{1}{\Delta t}\int_{t_n}^{t_{n+2}} (\Delta t - |s - t_{n+1}|)\,\xi_{tt}(\cdot, s)\, ds\right\| \\ &\leq \sum_{n=0}^{N-2}\frac{1}{\Delta t}\int_{t_n}^{t_{n+2}} \|(\Delta t - |s - t_{n+1}|)\,\xi_{tt}(\cdot, s)\|\, ds \\ &\leq \sum_{n=0}^{N-2}\int_{t_n}^{t_{n+2}} \|\xi_{tt}(\cdot, s)\|\, ds \\ &\leq 2\int_0^T \|\xi_{tt}(\cdot, s)\|\, ds \\ &\leq 2C_1 h^m \int_0^T |p_{tt}(\cdot, s)|_m\, ds.\end{aligned}$$

Now, we turn to II:

$$II = \sum_{n=0}^{N-2} \|\Delta\zeta^n\| = \sum_{n=0}^{N-2} \left\| \int_{t_n}^{t_{n+1}} \zeta_t(\cdot, s)\, ds \right\|.$$

Differentiating (63), we find that

$$\zeta_t(\cdot, s) = \frac{\Delta p_t(\cdot, s)}{\Delta t} - p_{tt}(\cdot, s) = \frac{1}{\Delta t}\int_s^{s+\Delta t} (s + \Delta t - \tau)\, p_{ttt}(\cdot, \tau)\, d\tau.$$

We insert this into II and make the bounds

$$\begin{aligned}
\sum_{n=0}^{N-2} \|\Delta\zeta^n\| &= \sum_{n=0}^{N-2} \left\| \int_{t_n}^{t_{n+1}} \frac{1}{\Delta t} \left(\int_s^{s+\Delta t} (s + \Delta t - \tau)\, p_{ttt}(\cdot, \tau)\, d\tau \right) ds \right\| \\
&\leq \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} \frac{1}{\Delta t} \left(\int_s^{s+\Delta t} (s + \Delta t - \tau)\, \|p_{ttt}(\cdot, \tau)\|\, d\tau \right) ds \\
&\leq \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} \left(\int_s^{s+\Delta t} \|p_{ttt}(\cdot, \tau)\|\, d\tau \right) ds.
\end{aligned}$$

To proceed, we will interchange the order of integration. However, the limits of integration on the inner integral depend on one of the variables, and this requires writing the new inner integral with two separate integrals. We have

$$\begin{aligned}
\sum_{n=0}^{N-2} \|\Delta\zeta^n\| &\leq \sum_{n=0}^{N-2} \left(\int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{s} \|p_{ttt}(\cdot, \tau)\|\, ds \right) d\tau + \int_{t_{n+1}}^{t_{n+2}} \left(\int_{s-\Delta t}^{t_{n+1}} \|p_{ttt}(\cdot, \tau)\|\, ds \right) d\tau \right) \\
&= \sum_{n=0}^{N-2} \left(\int_{t_n}^{t_{n+1}} (s - t_n)\, \|p_{ttt}(\cdot, \tau)\|_\varrho\, d\tau + \int_{t_{n+1}}^{t_{n+2}} (t_{n+2} - s)\, \|p_{ttt}(\cdot, \tau)\|_\varrho\, d\tau \right) \\
&\leq 2\Delta t \int_0^T \|p_{ttt}(\cdot, \tau)\| d\tau.
\end{aligned}$$

Combining these estimates gives that

$$\begin{aligned}
\sum_{n=0}^{N-2} \left\| \frac{\Delta^2\xi^n}{(\Delta t)^2} + \frac{\Delta\zeta^n}{\Delta t} \right\|_\varrho \Delta t &\leq 2C_1 h^m \sqrt{\varrho^*} \int_0^T |p_{tt}(\cdot, s)|_m\, ds \\
&\quad + 2\Delta t \sqrt{\varrho^*} \int_0^T \|p_{ttt}(\cdot, s)\|\, ds.
\end{aligned} \tag{74}$$

Similar techniques allow us to write

$$\begin{aligned}
\sum_{n=0}^{N-2} \left\| \frac{\Delta^2\eta^n}{(\Delta t)^2} + \frac{\Delta\psi^n}{\Delta t} \right\|_{\kappa^{-1}} \Delta t &\leq \frac{2C_2 h^m}{\sqrt{\kappa_*}} \int_0^T |u_{tt}(\cdot, s)|_m\, ds \\
&\quad + \frac{2\Delta t}{\sqrt{\kappa_*}} \int_0^T \|u_{ttt}(\cdot, s)\| ds.
\end{aligned} \tag{75}$$

This result allows us to give optimal-order estimates for the difference between the computed difference quotients and the true derivatives at each time step. We define

$$\varXi^n = \left\| \left(\frac{\Delta p_h^n}{\Delta t} - p_t^n, \frac{\Delta u_h^n}{\Delta t} - u_t^n \right) \right\|_{\mathcal{E}}, \tag{76}$$

and have the following optimal-order estimate.

Theorem 11 *If the assumptions of Lemma 7 hold, we have the estimate*

$$\begin{aligned} \max_{0 \le n \le N-1} \varXi^n \le & h^m \left[C_1 \sqrt{\frac{\varrho^*}{2}} \sup_{0 \le s \le T} |p_t(\cdot, s)|_m + \frac{C_2}{\sqrt{2\kappa_*}} \sup_{0 \le s \le T} |u_t(\cdot, s)|_m \right] \\ & + \Delta t \left[\sqrt{\frac{\varrho^*}{2}} \sup_{0 \le s \le T} \|p_{tt}(\cdot, s)\| + \frac{1}{\sqrt{2\kappa_*}} \sup_{0 \le s \le T} \|p_{tt}(\cdot, s)\| \right] \\ & + \max_{0 \le n \le N-1} \beta_n. \end{aligned} \tag{77}$$

Proof With the help of the triangle inequality, we write

$$\max_{0 \le n \le N-1} \varXi_n \le \max_{0 \le n \le N-1} \left\| (\zeta^n, \psi^n) \right\|_{\mathcal{E}} + \max_{0 \le n \le N-1} \left\| \left(\frac{\Delta \xi^n}{\Delta t}, \frac{\Delta \eta^n}{\Delta t} \right) \right\|_{\mathcal{E}} + \max_{0 \le n \le N-1} \beta_n.$$

The first of these terms satisfies

$$\max_{0 \le n \le N-1} \left\| (\zeta^n, \psi^n) \right\|_{\mathcal{E}} \le \sqrt{\frac{\varrho^*}{2}} \max_{0 \le n \le N-1} \left\| \zeta^n \right\| + \frac{1}{\sqrt{2\kappa_*}} \max_{0 \le n \le N-1} \left\| \psi^n \right\|.$$

We bound the ζ^n term by

$$\begin{aligned} \max_{0 \le n \le N-1} \left\| \zeta^n \right\| &\le \frac{1}{\Delta t} \max_{0 \le n \le N-1} \left\| \Delta p^n - \Delta t p_t^n \right\| \\ &= \frac{1}{\Delta t} \max_{0 \le n \le N-1} \left\| \int_{t_n}^{t_{n+1}} (s - t_n) \, p_{tt}(\cdot, s) \, ds \right\| \\ &\le \Delta t \sup_{0 \le s \le T} \|p_{tt}(\cdot, s)\|. \end{aligned}$$

Similarly, we have

$$\max_{0 \le n \le N-1} \left\| \psi^n \right\| \le \Delta t \sup_{0 \le s \le T} \|u_{tt}(\cdot, s)\|.$$

We also have that

$$\max_{0\leq n\leq N-1}\left\|\left(\frac{\Delta\xi^n}{\Delta t},\frac{\Delta\eta^n}{\Delta t}\right)\right\|_{\mathcal{E}}\leq\sqrt{\frac{\varrho^*}{2}}\max_{0\leq n\leq N-1}\left\|\frac{\Delta\xi^n}{\Delta t}\right\|+\frac{1}{\sqrt{2\kappa_*}}\max_{0\leq n\leq N-1}\left\|\frac{\Delta\eta^n}{\Delta t}\right\|.$$

The first term here satisfies

$$\begin{aligned}\max_{0\leq n\leq N-1}\left\|\frac{\Delta\xi^n}{\Delta t}\right\|&=\frac{1}{\Delta t}\max_{0\leq n\leq N-1}\left\|\int_{t_n}^{t_{n+1}}\xi_t(\cdot,s)\,ds\right\|\\&\leq C_1h^m\sup_{0\leq s\leq T}|p_t(\cdot,s)|_m,\end{aligned}$$

and by the same argument,

$$\max_{0\leq n\leq N-1}\left\|\frac{\Delta\eta^n}{\Delta t}\right\|\leq C_2h^m\sup_{0\leq s\leq T}|u_t(\cdot,s)|_m.$$

We collect these estimates to finish the proof.

We also have optimal-order estimates for the divergence.

Theorem 12 *If the assumptions of Lemma 7 hold and additionally* $\nabla\cdot u\in L^\infty(0,T;H^m(\Omega))$, *then for* $0\leq m\leq r+1$ *we have the error estimate*

$$\begin{aligned}\max_{0\leq n\leq N-1}\left\|\nabla\cdot(u^n-u_h^n)\right\|\leq&C_1h^m\left[\sup_{0\leq s\leq T}|\nabla\cdot u(\cdot,s)|_m+\varrho^*\sup_{0\leq s\leq T}|p_t(\cdot,s)|_m\right]\\&+\varrho^*\Delta t\sup_{0\leq s\leq T}\|p_{tt}(\cdot,s)\|_m+\sqrt{2\varrho^*}\max_{0\leq n\leq N-1}\beta_n.\end{aligned}\tag{78}$$

Proof We start by writing

$$\max_{0\leq n\leq N-1}\left\|\nabla\cdot(u^n-u_h^n)\right\|\leq\max_{0\leq n\leq N-1}\left\|\nabla\cdot\eta^n\right\|+\max_{0\leq n\leq N-1}\left\|\nabla\cdot\chi_h^n\right\|.$$

The first term is purely approximation-theoretic, and (10) gives

$$\max_{0\leq n\leq N-1}\left\|\nabla\cdot\eta^n\right\|\leq C_1h^m\sup_{0\leq s\leq T}|\nabla\cdot u(\cdot,s)|_m.$$

We bound the second term by relating it back to our estimate for β_n. Much like the estimate for $\|\nabla\cdot u_h^n\|$ in (61), we can use the error equations (65) to find

$$\left\|\nabla\cdot\chi_h^n\right\|\leq\varrho^*\left\|\frac{\Delta\xi^n}{\Delta t}+\zeta^n\right\|+\sqrt{2\varrho^*}\beta_n.$$

Using standard techniques completes the proof.

5 Other time discretizations

Here, we briefly comment on a few other possible time discretizations with interesting conservation properties. The Crank-Nicholson method, assuming no forcing terms, satisfies

$$\begin{aligned}\left(\varrho\frac{\Delta p_h^n}{\Delta t}, w_h\right) + \left(\nabla\cdot\left(\frac{u_h^{n+1}+u_h^n}{2}\right), w_h\right) &= 0,\\ \left(\kappa^{-1}\frac{\Delta u_h^n}{\Delta t}, v_h\right) - \left(\frac{p_h^{n+1}+p_h^n}{2}, \nabla\cdot v_h\right) &= 0.\end{aligned} \tag{79}$$

If we select $w_h = \frac{p_h^n+p_h^{n+1}}{2}$ and $v_h = \frac{u_h^n+u_h^{n+1}}{2}$ and add these equations the terms with spatial derivatives cancel, giving

$$\left\|(p_h^{n+1}, u_h^{n+1})\right\|_{\mathcal{E}} = \left\|(p_h^n, u_h^n)\right\|_{\mathcal{E}} \tag{80}$$

for each time step – exact energy conservation. However, this comes at the cost of a more complicated linear system for each time step. While forward and symplectic Euler involve only inverting mass matrices, we now have a skew symmetric perturbation of the mass matrices, for the system matrix will have the form

$$\begin{bmatrix} M^{\varrho} & \Delta t D \\ -\Delta t D^T & \widetilde{M}^{\kappa^{-1}} \end{bmatrix}. \tag{81}$$

The complication of the skew pertubation could be offset by a larger allowable time step and exact conservation provided an effective preconditioner were available, but this is a subject of further investigation.

In fact, the Crank-Nicholson method can be seen as the lowest-order instance of a family of continuous Galerkin methods in the time variable. Logg [19] shows that this entire family of methods is exactly conservative for Hamiltonian systems and develops methods with variable time stepping for individual components. Like

Crank-Nicholson, these methods are all implicit. On the other hand, we can also consider higher-order explicit methods, such as the Störmer-Verlet method [8], which is second-order accurate and preserves a perturbation of the system energy which is quadratic rather than linear in Δt. For brevity, we will present the method and discuss the energy conservation in the constant coefficient case of $\varrho = \kappa = 1$.

The Störmer-Verlet method comes from combining the two variants of symplectic Euler in an appropriate way. First, we take a half-time step of symplectic Euler with the p variable explicit and the u variable implicit

$$\begin{aligned}\left(\frac{p_h^{n+\frac{1}{2}} - p_h^n}{\frac{\Delta t}{2}}, w_h\right) + \left(\nabla \cdot u_h^n, w\right) = \left(f^n, w_h\right), \\ \left(\frac{u_h^{n+\frac{1}{2}} - u_h^n}{\frac{\Delta t}{2}}, v_h\right) - \left(p_h^{n+\frac{1}{2}}, \nabla \cdot v_h\right) = \left(g^{n+\frac{1}{2}}, v_h\right).\end{aligned}$$

Then, we advance from time level $n + \frac{1}{2}$ to $n + 1$ with p implicit and u explicit:

$$\begin{aligned}\left(\frac{u_h^{n+1} - u_h^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}, v_h\right) - \left(p_h^{n+\frac{1}{2}}, \nabla \cdot v_h\right) = \left(g^{n+\frac{1}{2}}, v_h\right), \\ \left(\frac{p_h^{n+1} - p_h^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}, w_h\right) + \left(\nabla \cdot u_h^{n+1}, w_h\right) = \left(f^{n+1}, w_h\right).\end{aligned}$$

The two equations for u in these half-steps are readily combined to give the overall method

$$\begin{aligned}\left(\frac{p_h^{n+\frac{1}{2}} - p_h^n}{\frac{\Delta t}{2}}, w_h\right) + \left(\nabla \cdot u_h^n, w_h\right) = \left(f^n, w_h\right), \\ \left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) - \left(p_h^{n+\frac{1}{2}}, \nabla \cdot v_h\right) = \left(g^{n+\frac{1}{2}}, v_h\right), \\ \left(\frac{p_h^{n+1} - p_h^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}, r_h\right) + \left(\nabla \cdot u_h^{n+1}, r_h\right) = \left(f^{n+1}, r_h\right),\end{aligned} \tag{82}$$

where we have renamed the test function in the last equation to emphasize its independence from the test function in the first.

A further simplification reducing the amount of computation is obtained by combining the first and third equations into a single rule advancing p between half time steps, but it is easier to analyze the conservation properties of the present form. To do so, we assume that the forcing terms f and g both vanish and select

$$\begin{aligned} w_h &= \frac{p_h^n + p_h^{n+\frac{1}{2}}}{4} \\ v_h &= \frac{u_h^n + u_h^{n+1}}{2} \\ r_h &= \frac{p_h^{n+\frac{1}{2}} + p_h^{n+1}}{4} \end{aligned}$$

and add the three equations together to find that

$$\begin{aligned} \frac{1}{2\Delta t}\left(\left\|p_h^{n+1}\right\|^2 - \|p_h^n\|^2\right) + \frac{1}{2\Delta t}\left(\left\|u_h^{n+1}\right\|^2 - \|u_h^n\|^2\right) + \left(\nabla\cdot u_h^n, \frac{p_h^{n+\frac{1}{2}} + p_h^n}{4}\right) \\ - \left(p_h^{n+\frac{1}{2}}, \nabla\cdot\left(\frac{u_h^{n+1}+u_h^n}{2}\right)\right) + \left(\nabla\cdot u_h^{n+1}, \frac{p_h^{n+1} + p_h^{n+\frac{1}{2}}}{4}\right) = 0. \end{aligned} \tag{83}$$

We can rewrite the last three terms on the left-hand side as

$$\frac{1}{4}\left(\nabla\cdot u_h^{n+1}, p_h^{n+1} - p_h^{n+\frac{1}{2}}\right) - \frac{1}{4}\left(\nabla\cdot u_h^n, p_h^{n+\frac{1}{2}} - p_n^n\right)$$

so that, multiplying (83) through by Δt, we have

$$\begin{aligned} \frac{1}{2}\left(\left\|p_h^{n+1}\right\|^2 - \|p_h^n\|^2\right) + \frac{1}{2}\left(\left\|u_h^{n+1}\right\|^2 - \|u_h^n\|^2\right) \\ + \frac{\Delta t^2}{8}\left(\nabla\cdot u_h^{n+1}, \frac{p_h^{n+1} - p_h^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}\right) - \frac{\Delta t^2}{8}\left(\nabla\cdot u_h^n, \frac{p_h^{n+\frac{1}{2}} - p_n^n}{\frac{\Delta t}{2}}\right) = 0. \end{aligned} \tag{84}$$

This does not quite have the form of some $\tilde{a}_{n+1}^2 - \tilde{a}_n^2 = 0$, but if we pick test functions $w_h = \nabla\cdot u_h^n$, $v_h = \frac{u_h^n + u_h^{n+1}}{2}$, and $r_h = \nabla\cdot u_h^{n+1}$ in the first and third equations of (82), again assuming zero forcing terms, we find that

$$\frac{1}{2}\left(\left\|p_h^{n+1}\right\|^2 - \|p_h^n\|^2\right) + \frac{1}{2}\left(\left\|u_h^{n+1}\right\|^2 - \|u_h^n\|^2\right) - \frac{\Delta t^2}{8}\left\|\nabla\cdot u_h^{n+1}\right\|^2 + \frac{\Delta t^2}{8}\left\|\nabla\cdot u_h^n\right\|^2 = 0, \tag{85}$$

so that if we define $\overline{a}_n = \frac{1}{2}\left(\|p_h^n\|^2 + \|u_h^n\|^2\right) - \frac{\Delta t^2}{8}\|\nabla \cdot u_h^n\|$, we see that

$$\overline{a}_{n+1} = \overline{a}_n$$

for each time step. Moreover, use of the inverse assumption gives an equivalence of the conserved quantity to the actual energy under a reasonable (i.e. $\Delta t = \mathcal{O}(h)$) assumption. The rest of the analysis will proceed just as for symplectic Euler, but with second-order error estimates in time.

6 Numerical results

In this section, we give some simple numerical results illustrating the convergence theory. We consider the constant coefficient wave equation on the unit square in two dimensions, both equipped with homogeneous Dirichlet boundary conditions. Since we only analyze a first order time discretization, we consider the lowest order mixed method. We used FEniCS [20] to perform our numerical simulations.

In two dimensions, the function

$$p(x, y, t) = \cos\left(\sqrt{2}\pi t\right)\sin(\pi x)\sin(\pi y)$$

exactly solves the second-order wave equation. We picked initial conditions $p(x, y, 0)$ consistent with this and initial velocity of zero. The corresponding velocity solution $u(x, y, t)$ is then obtained by antidifferentiating $-\nabla p$.

We divided the unit square into an $N \times N$ mesh of squares, each then subdivided into two right triangles using the `UnitSquareMesh` class in FEniCS. For each mesh, we solved the wave equation numerically until final time $T = \frac{1}{2}$. At this time, we measured the L^2 errors $\|p - p_h\|$, $\|u - u_h\|$, and $\|\nabla \cdot (u - u_h)\|$. Additionally, since energy conservation is an important motivation for using a symplectic time

N	$\|p - p_h\|$	$\|u - u_h\|$	$\|\nabla \cdot (u - u_h)\|$	$\max_n \frac{\lvert a_n^2 - a_0^2 \rvert}{a_0^2}$
16	2.63E-02	2.27E-02	1.18E-01	4.12E-02
32	1.31E-02	1.13E-02	5.84E-02	2.05E-02
64	6.50E-03	5.64E-03	2.91E-02	1.03E-02
128	3.24E-03	2.82E-03	1.45E-02	5.18E-03
256	1.62E-03	1.41E-03	7.24E-03	2.60E-03
512	8.08E-04	7.05E-04	3.62E-03	1.30E-03

integrator, we measured the maximum relative deviation from the initial energy in the system. That is, we computed the initial (exact) energy from the initial condition by

$$a_0^2 = \frac{1}{2} \left\| p_h^0 \right\|^2 + \frac{1}{2} \left\| u_h^0 \right\|^2,$$

and then the energy at each time level

$$a_n^2 = \frac{1}{2} \left\| p_h^n \right\|^2 + \frac{1}{2} \left\| u_h^n \right\|^2.$$

At each time level, we measured

$$\frac{\left| a_n^2 - a_0^2 \right|}{a_0^2},$$

and we report the maximum of this over all time steps for each mesh.

7 Conclusion and future directions

We have developed a method for the acoustic wave equation that preserves the essential structures of the spatial and temporal discretization. In addition to optimal estimates in the typical $L^\infty(L^2)$-based norms, our bootstrap techniques enable optimal estimates in stronger norms, as well. These results leave open many questions for further study.

Linear-algebraic questions related to handling the mass matrices for explicit methods (whether the symplectic Euler or high-order methods) and matrices like (81) for implicit ones will need to be addressed. On rectangular meshes, diagonalizing quadrature [14, 15, 26] would lead to explicit time-marching schemes at the cost of an additional perturbation to the conserved energy functional and restriction to highly structured geometry. In the low-order case, it may be possible to handle general quadrilaterals by the techniques in [27]. For simplicial meshes, Jenkins [16] reports that V_h mass matrices are easily handled by conjugate gradient algorithms.

Additionally, we need to study the applicability of our methodology to other kinds of equations. An application to curl-curl wave equations with edge elements should be straightforward, and current research is focused on mixed formulations of the Klein-Gordon equation. Finally, extension of the formulation and analysis beyond basic reflecting boundary conditions remains an open question of interest.

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