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SOLUTION OF PARABOLIC EQUATIONS BY BACKWARD EULER-MIXED FINITE ELEMENT METHODS ON A DYNAMICALLY CHANGING MESH*

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Abstract. We develop and analyze methods based on combining the lowest-order mixed finite element method with backward Euler time discretization for the solution of diffusion problems on dynamically changing meshes. The methods developed are shown to preserve the optimal rate error estimates that are well known for static meshes. The novel aspect of the scheme is the construction of a linear approximation to the solution, which is used in projecting the solution from one mesh to another. Extensions to advection-diffusion equations are discussed, where the advection is handled by upwinding. Numerical results validating the theory are also presented.

Key words. adaptive finite element methods, mixed finite element methods, upwinding, diffusion equations

AMS subject classifications. 35Q35, 35L65 65N30, 65N15

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1. Introduction.

Adaptive finite element methods, whereby the mesh changes dynamically during a simulation, have become important tools for approximating the solutions to partial differential equations (PDEs) efficiently and accurately. In this paper, we consider the solution of time-dependent, parabolic partial differential equations by a standard time-stepping procedure such as a backward Euler method, combined with the "lowest-order" Raviart–Thomas mixed finite element method in space [17]. In particular, we are concerned with deriving a priori error estimates for such procedures when the mesh changes with the time step. An equally important task is the development of a posteriori error estimates which give some indication of how to change the mesh.

Dynamically adaptive finite element methods have been considered by a number of authors. In particular, we refer the reader to the discontinuous Galerkin methods proposed and analyzed in series of papers by Eriksson and Johnson [9, 10, 11, 12], the moving space-time finite elements studied by Bank and Santos [2], the earlier work on mesh modification by Dupont [7], and the moving mesh methods of Miller and Miller [16, 15]. For applications to the mixed finite element method, see, for example, the work of Yang [21, 20].

When analyzing dynamically adaptive methods, one difficulty is in proving the same asymptotic order of accuracy observed when the mesh is static. For example, in the work of Eriksson and Johnson, the a priori error bound obtained is almost optimal, up to multiplication by a logarithmic factor of the time step. This estimate appears to be the closest to optimal that one can prove, under fairly general assumptions on the mesh. These estimates are obtained for a discontinuous Galerkin method in time combined with a continuous, piecewise linear Galerkin method in space.

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In this paper, we will first consider a standard mixed finite element-backward Euler method with dynamically changing mesh and will demonstrate that this method has optimal rate of convergence, assuming that the meshes between two consecutive time levels are obtained by very simple coarsening or refining procedures. This estimate, while not general, is very practical, as this approach is quite easy to implement. We will then consider a modification to the mixed method which preserves the optimal convergence rate under very general changes in the mesh. This approach involves first postprocessing the solution at the current time-step to obtain a discontinuous, piecewise linear approximation. This piecewise linear approximation is then projected onto the new mesh. The postprocessed solution is easily obtained, as the mixed finite element method gives an approximation to the gradient. Finally, we extend these estimates to convection-diffusion equations, where the convection operator is approximated using an upwind method [4, 5].

Our focus here on the mixed finite element method is dictated by the applications of interest, in particular, the solution of transport problems arising in porous media and surface water. For these applications, generally described by advection-diffusion-reaction equations, the mixed finite element method for diffusion combined with some type of upwinding or characteristic approach for advection has nice features, including local conservation of mass, minimal oscillation, and the ability to approximate sharp fronts. See, for example, [6, 18, 19] for applications to practical problems. A particular difficulty in these problems is the presence of chemical reactions, which are highly localized and can require extremely fine grid to resolve accurately. Therefore, we are investigating adaptive gridding as a way of handling these problems.

This paper is organized as follows. In the next section, we describe a standard mixed finite element method for the heat equation and analyze it in the case of a special type of dynamically changing mesh. In section 4, we introduce a novel modification to this method, which preserves the accuracy under very general assumptions on the mesh modifications. In section 5, we extend these estimates to an upwind-mixed method applied to advection-diffusion equations, such as those arising in typical transport problems. We also consider as a special case a constant coefficient advection-diffusion equation with diffusion coefficient possibly being zero. Finally, in section 6, we give some numerical results for a one-dimensional test problem.

2. Notation and assumptions. In this section, we describe a backward-Euler mixed finite element method for the solution of the heat equation on a dynamically changing mesh and derive an error estimate in the case of a special type of mesh modification.

We first give some notation and basic assumptions. Let Δt^n, $n = 1, 2, \dots, N^*$ denote a sequence of time-steps, $t^n = \sum_{k=1}^{n} \Delta t^k$, $T = \sum_{n=1}^{N^*} \Delta t^n$, and for $g = g(t)$, let $g^n = g(t^n)$. We will assume the time-steps Δt^n don't change too rapidly, that is, we assume there exist positive constants k_* and k^* such that

$$k_* \leq \frac{\Delta t^n}{\Delta t^{n-1}} \leq k^*, \tag{2.1}$$

independent of n and Δt, where $\Delta t = \max_n \Delta t^n$.

We assume Ω is a bounded domain in $\mathbb{R}^d$, $d = 1, 2$, or 3, with boundary $\partial\Omega$. We denote by $(\cdot,\cdot)_S$ and $||\cdot||_S$ the L^2 inner product and norm, respectively, defined over a set S. When $S = \Omega$, we omit the subscript. At each time level t^n, we construct a partition $\mathcal{T}^n$ of Ω, suitable for the mixed finite element method (for example, quadrilaterals and triangles in two dimensions, tetrahdra, hexahedra, and prisms in three dimensions), with maximum element diameter h^n. Boundary elements are allowed to have one curvilinear edge or nonflat face.

Let $h = \max_n h^n$. We will assume throughout that

$$\Delta t^n = \mathcal{O}(h^n); \tag{2.2}$$

thus the mesh diameter h^n must also satisfy an inequality of the form (2.1).

In our analysis, we will use the standard equality

$$a(a-b) = \frac{a^2 - b^2 + (a-b)^2}{2}, \quad a, b \in \mathbb{R}, \tag{2.3}$$

and inequality

$$|ab| \le \frac{1}{2\delta}a^2 + \frac{\delta}{2}b^2, \quad a, b, \delta \in \mathbb{R}, \delta > 0. \tag{2.4}$$

Also, throughout the paper K denotes a generic positive constant and ϵ a generic small positive constant.

3. The standard method. We begin by considering the heat equation, written in mixed form,

$$\left.\begin{array}{l} c_t + \nabla \cdot z = f \\ z = -\nabla c \end{array}\right\} \text{ on } \Omega \times (0, T] \tag{3.1}$$

with smooth initial condition,

$$c(x, 0) = c^0(x), \quad x \in \Omega, \tag{3.2}$$

and boundary condition,

$$c(x, t) = 0, \quad x \in \partial\Omega, t \in [0, T]. \tag{3.3}$$

The weak form of (3.1) of interest is

$$(c_t, w) + (\nabla \cdot z, w) = (f, w), \quad w \in L^2(\Omega), \tag{3.4}$$

and

$$(z, v) = (c, \nabla \cdot v), \quad v \in H(\Omega; \mathrm{div}), \tag{3.5}$$

where to obtain (3.5) we have integrated by parts and used (3.3).

Let $W_h^n \subset L^2(\Omega)$ and $V_h^n \subset H(\Omega; \mathrm{div})$ denote the lowest-order Raviart–Thomas spaces defined on the partition $\mathcal{T}^n$ of Ω. The space W_h^n is the space of functions which are constant on each element in $\mathcal{T}^n$, and V_h^n is the space of vector-valued functions whose components are linear on each element and whose normal component is continuous across interior element boundaries; see [17]. As discussed in Douglas and Roberts [14], for boundary elements with possibly one curved edge, the space is unchanged.

At each time level n, c^n and z^n are approximated by $C^n \in W_h^n$ and $Z^n \in V_h^n$. At initial time, we set

$$(C^0, w^0) = (c^0, w^0), \quad w^0 \in W_h^0. \tag{3.6}$$

Then for $n = 1, 2, \ldots,$

$$\left(\frac{C^n - C^{n-1}}{\Delta t^n}, w^n\right) + (\nabla \cdot Z^n, w^n) = (f^n, w^n), \quad w^n \in W_h^n, \tag{3.7}$$

$$(Z^n, v^n) = (C^n, \nabla \cdot v^n), \quad v^n \in V_h^n. \tag{3.8}$$

We note that, in (3.7), we must compute (C^{n-1}, w^n). That is, we compute the L^2 projection of C^{n-1}, which is a piecewise constant on $\mathcal{T}^{n-1}$, into piecewise constants on $\mathcal{T}^n$. Unlike in a Galerkin finite element method, this does not involve solving a system of equations.

Note that (3.7)–(3.8) gives a square system of equations. By setting $w^n = C^n$ and $v^n = Z^n$, the L^2 stability of this method is easily shown. Moreover, setting $f = 0$, one can show the uniqueness of the solution, and existence follows. To derive an a priori error estimate, we compare C^n to an L^2 projection $\Pi c^n \in W_h^n$, defined by

$$(c^n, w^n) = (\Pi c^n, w^n), \quad w^n \in W_h^n. \tag{3.9}$$

We compare Z^n to the well-known π-projection [17, 14], $\pi z^n \in V_h^n$, which satisfies

$$(\nabla \cdot (z^n - \pi z^n), w^n) = 0, \quad w^n \in W_h^n. \tag{3.10}$$

For $c^n \in H^1(\Omega)$, we have the error estimate

$$||c^n - \Pi c^n|| \le K h^n, \tag{3.11}$$

and for $z^n \in H^1(\Omega)$,

$$||z^n - \pi z^n|| \le K h^n. \tag{3.12}$$

Let $\psi_c = C - \Pi c$, $\psi_z = Z - \pi z$, $\theta_c = c - \Pi c$, and $\theta_z = z - \pi z$.

The true solution satisfies

$$\left(\frac{c^n - c^{n-1}}{\Delta t^n}, w^n\right) + (\nabla \cdot z^n, w^n) = (f^n, w^n) - (\rho^n, w^n), \quad w^n \in W_h^n, \tag{3.13}$$

$$(z^n, v^n) = (c^n, \nabla \cdot v^n), \quad v^n \in V_h^n, \tag{3.14}$$

where

$$\rho^n = c_t^n - \frac{c^n - c^{n-1}}{\Delta t^n}. \tag{3.15}$$

Thus, subtracting (3.13) from (3.7) and (3.14) from (3.8) we find

$$\begin{aligned} &\left(\frac{\psi_c^n - \psi_c^{n-1}}{\Delta t^n}, w^n\right) + (\nabla \cdot \psi_z^n, w^n) \\ &= \left(\frac{\theta_c^n - \theta_c^{n-1}}{\Delta t^n}, w^n\right) + (\rho^n, w^n), \quad w^n \in W_h^n; \end{aligned} \tag{3.16}$$

$$(\psi_z^n, v^n) = (\psi_c^n, \nabla \cdot v^n) + (\theta_z^n, v^n), \quad v^n \in V_h^n. \tag{3.17}$$

At this point, if we were to follow the technique in [10], we would multiply (3.16) by Δt^n and sum in time from $n = 1$ to the final time level N^* and sum the time difference terms by parts. We would then define w^n so as to satisfy an adjoint equation, namely,

$$\left(\frac{w^n - w^{n+1}}{\Delta t^n}, \chi^n\right) + (\nabla \cdot Y^n, \chi^n) = 0, \quad \chi^n \in W_h^n, \qquad n = N^* - 1, N^* - 2, \ldots, 1, \tag{3.18}$$

where $Y^n \in V_h^n$ satisfies

$$(Y^n, v^n) = (w^n, \nabla \cdot v^n), \quad v^n \in V_h^n, \tag{3.19}$$

and $w^{N^*} = \psi_c^{N^*}$. The trick is to find a tight bound for the term

$$\sum_n ||w^n - w^{n+1}|| \tag{3.20}$$

in terms of $||\psi_c^{N^*}||$. This bound is used in handling the first term on the right side of (3.16), which is the critical term in the estimate.

Rather than pursue this type of estimate, we study a special but very practical situation, and for simplicity we restrict our attention to triangular elements in $\mathbb{R}^2$. We assume the following:

(A1) $\Omega \subset \mathbb{R}^2$.

(A2) Each partition $\mathcal{T}^0$, $\mathcal{T}^1$, etc., consists of disjoint triangular elements such that no vertex of any triangle lies on the interior of a side of another triangle. Boundary triangles may have one curved edge.

(A3) Each mesh $\mathcal{T}^n$ is a refinement of some given coarse partition $\mathcal{T}$ of Ω. Moreover, $\mathcal{T}^n$ is obtained by at most one level of refinement or coarsening of the mesh $\mathcal{T}^{n-1}$. Thus, for example, if Ω_e^{n-1} is an element in $\mathcal{T}^{n-1}$, then it may be refined into four smaller triangles by joining the midpoints of the edges, and its neighbors must also be refined by joining the midpoints of the refined edges to the opposite vertex; see Figure 3.1. Furthermore, if Ω_e^{n-1} is part of a larger triangle which was previously refined, the mesh may be coarsened to the larger triangle. The extension to a triangle with a curved edge is obvious.

Setting $w^n = \psi_c^n$ in (3.16) and $v^n = \psi_z^n$ in (3.17), adding these equations, and using (2.3), we find

$$\frac{1}{2\Delta t^n}\left(||\psi_c^n||^2 - ||\psi_c^{n-1}||^2 + ||\psi_c^n - \psi_c^{n-1}||^2\right) + ||\psi_z^n||^2 = (\rho^n, \psi_c^n) + (\theta_z^n, \psi_z^n) + \left(\frac{\theta_c^n - \theta_c^{n-1}}{\Delta t^n}, \psi_c^n\right). \tag{3.21}$$

Let N be the time-step at which $||\psi_c^n||$ is maximized, that is,

$$||\psi_c^N||^2 = \max_{1 \le n \le N^*} ||\psi_c^n||^2. \tag{3.22}$$

Multiplying (3.21) by $2\Delta t^n$ and summing on n, $n = 1, \ldots, N$, we find

$$||\psi_c^N||^2 + \sum_{n=1}^{N} ||\psi_c^n - \psi_c^{n-1}||^2 + 2\sum_{n=1}^{N} ||\psi_z^n||^2 \Delta t^n = 2\sum_{n=1}^{N}\left[(\rho^n, \psi_c^n) + (\theta_z^n, \psi_z^n) + \left(\frac{\theta_c^n - \theta_c^{n-1}}{\Delta t^n}, \psi_c^n\right)\right]\Delta t^n. \tag{3.23}$$

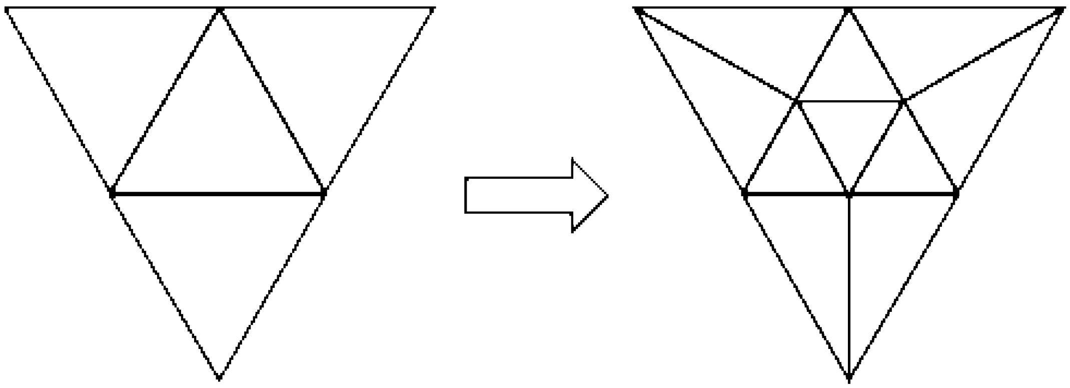

FIG. 3.1. *Refinement of an element* Ω_e^{n-1} *and its neighbors.*

The first two terms on the right side of (3.23) are easily shown to be bounded by

$$\frac{1}{2}||\psi_c^N||^2 + K\left(\sum_{n=1}^{N}||\rho^n||\Delta t^n\right)^2 + \frac{1}{4}\sum_{n=1}^{N}||\psi_z^n||^2\Delta t^n + K\sum_{n=1}^{N}||\theta_z^n||^2\Delta t^n. \tag{3.24}$$

Consider the third term on the right side of (3.23). Since, by the definition of θ_c^n,

$$(\theta_c^n, \psi_c^n) = 0, \tag{3.25}$$

this term becomes

$$-\sum_{n=1}^{N}(\theta_c^{n-1}, \psi_c^n). \tag{3.26}$$

On the mesh $\mathcal{T}^n$, let $\mathcal{E}_C$ denote the set of elements which are unchanged or result from coarsening of elements in $\mathcal{T}^{n-1}$. Let $\mathcal{E}_R$ denote those elements which are obtained by refining elements in $\mathcal{T}^{n-1}$. Then, dropping the subscript c momentarily in (3.26), consider

$$(\theta^{n-1}, \psi^n) = \sum_{e\in\mathcal{E}_C}(\theta^{n-1}, \psi^n)_{\Omega_e^n} + \sum_{e\in\mathcal{E}_R}(\theta^{n-1}, \psi^n)_{\Omega_e^n}. \tag{3.27}$$

On element Ω_e^n, $e \in \mathcal{E}_C$, ψ^n and Πc^{n-1} are both piecewise constants; thus the first sum vanishes by the definition of θ_c^{n-1}, and we need to consider only the second term. By our assumption on the relationship between meshes $\mathcal{T}^n$ and $\mathcal{T}^{n-1}$, a refined element Ω_e^n is one of four elements (or two elements) making up a triangle in $\mathcal{T}^{n-1}$. Assume

$$\Omega_e^{n-1} = \cup_{j=1}^4 \Omega_{e_j}^n; \tag{3.28}$$

see Figure 3.2. Then the second sum in (3.27) involves terms like

$$\begin{aligned}\sum_{j=1}^{4}(\theta^{n-1}, \psi^n)_{\Omega_{e_j}^n} &= (\theta^{n-1}, \psi^n)_{\Omega_e^{n-1}} \\ &= (\theta^{n-1}, \psi^n - \bar{\psi^n})_{\Omega_e^{n-1}},\end{aligned} \tag{3.29}$$

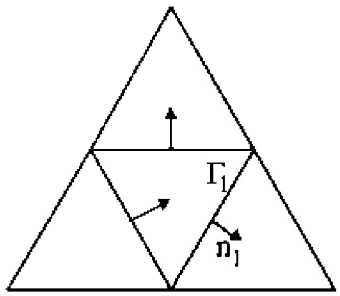

FIG. 3.2. *Elements and edges in* Ω_e^{n-1}.

where

$$\bar{\psi^n} = \frac{1}{m(\Omega_e^{n-1})}\int_{\Omega_e^{n-1}} \psi^n dx = \frac{1}{m(\Omega_e^{n-1})}\sum_{j=1}^{4} m(\Omega_{e_j}^n)\psi_j^n dx \tag{3.30}$$

with ψ_j^n representing the constant value of ψ^n on $\Omega_{e_j}^n$.

Let Γ_l, $l = 1, 2, 3$, denote the edges between elements $\Omega_{e_j}^n$ (see Figure 3.2), let ν_l be a unit vector normal to Γ_l, and let h_l denote the length of the edge. Let $[\psi^n]_{\Gamma_l}$ denote the jump in ψ^n across Γ_l in the direction of the normal vector. Then it is easily shown that

$$\int_{\Omega_e^{n-1}} |\psi^n - \bar{\psi}^n|^2 dx \le K h^n \sum_{l=1}^{3} |[\psi^n]_{\Gamma_l}|^2 h_l. \tag{3.31}$$

For any $v \in V_h^n$, v is specified in the interior of Ω by specifying $v \cdot \nu$ (which is constant) on each edge, where ν is a normal vector to the edge. On boundary elements having one curved edge, $v \cdot \nu$ is specified on the two straight edges, and $\nabla \cdot v$ is also specified, which then determines the average value of $v \cdot \nu$ on the curved edge. Let $\sigma \in V_h^n$ satisfy

$$\sigma \cdot \nu_l = [\psi^n]_{\Gamma_l} \quad \text{on } \Gamma_l \tag{3.32}$$

and let $\sigma \cdot \nu$ have average value zero on all other edges. In particular then $\sigma \equiv 0$ outside of Ω_e^{n-1}. Consider

$$(\psi^n, \nabla \cdot \sigma) = (\psi^n, \nabla \cdot \sigma)_{\Omega_e^{n-1}} = \sum_{j=1}^{4} \psi_j^n \int_{\Omega_{e_j}^n} \nabla \cdot \sigma = -\sum_{l=1}^{3} |[\psi^n]_{\Gamma_l}|^2 h_l, \tag{3.33}$$

where $h_l = m(\Gamma_l)$.

On the other hand, by (3.17) and the definition of σ,

$$-(\psi_c^n, \nabla \cdot \sigma) = -(\psi_z^n, \sigma) + (\theta_z^n, \sigma) \tag{3.34}$$

$$\leq \left(||\psi_z^n||_{\Omega_e^{n-1}} + ||\theta_z^n||_{\Omega_e^{n-1}}\right) ||\sigma||$$
$$\leq K(||\psi_z^n||_{\Omega_e^{n-1}} + ||\theta_z^n||_{\Omega_e^{n-1}}) \left(h^n \sum_l |[\psi^n]_{\Gamma_l}|^2 h_l\right)^{1/2}.$$

Thus by (3.33) and (3.34),

$$\sum_l |[\psi^n]_{\Gamma_l}|^2 h_l \leq K h^n (||\psi_z^n||^2_{\Omega_e^{n-1}} + ||\theta_z^n||^2_{\Omega_e^{n-1}}). \tag{3.35}$$

Combining (3.29)–(3.35) we find

$$\begin{aligned}(\psi_c^n - \bar{\psi}_c^n, \theta_c^{n-1})_{\Omega_e^{n-1}} &\leq \frac{\epsilon}{\Delta t^n} ||\psi_c^n - \bar{\psi}_c^n||^2_{\Omega_e^{n-1}} + K||\theta_c^{n-1}||^2_{\Omega_e^{n-1}} \Delta t^n \\ &\leq \frac{K\epsilon h^n}{\Delta t^n} \sum_l |[\psi_c^n]_{\Gamma_l}|^2 h_l + K||\theta_c^{n-1}||^2_{\Omega_e^{n-1}} \Delta t^n \\ &\leq \frac{K\epsilon (h^n)^2}{\Delta t^n} (||\psi_z^n||^2_{\Omega_e^{n-1}} + ||\theta_z^n||^2_{\Omega_e^{n-1}}) \\ &\quad + K||\theta_c^{n-1}||^2_{\Omega_e^{n-1}} \Delta t^n.\end{aligned} \tag{3.36}$$

Thus, returning to (3.26), we find

$$\begin{aligned}\sum_{n=1}^N (\theta_c^{n-1}, \psi_c^n) \leq K \sum_{n=1}^N \epsilon \left(\frac{h^n}{\Delta t^n}\right)^2 \left(||\psi_z^n||^2 + ||\theta_z^n||^2\right) \Delta t^n \\ + K \sum_{n=1}^N ||\theta_c^{n-1}||^2 \Delta t^n.\end{aligned} \tag{3.37}$$

Choosing ϵ sufficiently small and using (2.2), we find

$$\sum_{n=1}^N (\theta_c^{n-1}, \psi_c^n) \leq \frac{1}{4} \sum_{n=1}^N ||\psi_z^n||^2 \Delta t^n + K \sum_{n=1}^N \left(||\theta_z^n||^2 + ||\theta_c^{n-1}||^2\right) \Delta t^n. \tag{3.38}$$

Combining (3.38) with (3.21) and (3.24) we obtain

$$\begin{aligned}&||\psi_c^N||^2 + \sum_{n=1}^N ||\psi_z^n||^2 \Delta t^n \\ &\leq K \left(\sum_{n=1}^N ||\rho^n|| \Delta t^n\right)^2 + K \sum_{n=1}^N \left(||\theta_z^n||^2 + ||\theta_c^{n-1}||^2\right) \Delta t^n \\ &\leq K \left(\sum_{n=1}^{N^*} ||\rho^n|| \Delta t^n\right)^2 + K \sum_{n=1}^{N^*} \left(||\theta_z^n||^2 + ||\theta_c^{n-1}||^2\right) \Delta t^n.\end{aligned} \tag{3.39}$$

Applying the triangle inequality, we obtain the following theorem.

THEOREM 3.1. *Assume* (A1)–(A3) *hold and* Δt^n *satisfies* (2.2). *Then*

$$\begin{aligned}&\max_{1 \leq n \leq N^*} ||c^n - C^n||^2 \\ &\leq K \left(\sum_{n=1}^{N^*} ||c_t^n - \frac{c^n - c^{n-1}}{\Delta t^n}|| \Delta t^n\right)^2 + K \sum_{n=1}^{N^*} \left(||z^n - \pi z^n||^2 + ||c^n - \pi c^n||^2\right) \Delta t^n.\end{aligned} \tag{3.40}$$

Thus, if c is sufficiently smooth,

$$\max_{1\le n\le N^*} ||c^n - C^n|| \le Kh. \tag{3.41}$$

We note that this estimate has optimal rate for the lowest-order mixed method.

4. A more general approach. In this section, we outline a modification of the scheme (3.7) that preserves the estimate above, without the assumption (A3) on the mesh. This estimate also holds in $\mathbb{R}^d$, $d = 1, 2, 3$, and for more general elements. We still assume (2.1) and (2.2).

In order to motivate this approach, consider a simple one-dimensional example. Suppose we have three intervals $\Omega_1 = [0, 1]$, $\Omega_2 = [1, 2]$, and $\Omega_3 = [2, 3]$ at time t^{n-1} and the solution $C^{n-1} = 1$ in Ω_1, $1/2$ in Ω_2, and 0 in Ω_3. Based on some error indicator, we may decide to refine Ω_2 into two subintervals [1,1.5] and [1.5,2]. Using the method described above in section 3, the projected solution onto each of the refined subintervals has the value 1/2, which doesn't take into account the solution values in Ω_1 or Ω_3. In order to improve the projected solution, we construct a linear function on Ω_2 with average value 1/2, say,

$$\overline{C^{n-1}}(x) = .5 - (x - 1.5), \quad x \in \Omega_2,$$

which we note also satisfies $\overline{C^{n-1}}(1) = 1$ and $\overline{C^{n-1}}(2) = 0$. When this function is projected into the new mesh using L^2 projection, the new function has the value of 3/4 in the interval [1,1.5] and 1/4 in the interval [1.5,2], which intuitively is a better approximation of the behavior of the solution in this region.

Therefore, given $C^{n-1} \in W_h^{n-1}$, define a linear function $\overline{C^{n-1}}$ on element Ω_e^{n-1} by

$$\overline{C^{n-1}}|_{\Omega_e^{n-1}} = C^{n-1}(x_e^{n-1}) + (x - x_e^{n-1}) \cdot \delta C_e^{n-1}. \tag{4.1}$$

Here x_e^{n-1} is the midpoint (barycenter) of Ω_e^{n-1} and δC_e^{n-1} is a gradient or slope approximation, discussed below.

The modified scheme is as follows:

$$\left(\frac{C^n - \overline{C^{n-1}}}{\Delta t^n}, w^n\right) + (\nabla \cdot Z^n, w^n) = (f^n, w^n), \quad w^n \in W_h^n, \tag{4.2}$$

$$(Z^n, v^n) = (C^n, \nabla \cdot v^n), \quad v^n \in V_h^n. \tag{4.3}$$

We remark that the linear term in $\overline{C^{n-1}}$ only needs to be added when the mesh changes, for otherwise this term integrates to zero, that is,

$$(\overline{C^{n-1}}, w^n) = (C^{n-1}, w^n), \tag{4.4}$$

when $W_h^{n-1} = W_h^n$. Moreover, because this term integrates to zero, the scheme above is mass-preserving, that is,

$$\int_\Omega \overline{C^{n-1}} dx = \int_\Omega C^{n-1} dx. \tag{4.5}$$

The gradient δC_e^n could be calculated in a number of ways. One way is to recognize that for the heat equation above the mixed method gives an approximation to

∇C, namely, $-Z$. This choice also lends itself to our analysis. Therefore, we consider the case where

$$\delta C_e^n = -\frac{1}{m(\Omega_e^n)} \int_{\Omega_e^n} Z^n(x)dx, \tag{4.6}$$

that is, δC_e^n is the mean value of Z^n on Ω_e^n. Other types of approximations based on linear reconstruction [8, 3, 13] are equally inexpensive and lead to similar theoretical results; we further comment on this below.

The error estimate proceeds as follows. Using the same definitions as in the previous section, subtracting (3.13) from (4.2) and (3.14) from (4.3), we find

$$\begin{aligned} &\left(\frac{\psi_c^n - \psi_c^{n-1}}{\Delta t^n}, w^n\right) + (\nabla \cdot \psi_z^n, w^n) \\ &\quad = \left(\frac{\overline{\psi_c^{n-1}} - \psi_c^{n-1}}{\Delta t^n}, w^n\right) - \left(\frac{c^{n-1} - \overline{\Pi c^{n-1}}}{\Delta t^n}, w^n\right) + (\rho^n, w^n), \quad w^n \in W_h^n; \end{aligned} \tag{4.7}$$

$$(\psi_z^n, v^n) = (\psi_c^n, \nabla \cdot v^n) + (\theta_z^n, v^n), \quad v^n \in V_h^n. \tag{4.8}$$

Here $\overline{\psi_c^{n-1}} = \overline{C^{n-1}} - \overline{\Pi c^{n-1}}$, where we define

$$\overline{\Pi c^n}|_{\Omega_e^n} = \Pi c^n(x_e^n) - (x - x_e^n) \cdot \left(\frac{1}{m(\Omega_e^n)} \int_{\Omega_e^n} \pi z^n dy\right), \tag{4.9}$$

$n = 0, 1, \ldots$.

The first term on the right side of (4.7) is bounded as follows. First we note that

$$\begin{aligned} \left(\frac{\overline{\psi_c^{n-1}} - \psi_c^{n-1}}{\Delta t^n}, w^n\right) &\le \frac{1}{\Delta t^n} ||\overline{\psi_c^{n-1}} - \psi_c^{n-1}|| \, ||w^n|| \\ &\le \frac{\epsilon}{(\Delta t^n)^2} ||\overline{\psi_c^{n-1}} - \psi_c^{n-1}||^2 + K||w^n||^2. \end{aligned} \tag{4.10}$$

Consider for an arbitrary time t^n

$$\begin{aligned} ||\overline{\psi_c^n} - \psi_c^n||^2 &= \sum_e \int_{\Omega_e^n} |\overline{\psi_c^n} - \psi_c^n|^2 dx \\ &= \sum_e \int_{\Omega_e^n} \left|(x - x_e^n) \cdot \frac{1}{m(\Omega_e^n)} \int_{\Omega_e^n} \psi_z^n dy\right|^2 dx \\ &\le \sum_e (h_e^n)^2 ||\psi_z^n||^2_{\Omega_e^n} \\ &\le (h^n)^2 ||\psi_z^n||^2. \end{aligned} \tag{4.11}$$

Substituting (4.11) into (4.10) and using (2.1) and (2.2), we find

$$\begin{aligned} \left(\frac{\overline{\psi_c^{n-1}} - \psi_c^{n-1}}{\Delta t^n}, w^n\right) &\le \frac{\epsilon (h^{n-1})^2}{(\Delta t^n)^2} ||\psi_z^{n-1}||^2 + K||w^n||^2 \\ &\le K\epsilon ||\psi_z^{n-1}||^2 + K||w^n||^2. \end{aligned} \tag{4.12}$$

Next, consider the second term on the right side of (4.7),

$$\left(\frac{c^{n-1}-\overline{\Pi c^{n-1}}}{\Delta t^n}, w^n\right) \leq \frac{K}{(\Delta t^n)^2}||c^{n-1}-\overline{\Pi c^{n-1}}||^2 + K||w^n||^2. \tag{4.13}$$

By Taylor's series, for $x \in \Omega_e^n$ and c sufficiently smooth,

$$\begin{aligned} c^n(x) &= c^n(x_e^n) - (x-x_e^n)\cdot z^n(x_e^n) + \mathcal{O}((h_e^n)^2) \\ &= \Pi c^n(x_e^n) - (x-x_e^n)\cdot \frac{1}{m(\Omega_e^n)}\int_{\Omega_e^n} z^n dy + \mathcal{O}((h_e^n)^2). \end{aligned} \tag{4.14}$$

Here we have used

$$c^n(x_e^n) - \Pi c^n(x_e^n) = \mathcal{O}((h_e^n)^2)$$

and

$$z^n(x_e^n) - \frac{1}{m(\Omega_e^n)}\int_{\Omega_e^n} z^n dy = \mathcal{O}((h_e^n)^2)$$

by the midpoint rule of integration. Therefore

$$\begin{aligned} ||c^n-\overline{\Pi c^n}||^2 &= \sum_e \int_{\Omega_e^n} |c^n - \overline{\Pi c^n}|^2 dy \\ &= \sum_e \int_{\Omega_e^n} \left|(x-x_e^n)\cdot\frac{1}{m(\Omega_e^n)}\int_{\Omega_e^n}\theta_z^n dy + \mathcal{O}((h_e^n)^2)\right|^2 dx \\ &\leq K(h^n)^2||\theta_z^n||^2 + K(h^n)^4 \\ &\leq K(h^n)^4. \end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.13) and using (2.1) and (2.2), we find

$$\left(\frac{c^{n-1}-\overline{\Pi c^{n-1}}}{\Delta t^n}, w^n\right) \leq K(h^{n-1})^2 + K||w^n||^2. \tag{4.16}$$

Returning to (4.7) and (4.8), letting $w^n = \psi_c^n$ and $v^n = \psi_z^n$, and substituting (4.12) and (4.16), we find

$$\begin{aligned} &\frac{||\psi_c^n||^2 - ||\psi_c^{n-1}||^2}{2\Delta t^n} + ||\psi_z^n||^2 \\ &\quad \leq K\epsilon||\psi_z^{n-1}||^2 + K||\theta_z^n||^2 + K||\rho^n||^2 + K||\psi_c^n||^2 + K(h^{n-1})^2. \end{aligned} \tag{4.17}$$

Multiplying by Δt^n, summing on $n = 1, \ldots, N$, where here N is arbitrary, choosing ϵ sufficiently small, and hiding the first term on the right side of (4.17), we find

$$||\psi_c^N||^2 + \sum_{n=1}^{N} ||\psi_z^n||^2\Delta t^n \leq K\sum_{n=1}^{N}||\psi_c^n||^2\Delta t^n + K(h^2+\Delta t^2). \tag{4.18}$$

Finally, applying Gronwall's inequality, we obtain the following error estimate.

THEOREM 4.1. *Assume that Δt^n satisfies* (2.1) *and* (2.2). *Then assuming c is sufficiently smooth, there exists a constant K, independent of h and Δt, such that*

$$\max_n ||c^n - C^n|| \leq Kh. \tag{4.19}$$

Remark. For other choices of δC_e^n, in order to obtain the estimate above, it is necessary that an inequality of the form (4.11) can be shown and that the approximation property (4.15) holds. If, for example, one constructs δC_e^n by some type of finite difference approximation based on C^n, it is intuitive that one could demonstrate that these bounds hold, since ψ_z is related to finite differences of ψ_c through (4.8).

5. Extension to advection-diffusion equations.

5.1. A general case. In this section, we extend the method in the section above to include an advection term and allow for variable coefficients. In particular, we consider the equations

$$\left.\begin{array}{l} c_t + \nabla \cdot (g + z) = f \\ \tilde{z} = -\nabla c \\ z = D\tilde{z} \\ g = uc \end{array}\right\} \text{ on } \Omega \times (0, T], \tag{5.1}$$

$$c(x, 0) = c^0(x), \quad x \in \Omega, \tag{5.2}$$

$$c(x, t) = 0, \quad x \in \partial\Omega, t > 0. \tag{5.3}$$

Here $D = D(x, t)$ is assumed to be a symmetric, positive definite tensor, bounded below by a positive constant D_*, and $u = u(x, t)$ is a given velocity field. This model is typical of equations arising in transport problems. In previous papers [4, 5], we have proposed and analyzed so-called Godunov-mixed methods for approximating solutions to these equations. Here we will extend these methods to the case of dynamically changing meshes.

The weak form of (5.1) we consider is

$$\begin{aligned} (c_t + \nabla \cdot (g + z), w) &= (f, w), \quad w \in L^2(\Omega), \\ (\tilde{z}, v) &= (c, \nabla \cdot v), \quad v \in H(\Omega; \mathrm{div}), \\ (z, \chi) &= (D\tilde{z}, \chi), \quad \chi \in (L^2(\Omega))^d. \end{aligned}$$

The method can then be outlined as follows. At time level t^n, we approximate c^n by $C^n \in W_h^n$ and z, $\tilde{z}$, and the advective flux g by Z^n, $\tilde{Z}^n$, and G^n, all in V_h^n. Defining $\overline{C^{n-1}}$ as before (see (4.1) and (4.6)) using $\tilde{Z}$ instead of Z, these approximations are determined by the following system of equations:

$$\left(\frac{C^n - \overline{C^{n-1}}}{\Delta t^n}, w^n\right) + (\nabla \cdot (G^n + Z^n), w^n) = (f^n, w^n), \quad w^n \in W_h^n, \tag{5.4}$$

$$(\tilde{Z}^n, v^n) = (C^n, \nabla \cdot v^n), \quad v^n \in V_h^n, \tag{5.5}$$

$$(Z^n, v^n) = (D^n \tilde{Z}^n, v^n), \quad v^n \in V_h^n. \tag{5.6}$$

Here we are using the so-called "expanded" mixed finite element method, proposed by Arbogast, Wheeler, and Yotov, for elliptic equations [1], which gives us a gradient approximation $\tilde{Z}$ as well as an approximation to the diffusive flux z.

The advective flux approximation G^n is constructed from the solution C. There is a number of ways in which this can be determined, but we shall concentrate only on simple upwind methods. Since $g^n = 0$ on $\partial\Omega$ by (5.3), we set the integral average of $G^n \cdot \nu = 0$ on boundary edges, where ν is the unit outward normal to Ω. Suppose elements Ω_e^n and $\Omega_{e'}^n$ share an interior edge l, x_l is the midpoint of the edge, and ν_l points from Ω_e^n to $\Omega_{e'}^n$. Then one can define, for example,

$$G^n \cdot \nu_l = \begin{cases} C_e^n(u^n \cdot \nu_l)(x_l), & (u^n \cdot \nu_l)(x_l) \geq 0, \\ C_{e'}^n(u^n \cdot \nu_l)(x_l), & (u^n \cdot \nu_l)(x_l) < 0, \end{cases} \tag{5.7}$$

where C_e^n and $C_{e'}^n$ are the constant values of C^n on the elements. Here, we have defined G^n implicitly in terms of C^n. This complicates the solution of (5.4)–(5.5) by making the system of linear equations which arises nonsymmetric. Another approach would be to calculate G^n explicitly in the following way. Given $\overline{C^{n-1}}$, let $\tilde{C}^{n-1} \in W_h^n$ denote the L^2 projection of $\overline{C^{n-1}}$, that is,

$$(\overline{C^{n-1}} - \tilde{C}^{n-1}, w^n) = 0, \quad w^n \in W_h^n. \tag{5.8}$$

We can then modify the definition of G^n by

$$G^n \cdot \nu_l = \begin{cases} \tilde{C}_e^{n-1}(u^n \cdot \nu_l)(x_l), & (u^n \cdot \nu_l)(x_l) \geq 0, \\ \tilde{C}_{e'}^{n-1}(u^n \cdot \nu_l)(x_l), & (u^n \cdot \nu_l)(x_l) < 0. \end{cases} \tag{5.9}$$

Higher order approximations to g can be constructed by postprocessing $\tilde{C}^{n-1}$ to obtain a piecewise linear function on each element Ω_e^n, much in the same way that $\overline{C^{n-1}}$ is constructed. Since our overall method is at best first order, we will not pursue including these higher order approximations in our analysis, however, they can result in superior solutions, especially for advection-dominated problems, and we often include them when doing simulations.

The error estimate proceeds as in section 4. Define $\Pi\tilde{z} \in V_h^n$ to be the L^2 projection of $\tilde{z}$, that is,

$$(\Pi\tilde{z}^n - \tilde{z}^n, v^n) = 0, \quad v^n \in V_h^n, \tag{5.10}$$

and let $\tilde{\psi}_z = \tilde{Z} - \Pi\tilde{z}$ and $\tilde{\theta}_z = \tilde{z} - \Pi\tilde{z}$. Using the same definitions as in section 3 for the other terms, with the modification that

$$\overline{\Pi c^n}|_{\Omega_e^n} = \Pi c^n(x_e^n) - (x - x_e^n) \cdot \left(\frac{1}{m(\Omega_e^n)} \int_{\Omega_e^n} \Pi\tilde{z}^n dx \right), \tag{5.11}$$

we find

$$\begin{aligned} &\left(\frac{\psi_c^n - \psi_c^{n-1}}{\Delta t^n}, w^n \right) + (\nabla \cdot \psi_z^n, w^n) \\ &= \left(\frac{\overline{\psi_c^{n-1}} - \psi_c^{n-1}}{\Delta t^n}, w^n \right) - \left(\frac{c^{n-1} - \overline{\Pi c^{n-1}}}{\Delta t^n}, w^n \right) + (\rho^n, w^n) \\ &\quad + (\nabla \cdot (g^n - G^n), w^n), \quad w^n \in W_h^n, \end{aligned} \tag{5.12}$$

$$(\tilde{\psi}_z^n, v^n) = (\psi_c^n, \nabla \cdot v^n), \quad v^n \in V_h^n, \tag{5.13}$$

and

$$(\psi_z^n, v^n) = (D\tilde{\psi}_z^n, v^n) - (D\tilde{\theta}_z^n, v^n) + (\theta_z^n, v^n), \quad v^n \in V_h^n. \tag{5.14}$$

Setting $w^n = \psi_c^n$ in (5.12), $v^n = \psi_z^n$ in (5.13), and $v^n = \tilde{\psi}_z^n$ in (5.14), we find

$$\begin{aligned}\left(\frac{\psi_c^n - \psi_c^{n-1}}{\Delta t^n}, \psi_c^n\right) &+ (D\tilde{\psi}_z^n, \tilde{\psi}_z^n) \\ = \left(\frac{\overline{\psi_c^{n-1}} - \psi_c^{n-1}}{\Delta t^n}, \psi_c^n\right) &+ \left(\frac{c^{n-1} - \overline{\Pi c^{n-1}}}{\Delta t^n}, \psi_c^n\right) + (\rho^n, \psi_c^n) \\ &+ (\nabla \cdot (g^n - G^n), \psi_c^n) + (D\tilde{\theta}_z^n, \tilde{\psi}_z^n) - (\theta_z^n, \tilde{\psi}_z^n).\end{aligned} \tag{5.15}$$

The first and second terms on the right side of (5.15) are bounded as before; see (4.12) and (4.16), where now $\tilde{\psi}_z$ is playing the role of ψ_z. In particular,

$$\left(\frac{\overline{\psi_c^{n-1}} - \psi_c^{n-1}}{\Delta t^n}, \psi_c^n\right) \leq \frac{1}{8}(D\tilde{\psi}_z^{n-1}, \tilde{\psi}_z^{n-1}) + K(D_*^{-1})||\psi_c^n||^2 \tag{5.16}$$

and

$$\left(\frac{c^{n-1} - \overline{\Pi c^{n-1}}}{\Delta t^n}, \psi_c^n\right) \leq Kh^2 + K||\psi_c^n||^2. \tag{5.17}$$

Consider the fourth term on the right side of (5.15). Let $\pi g^n \in V_h^n$ denote the π projection of g^n, thus

$$(\nabla \cdot (\pi g^n - g^n), w^n) = 0, \quad w^n \in W_h^n, \tag{5.18}$$

and set $v^n = \pi g^n - G^n$ in (5.13); then

$$\begin{aligned}(\nabla \cdot (g^n - G^n), \psi_c^n) &= (\tilde{\psi}_z^n, \pi g^n - G^n) \\ &\leq \frac{1}{8}(D\tilde{\psi}_z^n, \tilde{\psi}_z^n) + K(D_*^{-1})||\pi g^n - G^n||^2.\end{aligned} \tag{5.19}$$

Let h_l denote the length of an interior edge Γ_l. Let ν_l denote a unit vector normal to edge l and x_l denote the midpoint of the edge, as before, and assume Ω_e^n and $\Omega_{e'}^n$ share edge l. By the properties of the π-projection [17, 14], we have

$$\int_{\Gamma_l} \pi g^n \cdot \nu_l = \int_{\Gamma_l} c^n (u^n \cdot \nu_l) ds, \tag{5.20}$$

and it is easily seen that for g^n sufficiently smooth,

$$\frac{1}{h_l}\int_{\Gamma_l} \pi g^n \cdot \nu_l - ((u^n \cdot \nu_l)c^n)(x_l) = \mathcal{O}(h_l^2) \tag{5.21}$$

by the midpoint rule of integration. Suppose G^n is defined by (5.7) and assume without loss of generality that $(u^n \cdot \nu_l)(x_l) \geq 0$; then

$$\frac{1}{h_l}\int_{\Gamma_l} (\pi g^n - G^n) \cdot \nu_l = (u^n \cdot \nu_l)(x_l)(c^n(x_l) - C_e^n) + \mathcal{O}(h_l^2). \tag{5.22}$$

Moreover,

$$c^n(x_l) - C_e^n = c^n(x_l) - c^n(x_e^n) + c^n(x_e^n) - \Pi c^n(x_e^n) + \Pi c^n(x_e^n) - C_e^n.$$

Therefore, for c^n sufficiently smooth,

$$|c^n(x_l) - C_e^n| \leq |\psi_c(x_e^n)| + \mathcal{O}(h^n). \tag{5.23}$$

Thus, by (5.21)–(5.23),

$$||\pi g^n - G^n||^2 \leq K||\psi_c^n||^2 + K(h^n)^2. \tag{5.24}$$

Next, consider the case where G^n is defined by (5.9). In this case, following the steps that led to (5.24), we find

$$||\pi g^n - G^n||^2 \leq K||\tilde{C}^{n-1} - \Pi c^n||^2 + K(h^n)^2. \tag{5.25}$$

Consider the first term on the right side of (5.25). By (5.8) and the definition of Πc we have

$$\begin{aligned}(\tilde{C}^{n-1} - \Pi c^n, \tilde{C}^{n-1} - \Pi c^n) &= (\overline{C^{n-1}} - c^n, \tilde{C}^{n-1} - \Pi c^n)\\ &\leq ||\overline{C^{n-1}} - c^n||\, ||\tilde{C}^{n-1} - \Pi c^n||,\end{aligned}$$

thus

$$||\tilde{C}^{n-1} - \Pi c^n|| \leq ||\overline{C^{n-1}} - c^n||. \tag{5.26}$$

Moreover

$$\begin{aligned}||\overline{C^{n-1}} - c^n|| &\leq ||\overline{C^{n-1}} - \overline{\Pi c^{n-1}}|| + ||\overline{\Pi c^{n-1}} - c^{n-1}|| + ||c^{n-1} - c^n||\\ &\leq ||\psi_c^{n-1}|| + Kh^{n-1}||\tilde{\psi}_z^{n-1}|| + K(h^{n-1} + \Delta t^n).\end{aligned} \tag{5.27}$$

Combining (5.25)–(5.27), we find

$$||\pi g^n - G^n||^2 \leq K||\psi_c^{n-1}||^2 + K(h^{n-1})^2||\tilde{\psi}_z^{n-1}||^2 + K(h^2 + \Delta t^2). \tag{5.28}$$

Combining (5.24) or (5.28) with (5.15), bounding the other terms on the right side of (5.15), and using estimates for θ_z and $\tilde{\theta}_z$, we obtain

$$\begin{aligned}&\left(\frac{\psi_c^n - \psi_c^{n-1}}{\Delta t^n}, \psi_c^n\right) + (D\tilde{\psi}_z^n, \tilde{\psi}_z^n)\\ &\leq \frac{1}{4}(D\tilde{\psi}_z^{n-1}, \tilde{\psi}_z^{n-1}) + \frac{1}{4}(D\tilde{\psi}_z^n, \tilde{\psi}_z^n) + K||\psi_c^n||^2 + K(h^{n-1})^2||\tilde{\psi}_z^{n-1}||^2\\ &\quad + K||\psi_c^{n-1}||^2 + K(h^2 + \Delta t^2).\end{aligned} \tag{5.29}$$

If we multiply (5.29) by Δt^n, sum on n, hide $\tilde{\psi}_z$ terms assuming h is sufficiently small, note that $\tilde{\psi}_z^0 = 0$ by (5.13) since $\psi_c^0 = 0$, and apply Gronwall's inequality, we have the following result.

THEOREM 5.1. *Assume that* Δt^n *satisfies* (2.1) *and* (2.2). *Then assuming* c *and* u *are sufficiently smooth and* h *is sufficiently small, there exists a constant* K, *independent of* h *and* Δt, *such that*

$$\max_n ||c^n - C^n|| \leq Kh. \tag{5.30}$$

5.2. A special case. We conclude this section by considering a special case which allows us to prove an error estimate for the method above, assuming that the diffusion coefficient D is only nonnegative. This is the first convergence proof for an upwind-mixed method with possibly zero diffusion.

Assume the following:

(A4) u and D are constant, and D is a nonnegative scalar.

(A5) The mesh is modified at most M times, and $M \leq M^*$, where M^* is independent of h and Δt.

(A6) We set $\overline{C^{n-1}} = C^{n-1}$.

(A7) G^n is defined by (5.7).

(A8) The domain Ω is polygonal, so that all elements have straight edges. Thus, the normal vector to an edge is constant over that edge.

Under these assumptions, the first term on the right side of (5.15) is zero. We can rearrange terms to obtain

$$
\begin{aligned}
&\left(\frac{\psi_c^n - \psi_c^{n-1}}{\Delta t^n}, \psi_c^n\right) + D||\tilde{\psi}_z^n||^2 + (\nabla \cdot (G^n - \pi g^n), \psi_c^n) \\
&= \left(\frac{c^{n-1} - \Pi c^{n-1}}{\Delta t^n}, \psi_c^n\right) + (\rho^n, \psi_c^n) + (\nabla \cdot (g^n - \pi g^n), \psi_c^n) \\
&\quad + D(\tilde{\theta}_z^n, \tilde{\psi}_z^n) - (\theta_z^n, \tilde{\psi}_z^n).
\end{aligned} \tag{5.31}
$$

Here we are defining πg^n differently than above. We define πg^n analogously to the definition of G^n in (5.7), with Πc^n playing the role of C^n.

In order to analyze this equation, we define some additional terms. On an edge Γ_l in the mesh at time t^n, let ν_l denote a unit normal vector as before. For $x \in \Gamma_l$ let

$$\psi_c^-(x) = \lim_{s \to 0-} \psi_c(x + s\nu_l)$$

and

$$\psi_c^+(x) = \lim_{s \to 0+} \psi_c(x + s\nu_l).$$

As in section 3, we define

$$[\psi_c] = \psi_c^+ - \psi_c^-$$

with the understanding that for an edge on the boundary of the domain, $\psi_c^+ = 0$. Let ψ_c^u denote the upwind value of ψ_c, as determined by the sign of $u \cdot \nu_l$, and ψ_c^d denote the downwind value. Let

$$\psi_c^a = (\psi_c^- + \psi_c^+)/2 = (\psi_c^u + \psi_c^d)/2.$$

Consider the third term on the left side of (5.31). We will drop the superscript n momentarily for convenience.

$$
\begin{aligned}
(\nabla \cdot (G - \pi g), \psi_c) &= \sum_e \int_{\Omega_e} \nabla \cdot (G - \pi g) \psi_c dx \\
&= -\sum_l \int_{\Gamma_l} (G - \pi g) \cdot \nu_l [\psi_c] ds \\
&= -\sum_l \int_{\Gamma_l} u \cdot \nu_l \psi_c^u [\psi_c] ds
\end{aligned} \tag{5.32}
$$

$$= -\frac{1}{2}\sum_l \int_{\Gamma_l} u\cdot\nu_l(\psi_c^u - \psi_c^d)[\psi_c]ds$$
$$-\frac{1}{2}\sum_l \int_{\Gamma_l} (u\cdot\nu_l)\psi_c^a[\psi_c]ds.$$

If $u\cdot\nu_l > 0$, then $(\psi_c^u - \psi_c^d)[\psi_c] = -[\psi_c]^2$. If $u\cdot\nu_l \le 0$, then this term has the opposite sign. Thus, the first term on the right side of (5.32) satisfies

$$-\frac{1}{2}\sum_l \int_{\Gamma_l} u\cdot\nu_l(\psi_c^u - \psi_c^d)[\psi_c]ds = \frac{1}{2}\sum_l \int_{\Gamma_l} |u\cdot\nu_l|[\psi_c]^2 ds. \tag{5.33}$$

The second term on the right side of (5.32) satisfies

$$\begin{aligned}-\frac{1}{2}\sum_l \int_{\Gamma_l} u\cdot\nu_l\psi_c^a[\psi_c]ds &= -\frac{1}{2}\sum_l \int_{\Gamma_l} u\cdot\nu_l\left\{(\psi_c^+)^2 - (\psi_c^-)^2\right\}ds\\ &= \frac{1}{2}\sum_e \int_{\Omega_e} (\psi_c)^2\nabla\cdot u\,dx\\ &= 0 \qquad (5.34)\end{aligned}$$

since u is constant, and hence divergence free, and ψ_c is constant on each element. Therefore, by (5.31)–(5.33), we find that

$$(\nabla\cdot(G^n - \pi g^n), \psi_c^n) = \frac{1}{2}\sum_l \int_{\Gamma_l} |u\cdot\nu_l|[\psi_c^n]^2 ds \ge 0. \tag{5.35}$$

As we did earlier in the paper, let

$$||\psi_c^N|| = \max_{1\le n\le N^*} ||\psi_c^n||.$$

By assumption (A5) and (3.11),

$$\begin{aligned}\sum_{n=1}^{N}\left(c^{n-1} - \Pi c^{n-1}, \psi_c^n\right) &\le KM^*h||\psi_c^N||\\ &\le K(M^*h)^2 + \frac{1}{4}||\psi_c^N||^2.\end{aligned} \tag{5.36}$$

Furthermore

$$\begin{aligned}\sum_{n=1}^{N}(\rho^n, \psi_c^n)\Delta t^n &\le \frac{1}{4}||\psi_c^N||^2 + K\left(\sum_{n=1}^{N}||\rho^n||\Delta t^n\right)^2\\ &\le \frac{1}{4}||\psi_c^N||^2 + K\Delta t^2.\end{aligned} \tag{5.37}$$

Next, consider

$$\begin{aligned}(\nabla\cdot(g - \pi g), \psi_c) &= \sum_l \int_{\Gamma_l} u\cdot\nu_l(c - \Pi c^u)[\psi_c]ds\\ &\le K\sum_l \int_{\Gamma_l} |u\cdot\nu_l||c - \Pi c^u|^2 ds + \frac{1}{4}\sum_l \int_{\Gamma_l} |u\cdot\nu_l|[\psi_c]^2 ds\\ &\le Kh + \frac{1}{4}\sum_l \int_{\Gamma_l} |u\cdot\nu_l|[\psi_c]^2 ds\end{aligned} \tag{5.38}$$

since $|c - \Pi c^u| = \mathcal{O}(h)$ and $\sum_l \int_{\Gamma_l} ds = \mathcal{O}(h^{-1})$.

Moreover, since D is assumed to be constant,

$$(D\tilde{\theta}_z^n, \tilde{\psi}_z^n) = D(\tilde{\theta}_z^n, \tilde{\psi}_z^n) = 0 \tag{5.39}$$

by the definition of $\Pi\tilde{z}$. Moreover,

$$\begin{aligned}(\theta_z^n, \tilde{\psi}_z^n) &= (z^n - \pi z^n, \tilde{\psi}_z^n)\\ &= D(\tilde{z}^n - \pi\tilde{z}^n, \tilde{\psi}_z^n),\end{aligned}$$

where $\pi\tilde{z}^n \in V_h^n$ is the π-projection of $\tilde{z}^n$. Thus

$$\begin{aligned}(\theta_z^n, \tilde{\psi}_z^n) &\leq KD||\tilde{z}^n - \pi\tilde{z}^n||^2 + \frac{D}{2}||\tilde{\psi}_z^n||^2\\ &\leq KDh^2 + \frac{D}{2}||\tilde{\psi}_z^n||^2.\end{aligned} \tag{5.40}$$

Multiplying (5.31) by Δt^n and summing on n, and substituting (5.32)–(5.40), hiding terms as we go, we find

$$||\psi_c^N||^2 + D\sum_{n=1}^{N}||\tilde{\psi}_z^n||^2\Delta t^n \leq K\left(1 + \left(D + (M^*)^2\right)h\right)h + K\Delta t^2. \tag{5.41}$$

Finally, we obtain the following theorem.

THEOREM 5.2. *Assume that* (A4)–(A8) *hold. Then assuming c is sufficiently smooth, there exists a constant K, independent of h, Δt such that*

$$\max_n ||c^n - C^n|| \leq K(h^{1/2} + \Delta t) + K(D, M^*)h. \tag{5.42}$$

Moreover, this estimate holds when $D = 0$, as long as the solution remains smooth.

Remark. It is possible to extend this result to the case of u smooth but nonconstant, as long as $\nabla \cdot u = 0$. The assumption that D is constant allows us to handle difficulties with the convergence of the expanded mixed finite element method when the diffusion coefficient goes to zero.

6. Experimental results. In this section, we present numerical results for the methods analyzed above. We consider a one-dimensional problem

$$c_t + c_x - c_{xx} = f, \quad 0 < x < 1, \tag{6.1}$$

with f and initial and boundary conditions chosen so that $c(x,t) = xe^{-xt}$. In the tables below, we present results for both methods with and without the addition of the gradient term δC_e in computing $\overline{C^{n-1}}$. Setting this term to zero coincides with the method discussed in section 3. In the tables, the L^2 error at $t = 1$ is given. Comparisons are given for changing the mesh every p steps. We also give errors for static meshes. The mesh modifications are obtained by refining and coarsening alternate elements in the mesh, by at most one level of refinement or coarsening. This is the one-dimensional analogue of the case considered in section 3. In all runs $\Delta t^n = h^n$.

In Table 6.1 we present results for the method presented in section 5, with the gradient term included in $\overline{C^{n-1}}$.

Note that asymptotically, in all cases the error decreases by a factor of two as h decreases by the same factor, indicative of first order convergence.

TABLE 6.1

h	Static mesh	$p = 5$	$p = 10$
.0625	3.8347e-03	2.1819e-03	2.1733e-03
.03125	2.1563e-03	1.3739e-03	1.3181e-03
.015625	1.1431e-03	7.3857e-04	7.3694e-04
.007812	5.8843e-04	3.8626e-04	3.8510e-04
.003906	2.9851e-04	1.9745e-04	1.9717e-04

TABLE 6.2

h	$p = 5$	$p = 10$
.0625	2.4330e-03	2.1818e-03
.03125	3.5402e-03	1.3862e-03
.015625	7.6188e-04	7.4478e-04
.007812	3.9698e-04	3.9085e-04
.003906	2.0365e-04	1.9984e-04

In Table 6.2 we present results for the same problem without including the gradient term in $\overline{C^{n-1}}$, that is, the generalization of the method in section 3 to include an advection term.

Asymptotically the convergence is again first order, as predicted by the theory. However, we also notice that in the $p = 5$ case, the error oddly increases as the size of the mesh decreases from $h = .0625$ to $h = .03125$, before settling down to the expected convergence rate. We expect this is an anomaly and would not occur if we were refining the mesh based on a reasonable error indicator.

7. Conclusions. In this paper, we have taken a first step at developing and analyzing mixed and upwind-mixed methods for diffusion equations, when the mesh changes dynamically. The next step is to develop a posteriori error estimates which can be used to indicate where and when mesh modification and time-step control are needed. This will be the topic of subsequent work.

REFERENCES

[1] T. ARBOGAST, M. WHEELER, AND I. YOTOV, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal., 34 (1997), pp. 828–852.

[2] R. BANK AND R. SANTOS, *Analysis of some moving space-time finite element methods*, SIAM J. Numer. Anal., 30 (1993), pp. 1–18.

[3] B. COCKBURN AND C. W. SHU, *TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservations laws* IV*: The multidimensional case*, Math. Comp., 54 (1990), pp. 545–581.

[4] C. N. DAWSON, *Godunov-mixed methods for advective flow problems in one space dimension*, SIAM J. Numer. Anal., 28 (1991), pp. 1282–1309.

[5] C. N. DAWSON, *Godunov-mixed methods for advection-diffusion equations in multidimensions*, SIAM J. Numer. Anal., 30 (1993), pp. 1315–1332.

[6] C. N. DAWSON, C. J. VAN DUIJN, AND R. E. GRUNDY, *Large-time asymptotics in contaminant transport in porous media*, SIAM J. Appl. Math., 56 (1996), pp. 965–993.

[7] T. F. DUPONT, *Mesh modification for evolution equations*, Math. Comp., 39 (1982), pp. 85–108.

[8] L. DURLOVSKY, B. ENGQUIST, AND S. OSHER, *Triangle based adaptive stencils for the solution of hyperbolic conservation laws*, J. Comput. Phys., 98 (1992), pp. 64–73.

[9] K. ERIKSSON AND C. JOHNSON, *Adaptive finite element methods for parabolic problems* I*: A linear model problem*, SIAM J. Numer. Anal., 28 (1991), pp. 43–77.

[10] K. ERIKSSON AND C. JOHNSON, *Adaptive finite element methods for parabolic problems* II*: Optimal error estimates in* $L^\infty(L^2)$ *and* $L^\infty(L^\infty)$, SIAM J. Numer. Anal., 32 (1995), pp. 706–740.

[11] K. ERIKSSON AND C. JOHNSON, *Adaptive finite element methods for parabolic problems* IV*:*

Nonlinear problems, SIAM J. Numer. Anal., 32 (1995), pp. 1729–1749.
[12] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems* V*: Long-time integration*, SIAM J. Numer. Anal., 32 (1995), pp. 1750–1763.
[13] A. Harten and S. R. Chakravarthy, *Multi-dimensional ENO Schemes for General Geometries*, Tech. report 91-76, ICASE, Hampton, VA, 1991.
[14] J. Jim Douglas and J. E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp., 44 (1985), pp. 39–52.
[15] K. Miller, *Moving finite elements* II, SIAM J. Numer. Anal., 18 (1981), pp. 1033–1057.
[16] K. Miller and R. N. Miller, *Moving finite elements* I, SIAM J. Numer. Anal., 18 (1981), pp. 1019–1032.
[17] P. A. Raviart and J. M. Thomas, *A mixed finite element method for second order elliptic problems*, in Mathematical Aspects of Finite Element Methods, I. Galligani and E. Magenes, eds., Lecture Notes in Math. 606, Springer-Verlag, Berlin, 1977, pp. 292–315.
[18] C. J. van Duijn, R. E. Grundy, and C. N. Dawson, *Limiting profiles in reactive solute transport*, Transport in Porous Media, 27 (1997), pp. 57–84.
[19] M. F. Wheeler, T. Arbogast, S. Bryant, C. Dawson, F. Saaf, and C. Wang, *Computational methods for multiphase flow and reactive transport problems arising in subsurface contaminant remediation*, J. Comput. Appl. Math., 74 (1996), pp. 19–32.
[20] D. Q. Yang, *Mixed methods with dynamic finite element spaces for miscible displacement in porous media*, J. Comput. Appl. Math., 30 (1990), pp. 313–328.
[21] D. Q. Yang, *A characteristic mixed method with dynamic finite element space for convection-dominated diffusion problems*, J. Comput. Appl. Math., 43 (1992), pp. 343–353.