Nonclassical Jacobi Polynomials and Sobolev Orthogonality

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Abstract. In this paper, we consider the second-order Jacobi differential expression

$$\ell_{\alpha,\beta}[y](x) = \frac{-1}{(1-x)^a(1+x)^{-1}} \left( (1-x)^{\alpha+1} y'(x) \right)' \quad (x \in (-1,1));$$

here, the Jacobi parameters are $\alpha > -1$ and $\beta = -1$. This is a non-classical setting since the classical setting for this expression is generally considered when $\alpha, \beta > -1$. In the classical setting, it is well-known that the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ are (orthogonal) eigenfunctions of a self-adjoint operator $T_{\alpha,\beta}$, generated by the Jacobi differential expression, in the Hilbert space $L^2((-1,1); (1-x)^a (1+x)^{-1})$. When $\alpha > -1$ and $\beta = -1$, the Jacobi polynomial of degree 0 does not belong to the Hilbert space $L^2((-1,1); (1-x)^a (1+x)^{-1})$. However, in this paper, we show that the full sequence of Jacobi polynomials $\{P_n^{(\alpha,-1)}\}_{n=0}^{\infty}$ forms a complete orthogonal set in a Hilbert–Sobolev space $W_\alpha$, generated by the inner product

$$\phi(f,g) := f(-1)\overline{g(-1)} + \int_{-1}^{1} f'(t)\overline{g'(t)}(1-t)^{\alpha+1} dt.$$ 

We also construct a self-adjoint operator $T_\alpha$, generated by $\ell_{\alpha,-1}[\cdot]$ in $W_\alpha$, that has the Jacobi polynomials $\{P_n^{(\alpha,-1)}\}_{n=0}^{\infty}$ as eigenfunctions.

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1. Introduction

For $\alpha, \beta > -1$, the special functions properties as well as the spectral properties of the classical Jacobi differential expression

$$l_{\alpha, \beta}[y](x) := \frac{1}{\omega_{\alpha, \beta}(x)} \left[ \left( - (1-x)^{\alpha+1} (1+x)^{\beta+1} \right) y'(x) \right]' + k (1-x)^{\alpha} (1+x)^{\beta} y(x)$$

$$= - (1-x^2) y''(x) + (\alpha - \beta + (\alpha + \beta + 2)x) y'(x) + ky(x)$$

where $k \geq 0$ is fixed, $x \in (-1, 1)$ and $\omega_{\alpha, \beta}(x) = (1-x)^{\alpha} (1+x)^{\beta}$ are well known. In this case, the $n$th degree Jacobi polynomial $y = \mathcal{P}_{n}^{\alpha, \beta}(x)$ is a solution of the equation

$$l_{\alpha, \beta}[y](x) = (n(n + \alpha + \beta + 1) + k) y(x) \quad (n \in \mathbb{N}_0);$$

details of the properties of these polynomials can be found in [7,16]. The right-definite spectral analysis has been studied in [1,9]. Through the Glazman–Krein–Naimark (GKN) theory it has been known that there exists a self-adjoint operator $A^{(\alpha, \beta)}$ generated from the Jacobi differential expression in the Hilbert space $L^2((-1, 1); (1-x)^{\alpha} (1+x)^{\beta})$ having the Jacobi polynomials as a complete set of eigenfunctions.

For $\alpha, \beta \geq -1$, let

$$L^2_{\alpha, \beta}(-1, 1) := L^2((-1, 1); (1-x)^{\alpha} (1+x)^{\beta})$$

be the weighted Hilbert space with usual inner product

$$(f,g)_{\alpha, \beta} = \int_{-1}^{1} f(x) g(x) (1-x)^{\alpha} (1+x)^{\beta} dx$$

and related norm $\| \cdot \|_{\alpha, \beta}$. We will study the nonclassical Jacobi case when $\alpha > -1$ and $\beta = -1$. One main difference to the classical case is that the constant Jacobi polynomial $P_{0}^{\alpha, -1}(x)$ does not belong to the Hilbert space $L^2_{\alpha, -1}(-1, 1)$. However, it is true that the Jacobi polynomials $\{P_{n}^{\alpha, -1}\}_{n=1}^{\infty}$ still form a complete orthogonal set in $L^2_{\alpha, -1}(-1, 1)$. Moreover, by the GKN theory, there exists a self-adjoint differential operator $A^{(\alpha, -1)}$, generated by $l_{\alpha, -1}[\cdot]$, that is positively bounded below in $L^2_{\alpha, -1}(-1, 1)$, that has the set $\{P_{n}^{(\alpha, -1)}\}_{n=1}^{\infty}$ as eigenfunctions. Furthermore, in [12], Kwon and Littlejohn showed that the full sequence of Jacobi polynomials $\{P_{n}^{(\alpha, -1)}\}_{n=0}^{\infty}$ are orthogonal in a Hilbert–Sobolev function space $W_{\alpha}$ with inner product

$$\phi(f,g) := f(-1)g(-1) + \int_{-1}^{1} f'(x)g'(x)(1-x)^{\alpha+1} dx.$$
In this paper we prove that the entire sequence of Jacobi polynomials \( \{P_n^{(\alpha,-1)}\}_{n=0}^{\infty} \) are, in fact, complete in \( W_\alpha \). More importantly, we construct a self-adjoint, positively bounded below operator \( T_\alpha \), generated from \( l_{\alpha,-1}[\cdot] \), in \( W_\alpha \) having the entire set of Jacobi polynomials \( \{P_n^{(\alpha,-1)}\}_{n=0}^{\infty} \) as a complete set of eigenfunctions. The general left-definite theory, that was recently developed by Littlejohn and Wellman [13] is, surprisingly, of paramount importance in the construction of this self-adjoint operator.

We note that, for \( m \in \mathbb{N} \), the Jacobi polynomials \( \{P_n^{(\alpha,-m)}\}_{n=0}^{\infty} \) are orthogonal with respect to inner products of the form (1.3) but whose integrand involves the \( m \)th derivative of the functions. In this respect, we refer the reader to [2–4,11,12] where general results on the Sobolev orthogonality of the Jacobi or Gegenbauer polynomials when one or both parameters \( \alpha \) and \( \beta \) are negative integers.

2. Preliminaries: Properties of the Jacobi Polynomials

For \( \alpha, \beta > -1 \), the Jacobi polynomials \( \{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty} \) are defined by

\[
P_n^{(\alpha,\beta)}(x) := k_n^{\alpha,\beta} \sum_{j=0}^{n} \frac{(1+x)^{n-j}}{j!(n-j)!} \frac{(1+\alpha+j)(1+\alpha+\beta+j)}{(1-x)^j} \, (2.1)
\]

where

\[
k_n^{\alpha,\beta} := \frac{(n!)^{1/2}(1+\alpha+\beta+2n)^{1/2}}{2^{\alpha+\beta+1/2}(\Gamma(\alpha+n+1))^{1/2}(\Gamma(\beta+n+1))^{1/2}};
\]

see [7,15,16] for detailed discussions of these polynomials. For each \( n \in \mathbb{N}_0 \), \( y = P_n^{(\alpha,\beta)}(x) \) is a solution of the Jacobi differential equation

\[
l_{\alpha,\beta}[y](x) = (n(n+\alpha+\beta+1)+k) \, y(x).
\]

The Jacobi polynomials can always be normalized, in any number of ways, and we will assume that the polynomials are normalized in various spaces throughout this paper. They form a complete orthogonal set in the Hilbert space \( L^2_{\alpha,\beta}(-1,1) \). In fact, they satisfy the orthogonality relation

\[
\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_r^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \, dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!} \delta_{n,r}
\]

for \( \alpha, \beta > -1 \). The derivatives of the Jacobi polynomials satisfy the identity

\[
\frac{d^j}{dx^j} P_n^{(\alpha,\beta)}(x) = a^{(\alpha,\beta)}(n,j) P_{n-j}^{(\alpha+j,\beta+j)}(x) \quad (n,j \in \mathbb{N}_0),
\]
where
\[ a^{(\alpha,\beta)}(n, j) = \frac{(n!)^{1/2} (\Gamma(\alpha + \beta + n + 1 + j))^{1/2}}{((n-j)!)^{1/2} (\Gamma(\alpha + \beta + n + 1))^{1/2}} \quad (j = 0, 1, \ldots, n), \]
and \( a^{(\alpha,\beta)}(n, j) = 0 \) if \( j > n \). Furthermore, for \( n, r, j \in \mathbb{N}_0 \), we have the orthogonality relation
\[ \int_{-1}^{1} \frac{d^j}{dx^j} P^{(\alpha,\beta)}_n(x) \frac{d^j}{dx^j} P^{(\alpha,\beta)}_r(x) w_{\alpha+j,\beta+j}(x) dx = \frac{n! \Gamma(\alpha + \beta + n + 1 + j)}{(n-j)! \Gamma(\alpha + \beta + n + 1)} \delta_{n,r}. \tag{2.3} \]

When \( \alpha > -1 \) and \( \beta = -1 \), the natural setting for the Jacobi polynomials \( \{P^{(\alpha,-1)}_n\} \) is still the Hilbert space \( L^2_{\alpha,-1}(-1,1) \). However, because of the singularity in the weight function, \( P^{(\alpha,-1)}_0 \notin L^2_{\alpha,-1}(-1,1) \). Moreover, with the definition of the Jacobi polynomials as given in, say [7],
\[ P^{(\alpha,-1)}_n(x) := \sum_{j=0}^{n} \binom{n + \alpha}{n+j} \binom{n-1}{n-j} \left( \frac{x - 1}{2} \right)^j \left( \frac{x + 1}{2} \right)^{n-j}, \]
notice that the first Jacobi polynomial \( P^{(\alpha,-1)}_1(x) \) is degenerate. However, any multiple of the first degree polynomial \( y = x+1 \) will be a solution of the Jacobi differential equation
\[ l^{(\alpha,-1)}[y](x) = (n(n + \alpha) + k)y(x). \]
Therefore, by renormalizing the sequence of Jacobi polynomials as follows:
\[ P^{(\alpha,-1)}_1(x) = \sqrt{\frac{(\alpha + 1)(\alpha + 2)}{2^{\alpha+2}}} (x + 1), \]
and, for \( n \geq 2 \),
\[ P^{(\alpha,-1)}_n(x) = \sqrt{\frac{n(2n + \alpha)}{2^{\alpha}(n + \alpha)}} \sum_{j=0}^{n} \binom{n + \alpha}{n+j} \binom{n-1}{n-j} \left( \frac{x - 1}{2} \right)^j \left( \frac{x + 1}{2} \right)^{n-j}, \]
the sequence \( \{P^{(\alpha,-1)}_n\}_{n=1}^{\infty} \) is an orthonormal set in \( L^2_{\alpha,-1}(-1,1) \). We remark that the full set of Jacobi polynomials \( \{P^{(\alpha,-1)}_n\}_{n=0}^{\infty} \) cannot be orthogonal on the real line with respect to any bilinear form of type
\[ \int_{\mathbb{R}} f \overline{g} d\mu, \]
where \( \mu \) is a (possibly signed) measure. This is an application of Favard’s classic theorem in orthogonal polynomials. However,
Lemma 1. The sequence \( \left\{ P_n^{(\alpha,-1)}(x) \right\}_{n=1}^\infty \) of Jacobi polynomials of degree \( n \geq 1 \) forms a complete orthogonal set in the Hilbert space \( L^2_{\alpha,-1}(-1,1) \). Equivalently, the set of all polynomials \( p \in P[-1,1] \) of degree \( \geq 1 \) satisfying \( p(-1) = 0 \) is dense in the space \( L^2_{\alpha,-1}(-1,1) \).

Proof. We have

\[
\int_{-1}^{1} |f(x)|^2 (1-x)^\alpha (1+x)^{-1} \, dx = \int_{-1}^{1} |(1+x)^{-1} f(x)|^2 (1-x)^\alpha (1+x) \, dx,
\]
that is to say,

\[
f \in L^2_{\alpha,-1}(-1,1) \iff (1+x)^{-1} f \in L^2_{\alpha,1}(-1,1),
\]
and in this case,

\[
\|f\|_{L^2_{\alpha,-1}} = \|(1+x)^{-1} f\|_{L^2_{\alpha,1}}.
\]

Let \( f \in L^2_{\alpha,-1}(-1,1) \) and let \( \epsilon > 0 \). Then

\[
(1+x)^{-1} f \in L^2_{\alpha,1}(-1,1);
\]

since the Jacobi polynomials \( \left\{ P_n^{(\alpha,1)}(x) \right\}_{n=0}^\infty \) forms a complete orthogonal set in \( L^2_{\alpha,1}(-1,1) \), there exists \( q \in P[-1,1] \) such that

\[
\|(1+x)^{-1} f - q\|_{L^2_{\alpha,1}} < \epsilon.
\]

Let \( p(x) := (1+x)q(x) \), so \( \deg(p) \geq 1, p(-1) = 0 \), and \( q(x) = (1+x)^{-1} p(x) \). Then

\[
\epsilon > \|(1+x)^{-1} f - (1+x)^{-1} p\|_{L^2_{\alpha,1}} = \|(1+x)^{-1} (f-p)\|_{L^2_{\alpha,1}} = \|f-p\|_{L^2_{\alpha,-1}}.
\]

\[
\square
\]

3. An Operator Inequality

The following result, due to Chisholm, Everitt, and Littlejohn [6], is important in establishing our main results. The authors in [6] proved a more general result for conjugate indices \( p \) and \( q \); for the purposes of this paper, we only state the result when \( p = q = 2 \).

Theorem 1. Suppose \( I = (a,b) \) is an open interval of the real line, where \( -\infty \leq a < b \leq \infty \). Suppose \( w \) is a positive Lebesgue measurable function on \( (a,b) \) and \( \varphi, \psi \) are functions satisfying the three conditions:

(i) \( \varphi \in L^2_{\text{loc}}((a,b);w) \) and \( \psi \in L^2_{\text{loc}}((a,b);w) \);

(ii) for some \( c \in (a,b) \), suppose \( \varphi \in L^2((a,c);w) \) and \( \psi \in L^2((c,b);w) \);
(iii) for all \([\alpha, \beta] \subset (a, b)\),
\[
\int_a^\alpha |\varphi(t)|^2 w(t) dt > 0 \quad \text{and} \quad \int_\beta^b |\psi(t)|^2 w(t) dt > 0.
\]

Define the linear operators \(A\) and \(B\) on \(L^2((a, b); w)\) and \(L^2((a, b); w)\), respectively, by
\[
(Ag)(x) := \varphi(x) \int_x^b \psi(t) g(t) w(t) dt \quad (x \in (a, b) \text{ and } g \in L^2((a, b); w))
\]
\[
(Bg)(x) := \psi(x) \int_a^x \varphi(t) g(t) w(t) dt \quad (x \in (a, b) \text{ and } g \in L^2((a, b); w));
\]
then
\[
A : L^2((a, b); w) \to L^2_{\text{loc}}((a, b); w)
\]
\[
B : L^2((a, b); w) \to L^2_{\text{loc}}((a, b); w).
\]

Define \(K(\cdot) : (a, b) \to (0, \infty)\) by
\[
K(x) := \left\{ \int_a^x |\varphi(t)|^2 w(t) dt \right\}^{1/2} \left\{ \int_x^b |\psi(t)|^2 w(t) dt \right\}^{1/2} \quad (x \in (a, b))
\]
and the number \(K \in (0, \infty]\)
\[
K := \sup \{K(x) \mid x \in (a, b)\}.
\]

Then a necessary and sufficient condition that \(A\) and \(B\) are bounded linear operators on \(L^2((a, b); w)\) is that the number \(K\) is finite, i.e.
\[
K \in (0, \infty).
\]

4. Right-Definite Analysis of the Jacobi Expression When \(\alpha > -1\) and \(\beta = -1\)

Recall that, for \(\alpha > -1\) and \(\beta = -1\), the second-order Jacobi differential expression is defined to be
\[
\ell_{\alpha,-1}[y](x) := (1-x^2)y''(x) - (\alpha + 1)(x+1)y'(x) + ky(x)
\]
\[
= \frac{1}{w_{\alpha,-1}(x)}\left[ (1-x)^{\alpha+1}y'(x) + k(1-x)^\alpha(1+x)^{-1}y \right], (4.1)
\]
where \(w_{\alpha,-1}(x) := (1-x)^\alpha(1+x)^{-1}, x \in (-1, 1)\) and \(k\) is a fixed, non-negative constant.

Both points \(x = \pm 1\) are regular singular endpoints of \(l_{\alpha,-1}[\cdot]\) in the sense of Frobenius. In fact, in the case \(\alpha \geq 1\) and \(\beta = -1\), both endpoints \(x = \pm 1\)
are limit-point in $L^2_{\alpha,-1}(-1,1)$, defined in (1.1), and thus the right-definite GKN self-adjoint operator is unique and there are no boundary conditions prescribed in the domain of $A^{(\alpha,-1)}$. For $-1 < \alpha < 1$, the endpoint $x = -1$ is in the limit-point condition, whereas $x = +1$ is in the limit-circle case. Therefore, when $-1 < \alpha < 1$, one boundary condition is necessary to define the self-adjoint operator $A^{(\alpha,-1)}$ having the Jacobi polynomials $\{P_n^{(\alpha,-1)}\}_{n=1}^{\infty}$ as eigenfunctions.

For $\alpha > -1$, the maximal domain $\Delta_{\alpha,-1}$ of $\ell_{\alpha,-1}[\cdot]$ in the Hilbert space $L^2_{\alpha,-1}(-1,1)$ is defined to be

$$\Delta_{\alpha,-1} := \{ f : (-1,1) \to \mathbb{C} | f, f' \in AC_{\text{loc}}(-1,1); f, \ell_{\alpha,-1}[f] \in L^2_{\alpha,-1}(-1,1) \}$$

$$= \{ f : (-1,1) \to \mathbb{C} | f, f' \in AC_{\text{loc}}(-1,1); \frac{(1-x)^{\alpha/2}}{(1+x)^{1/2}} f \in L^2(-1,1); (1-x)^{-\alpha/2}(1+x)^{1/2} ((1-x)^{\alpha+1} f'(x))' \in L^2(-1,1) \}.$$ 

The maximal operator $T^{(\alpha,-1)}_{\text{max}}$ associated with $\ell_{\alpha,-1}[\cdot]$ is given by

$$T^{(\alpha,-1)}_{\text{max}}(f) := \ell_{\alpha,-1}[f]$$

$$\mathcal{D}(T^{(\alpha,-1)}_{\text{max}}) := \Delta_{\alpha,-1}.$$ 

The minimal operator is defined as $T^{(\alpha,-1)}_{\text{min}} := (T^{(\alpha,-1)}_{\text{max}})^*$, the Hilbert space adjoint of $T^{(\alpha,-1)}_{\text{max}}$. The operator $T^{(\alpha,-1)}_{\text{min}}$ is closed, symmetric and satisfies

$$(T^{(\alpha,-1)}_{\text{min}})^* = T^{(\alpha,-1)}_{\text{max}}.$$ 

The deficiency index $d(T^{(\alpha,-1)}_{\text{min}})$ of $T^{(\alpha,-1)}_{\text{min}}$ is

$$d(T^{(\alpha,-1)}_{\text{min}}) = \begin{cases} (0,0) & \text{if } \beta = -1, \; \alpha \geq 1 \\ (1,1) & \text{if } \beta = -1, \; -1 < \alpha < 1. \end{cases}$$ 

This can be seen from the limit-point/limit-circle classification of the singular endpoints $x = \pm 1$:

(i) $x = \pm 1$ are limit-point if $\beta = -1, \; \alpha \geq 1$ and

(ii) $x = -1$ is limit-point, $x = 1$ is limit-circle if $\beta = -1, \; -1 < \alpha < 1$. 

Consequently, by von Neumann’s theory of self-adjoint extensions of symmetric operators ([8], chapter XII), $T^{(\alpha,-1)}_{\text{min}}$ has self-adjoint extensions in $L^2_{\alpha,-1}(-1,1)$ for $\alpha > -1$ and $\beta = -1$.

We define the operator $A^{(\alpha,-1)} : \mathcal{D}(A^{(\alpha,-1)}) \subset L^2_{\alpha,-1}(-1,1) \to L^2_{\alpha,-1}(-1,1)$ as follows:

$$\begin{cases} (A^{(\alpha,-1)} f)(x) := \ell_{\alpha,-1}[f](x) & (\text{a.e. } x \in (-1,1)) \\ f \in \mathcal{D}(A^{(\alpha,-1)}), \end{cases}$$

(4.2)
where \( \mathcal{D}(A^{(\alpha,-1)}) \) is defined by

\[
\mathcal{D}(A^{(\alpha,-1)}) := \begin{cases} \{ f \in \Delta_{\alpha,-1} \mid \lim_{x \to 1-} (1-x)^{\alpha+1} f'(x) = 0 \} & \text{if } -1 < \alpha < 1 \\
\Delta_{\alpha,-1} & \text{if } \alpha \geq 1. \end{cases}
\]

(4.3)

From the GKN theory, \( A^{(\alpha,-1)} \) is self-adjoint in \( L^2_{\alpha,-1}(-1,1) \). More importantly, from our point of view, the Jacobi polynomials \( \{ P_n^{(\alpha,-1)} \}_{n=1}^{\infty} \) of degree \( \geq 1 \) form a complete set of eigenfunctions of \( A^{(\alpha,-1)} \); see Lemma 1. Furthermore, the spectrum of \( A^{(\alpha,-1)} \) is discrete and given by

\[ \sigma(A^{(\alpha,-1)}) = \{ n(n+\alpha) \mid n \in \mathbb{N} \}. \]

In order to develop left-definite properties of \( A^{(\alpha,-1)} \) which, in turn, are important to construct the self-adjoint operator \( T_\alpha \) in the Sobolev space \( W_\alpha \), we turn our attention to developing properties of functions in the spaces \( \Delta_{\alpha,-1} \) and \( \mathcal{D}(A^{(\alpha,-1)}) \).

Observe that if \( f, g \in \Delta_{\alpha,-1} \), then \( \ell_{\alpha,-1}[f] \bar{g} \in L^2_{\alpha,-1}(-1,1) \). Furthermore, for \( f, g \in \Delta_{\alpha,-1} \), Green’s formula gives us

\[
\int_0^x (1-t)^{\alpha+1} f'(t) \bar{g}(t) dt = (1-x)^{\alpha+1} f'(x) \bar{g}(x) - f'(0) \bar{g}(0)
\]

\[
- \int_0^x \ell_{\alpha,-1}[f](t) \bar{g}(t)(1-t)^{\alpha}(1+t)^{-1} dt
\]

\[
+ k \int_0^x f(t) \bar{g}(t)(1-t)^{\alpha}(1+t)^{-1} dt \quad (-1 < x < 1).
\]

(4.4)

**Lemma 2.** For \( \alpha > -1 \) and \( f \in \Delta_{\alpha,-1} \), we have \( f' \in L^2(-1,0) \). In particular, we can define \( f \) at \( x = -1 \) such that \( f \in AC[-1,0] \). Moreover, it is necessary that \( f(-1) = 0 \).

**Proof.** Let \( f \in \Delta_{\alpha,-1} \). Note that

\[
f'(x) = -(1-x)^{-\alpha-1} \int_0^x \frac{(1-t)^{-\alpha/2}(1+t)^{1/2}}{(1-t)^{-\alpha/2}(1+t)^{1/2}}((1-t)^{\alpha+1} f'(t))' dt
\]

\[
+(1-x)^{-\alpha-1} f'(0) \quad (x \in (-1,0]).
\]

From the definition of \( \Delta_{\alpha,-1} \), we see that \( (1-t)^{-\alpha/2}(1+t)^{1/2}((1-t)^{\alpha+1} f'(t))' \in L^2(-1,1) \); consequently, we apply the inequality in Theorem 1 on the interval \((-1,0]\) with

\[ \varphi(x) = (1-x)^{-\alpha-1}, \quad \psi(x) = \frac{1}{(1-x)^{-\alpha/2}(1+x)^{1/2}} \quad (x \in (-1,0]). \]
Since the function $1 - t$ is bounded on $(-1,0]$, there exists a constant $M$ such that
\[
\int_{-1}^{x} \varphi^2(t)dt \int_{0}^{x} \psi^2(t) dt = \int_{-1}^{x} (1-t)^{-2\alpha-2} dt \int_{0}^{x} (1-t)^{\alpha}(1+t)^{-1} dt \\
\leq M \int_{-1}^{x} dt \int_{0}^{x} (1+t)^{-1} dt = -M(x+1)\ln(1+x);
\]
since $(x+1)\ln(1+x)$ is bounded on $(-1,0]$, the inequality from Theorem 1 implies that $f' \in L^2(-1,0)$ as claimed. From the identity
\[
f(x) = f(0) - \int_{x}^{0} f'(t) dt \quad (-1 < x \leq 0),
\]
define
\[
f(-1) := f(0) - \int_{-1}^{0} f'(t) dt.
\]
With $f$ defined this way, we see that $f \in AC[-1,0]$. We now show that $f(-1) = 0$. We suppose that $f$ is real-valued and, by way of contradiction and without loss of generality, we assume that $f(-1) > 0$. Then there exists $x^* \in (-1,0]$ such that $f(t) \geq c$ for all $t \in (-1,x^*]$, where $c$ is some positive number. Since $(1-x)^{\alpha}$ is bounded below on $(-1,0]$ by a constant $K > 0$, we see that
\[
-\infty > \int_{-1}^{1} |f(t)|^2 (1-t)^{\alpha}(1+t)^{-1} dt \\
\geq \int_{-1}^{x^*} |f(t)|^2 (1-t)^{\alpha}(1+t)^{-1} dt \geq Kc^2 \int_{-1}^{x^*} (1+t)^{-1} dt = \infty,
\]
a contradiction. Hence $f(-1) = 0$. \hfill \Box

Lemma 3. For $\alpha > -1$ and $f \in \Delta_{\alpha,-1}$, we have $(1-x)^{(\alpha+1)/2} f' \in L^2(-1,0)$.

Proof. Suppose, to the contrary, that
\[
\int_{-1}^{0} |f'(t)|^2 (1-t)^{\alpha+1} dt = \infty.
\]
We assume, without loss of generality, that $f$ is real-valued. From (4.4), we see that
\[
\int_0^x (1-t)^{\alpha+1}(f'(t))^2\,dt = f'(0)f(0) - (1-x)^{\alpha+1}f'(x)f(x)
\]
\[
- \int_0^x \ell_{\alpha,-1}(t)f(t)(1-t)^\alpha(1+t)^{-1}\,dt
\]
\[
+ k \int_0^{x^*} (f(t))^2(1-t)^\alpha(1+t)^{-1}\,dt \quad (-1 < x \leq 0);
\]
(4.5)

consequently, it must be the case that
\[
\lim_{x \to -1^+} (1-x)^{\alpha+1}f'(x)f(x) = -\infty,
\]
and hence
\[
\lim_{x \to -1^+} f'(x)f(x) = -\infty.
\]
It follows that there exists $x^* \in (-1,0], f'(x)f(x) \leq -1$ for $x \in (-1,x^*].$

Integrating, we see that
\[
\frac{(f(x^*))^2}{2} - \frac{(f(x))^2}{2} = \int_x^{x^*} f'(t)f(t)\,dt \leq -1 \int_x^{x^*} dt = x - x^*
\]
so that
\[
(f(x))^2 \geq -2x + (f(x^*))^2 + 2x^* \geq -2(x-x^*) \quad (x \in (-1,x^*)].
\]
(4.6)

With
\[
M_{\alpha} = \min_{t \in (-1,x^*]} (1-t)^\alpha,
\]

after integrating the inequality in (4.6), we see that
\[
\int_x^{x^*} (f(t))^2(1-t)^\alpha(1+t)^{-1}\,dt \geq -2 \int_x^{x^*} (t-x^*)(1-t)^\alpha(1+t)^{-1}\,dt
\]
\[
\geq -2M_{\alpha} \int_x^{x^*} (t-x^*)(1+t)^{-1}\,dt
\]
\[
\to \infty \quad \text{as } x \to -1^+.
\]
However, this contradicts the fact that \( f \in \Delta_{\alpha,-1} \), and in particular, that
\[
\int_{-1}^{1} |f(t)|^2 (1-t)^\alpha (1+t)^{-1} dt < \infty.
\]
\[
\square
\]

**Lemma 4.** Let \( f, g \in \Delta_{\alpha,-1} \). Then
\[
\lim_{x \to -1^+} (1-x)^{\alpha+1} f'(x) g(x) = 0.
\]

**Proof.** Let \( f, g \in \Delta_{\alpha,-1} \); assume that both \( f \) and \( g \) are real-valued. We note that, from Lemma 2, that \( f(-1) = g(-1) = 0 \). From (4.5) and Lemma 3, we see that
\[
\lim_{x \to -1^+} (1-x)^{\alpha+1} f'(x) g(x)
\]
exists and is finite. Suppose, then, that
\[
\lim_{x \to -1^+} (1-x)^{\alpha+1} f'(x) g(x) = c > 0.
\]
Then we can assume there exists \( x^* \in (-1,0) \) such that
\[
f'(x) > 0, \quad g(x) > 0, \quad \text{and} \quad (1-x)^{\alpha+1} f'(x) \geq \frac{c}{2g(x)} \quad (x \in (-1,x^*)).
\]
Multiply this inequality by \( |g'(x)| \) to get
\[
(1-x)^{\alpha+1} f'(x) |g'(x)| \geq \frac{c |g'(x)|}{2g(x)} \quad (x \in (-1,x^*)).
\]
Now integrate to obtain
\[
\infty > \int_{-1}^{1} (1-t)^{\alpha+1} f'(t) |g'(t)| dt \quad \text{(using Lemma 3)}
\]
\[
\geq \int_{-x}^{-1} (1-t)^{\alpha+1} f'(t) |g'(t)| dt \geq \frac{c}{2} \int_{x}^{x^*} \frac{|g'(t)|}{g(t)} dt \geq \frac{c}{2} \int_{x}^{x^*} \frac{g'(t)}{g(t)} dt
\]
\[
= \frac{c}{2} |\ln g(x^*) - \ln g(x)| \to \infty \text{ as } x \to -1^+ \text{ since } g(-1) = 0.
\]
This contradiction gives us the required result. \( \square \)

We summarize the results from Lemmas 2, 3, and 4.

**Theorem 2.** Assume \( \alpha > -1 \) and \( \beta = -1 \). Let \( f, g \in \Delta_{\alpha,-1} \). Then
(a) \( f' \in L^2(-1,0) \);
(b) \( f \in AC[-1,0] \);
(c) The Jacobi differential expression (4.1) is strong limit point at $x = -1$; that is to say,
\[ \lim_{x \to -1^+} (1 - x)^{\alpha+1} f'(x)g(x) = 0; \]

(d) The Jacobi differential expression (4.1) is Dirichlet at $x = -1$; that is to say,
\[ \int_{-1}^{0} (1 - x)^{\alpha+1} |f'(x)|^2 dx < \infty. \]

As a consequence of this theorem, note that for $f, g \in \Delta_{\alpha,-1}$, and for 
$-1 < x \leq 0$,
\[ \int_{-1}^{0} (1 - t)^{\alpha+1} f'(t)g'(t) dt = f'(0)g(0) - \int_{x}^{0} \ell_{\alpha,-1}[f](t)g(t)(1 - t)^{\alpha}(1 + t)^{-1} dt. \]

We now turn our attention to establishing that the Jacobi differential expression (4.1) is both strong-limit point and Dirichlet at $x = 1$ on $\mathcal{D}(A^{(\alpha,-1)})$, defined in (4.3).

**Lemma 5.** Let $\alpha > -1$. Then, for all $f \in \mathcal{D}(A^{(\alpha,-1)})$,
\[ \lim_{x \to 1^-} (1 - x)^{\alpha+1} f'(x) = 0. \]

**Proof.** From the definition of $\mathcal{D}(A^{(\alpha,-1)})$ in (4.3), this result is automatically true for $-1 < \alpha < 1$. So we assume $\alpha \geq 1$. Let $f \in \mathcal{D}(A^{(\alpha,-1)})$. Since the Jacobi expression is limit point at $x = 1$ in this case, we know that
\[ \lim_{x \to 1^-} (1 - x)^{\alpha+1} [f'(x)g(x) - f(x)g'(x)] = 0 \quad (g \in \Delta_{\alpha,-1}). \]

Construct a real-valued $\tilde{g} \in C^2[-1,1]$ such that
\[ \tilde{g}(x) = \begin{cases} 
0 & \text{for } x \text{ near } -1 \\
1 & \text{for } x \text{ near } +1;
\end{cases} \]

for example, we could take
\[ \tilde{g}(x) = \begin{cases} 
0 & \text{if } -1 < x \leq 0 \\
-16x^3 + 12x^2 & \text{if } 0 < x < 1/2 \\
1 & \text{if } 1/2 \leq x \leq 1. 
\end{cases} \]

Clearly $\tilde{g} \in \Delta_{\alpha,-1}$; moreover, with this choice of $\tilde{g}$, we have
\[ 0 = \lim_{x \to 1^-} (1 - x)^{\alpha+1} [f'(x)\tilde{g}(x) - f(x)\tilde{g}'(x)] = \lim_{x \to 1^-} (1 - x)^{\alpha+1} f'(x). \]
The proof of the following lemma is similar to the proof given in Lemma 4; we omit the details. This result shows that the Jacobi expression is Dirichlet at $x = 1$ on $D(\alpha,-1)$.

**Lemma 6.** Suppose $\alpha > -1$. Then, for any $f \in D(\alpha,-1)$, we have

$$(1 - x)^{\alpha+1/2} f' \in L^2(0,1).$$

We are now in position to prove that the Jacobi expression (4.1) is strong limit point at $x = 1$ on $D(\alpha,-1)$. The proof is similar to the one of Lemma 4 but different enough to warrant a demonstration.

**Lemma 7.** Suppose $\alpha > -1$. Then for $f, g \in D(\alpha,-1)$, we have

$$\lim_{x \to 1^-} (1 - x)^{\alpha+1} f'(x) g(x) = 0.$$

**Proof.** Let $f, g \in D(\alpha,-1)$. From (4.4), the definition of $D(\alpha,-1)$, and Lemma 6, we see that

$$\lim_{x \to 1^-} (1 - x)^{\alpha+1} f'(x) g(x)$$

exists and is finite. Without loss of generality, assume that both $f$ and $g$ are real-valued on $(-1,1)$; furthermore, by way of contradiction, we assume that

$$\lim_{x \to 1^-} (1 - x)^{\alpha+1} f'(x) g(x) = c > 0.$$

Again, we assume that there exists $x^* \in [0,1)$ such that

$$(1 - x)^{\alpha+1} f'(x) > 0, \ g(x) > 0, \ (1 - x)^{\alpha+1} f'(x) g(x) \geq c/2 \ (x \in [x^*,1)).$$

Hence it follows that

$$\left| \frac{(1 - x)^{\alpha+1} f'(x)}{(1 - x)^{\alpha+1} f'(x)} \right| g(x) \geq \frac{c}{2} \left( \frac{(1 - x)^{\alpha+1} f'(x)}{(1 - x)^{\alpha+1} f'(x)} \right) (x \in [x^*,1)).$$

Integrate over $[x^*, x]$ to get

$$\infty > \frac{1}{\ell_{\alpha,-1}} \int_{-1}^1 [f(t)] g(t) (1-t)^{\alpha} (1+t)^{-1} dt - k \int_{-1}^1 (f(t))^2 (1-t)^{\alpha} (1+t)^{-1} dt$$

$$= \int_{-1}^1 \left| (1-t)^{\alpha+1} f'(t) \right| g(t) dt \geq \int_{x}^{x^*} \left| (1-t)^{\alpha+1} f'(t) \right| g(t) dt$$

$$\geq \frac{c}{2} \int_{x}^{x^*} \frac{(1-t)^{\alpha+1} f'(t)}{(1-t)^{\alpha+1} f'(t)} dt \geq \frac{c}{2} \int_{x}^{x^*} \frac{(1-t)^{\alpha+1} f'(t)}{(1-t)^{\alpha+1} f'(t)} dt$$

$$= \frac{c}{2} \ln((1 - x^*)^{\alpha+1} f'(x^*)) - \ln((1 - x)^{\alpha+1} f'(x)) \to \infty \text{ as } x \to 1^-$$

by Lemma 5. This contradiction gives us the required result. \qed
Summarizing Lemmas 6 and 7, we have the following theorem.

**Theorem 3.** For \(\alpha > -1\) and \(\beta = -1\), the Jacobi differential expression \(\ell_{\alpha,-1}[]\), given in (4.1), is strong limit point and Dirichlet at \(x = 1\) on \(D(A^{(\alpha,-1)})\). That is to say,

(a) \(\lim_{x \to 1^-} (1 - x)^{\alpha+1} f'(x)g(x) = 0\) for all \(f, g \in D(A^{(\alpha,-1)})\), and

(b) \(\int_0^1 (1 - x)^{\alpha+1} |f'(x)|^2 \, dx < \infty\) for all \(f \in D(A^{(\alpha,-1)})\).

Combining Theorems 2 and 3, we see that Dirichlet’s formula, for \(f, g \in D(A^{(\alpha,-1)})\), over the interval \((-1, 1)\) reads:

\[
(A^{(\alpha,-1)} f, g)_{\alpha,-1} = \int_{-1}^1 [(1 - t)^{\alpha+1} f'(t)\overline{g}(t) + k f(t)\overline{g}(t)(1 - t)^{\alpha}(1 + t)^{-1}] \, dt.
\]

(4.8)

In particular, we see that

\[
(A^{(\alpha,-1)} f, f)_{\alpha,-1} \geq k (f, f)_{\alpha,-1} \quad (f \in D(A^{(\alpha,-1)}));
\]

(4.9)

that is to say, \(A^{(\alpha,-1)}\) is bounded below by \(kI\) in \(L^2_{\alpha,-1}(-1, 1)\). The consequence of the inequality in (4.9) is that the left-definite theory, developed by Littlejohn and Wellman, can be applied to \(A^{(\alpha,-1)}\). We now briefly review this theory.

5. **General Left-Definite Theory**

In [13], Littlejohn and Wellman developed a general abstract left-definite theory for a self-adjoint operator \(A\) that is bounded below in a Hilbert space \((H, (\cdot, \cdot))\).

Let \(V\) be a vector space over \(\mathbb{C}\) with inner product \((\cdot, \cdot)\) such that \(H := (V, (\cdot, \cdot))\) is a Hilbert space. Let \(r > 0\) and suppose that \(V_r\) is a vector subspace of \(V\) with inner product \((\cdot, \cdot)_r\); denote this inner product space by \(W_r := (V_r, (\cdot, \cdot)_r)\). Let \(A : D(A) \subset H \to H\) be a self-adjoint operator that is bounded below by \(rI\) for some \(r > 0\), that is to say

\[(Ax, x) \geq r(x, x) \quad (x \in D(A)).\]

Then, for any \(s > 0\), the operator \(A^s\) is self-adjoint and bounded below in \(H\) by \(r^sI\).

**Definition 1.** Let \(s > 0\) and let \(V_s\) be a vector subspace of the Hilbert space \(H = (V, (\cdot, \cdot))\) with inner product \((\cdot, \cdot)_s\). We say that \(W_s = (V_s, (\cdot, \cdot)_s)\) is an \(s^{th}\) left-definite space associated with the pair \((H, A)\) if

(i) \(W_s\) is a Hilbert space
(ii) \(D(A^s)\) is a vector subspace of \(V_s\)
(iii) \(D(A^s)\) is dense in \(W_s\)
(iv) \((x, x)_s \geq r^s(x, x)\) for all \(x \in V_s\)
(v) \((x,y)_s = (A^s x, y)\) for all \(x \in \mathcal{D}(A^s), y \in V_s\).

Littlejohn and Wellman in [13, Theorem 3.1] prove the following result.

**Theorem 4.** Let \(A : \mathcal{D}(A) \subset H \to H\) be a self-adjoint operator that is bounded below by \(rI\) for some \(r > 0\). Let \(s > 0\) and define \(W_s := (V_s, (\cdot, \cdot)_s)\) by

\[ V_s = \mathcal{D}(A^{s/2}) \]

and

\[ (x,y)_s = (A^{s/2} x, A^{s/2} y) \quad (x,y \in V_s). \]

Then \(W_s\) is the unique left-definite space associated with the pair \((H,A)\).

**Definition 2.** For \(s > 0\), let \(W_s := (V_s, (\cdot, \cdot)_s)\) be the \(s^{th}\) left-definite space associated with \((H,A)\). If there exists a self-adjoint operator \(B_s : \mathcal{D}(B_s) \subset W_s \to W_s\) satisfying

\[ B_s f = Af \quad (f \in \mathcal{D}(B_s) \subset \mathcal{D}(A)), \]

we call such an operator an \(s^{th}\) left-definite operator associated with the pair \((H,A)\).

In [13, Theorem 3.2], the authors prove the following existence/uniqueness result.

**Theorem 5.** Let \(A\) be a self-adjoint operator in a Hilbert space \(H\) that is bounded below by \(rI\) for some \(r > 0\). For any \(s > 0\), let \(W_s := (V_s, (\cdot, \cdot)_s)\) denote the \(s^{th}\) left-definite space associated with \((H,A)\). Then there exists a unique left-definite operator \(B_s\) in \(W_s\) associated with \((H,A)\). Furthermore,

\[ \mathcal{D}(B_s) = V_{s+2} \subset \mathcal{D}(A). \]

We refer the reader to other results established in [13]. We note if, in addition to the hypotheses assumed in this section, that \(A\) is bounded then the left-definite theory is trivial; specifically, \(V = V_s\) and \(A = A_s\) for all \(s > 0\). Only in the unbounded case will there be a non-trivial left-definite theory associated with \((H,A)\). Moreover, we note that the authors in [13] prove \(\sigma(A) = \sigma(A_s)\) for all \(s > 0\); in fact, the point spectrum \(\sigma_p\) and continuous spectrum \(\sigma_c\) match up for \(A\) and each left-definite operator \(A_s\).

6. Left-Definite Theory of the Nonclassical Jacobi Differential Equation

Let \(k > 0\). For each \(n \in \mathbb{N}\), define

\[ V_n^{(\alpha,-1)} := \{ f : (-1,1) \to \mathbb{C} | f \in AC_{loc}^{(n-1)}; f^{(j)} \in L_{\alpha+j,j-1}^2(-1,1), j = 0, \ldots, n \} \]
and let $(\cdot, \cdot)^{(\alpha,-1)}_n$ and $\|\cdot\|^{(\alpha,-1)}_n$ denote, respectively, the Sobolev inner product
\[
(f, g)^{(\alpha,-1)}_n := \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n, k) \int_{-1}^{1} f^{(j)}(t) g^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{j-1} dt
\]
and norm $\|f\|^{(\alpha,-1)}_n := (f, f)^{(\alpha,-1)}_n^{1/2}$, where the numbers $c_j^{(\alpha,-1)}(n, k)$ are defined as in [9] by
\[
c_j^{(\alpha,-1)}(n, k) := \begin{cases} 
0 & \text{if } k = 0 \\
k^n & \text{if } k > 0
\end{cases} ,
\]
and, for $j \in \{1, 2, \ldots, n\}$,
\[
c_{j}^{(\alpha,-1)}(n, k) := \begin{cases} 
P^{(\alpha,-1)} S_{n}^{(j)} & \text{if } k = 0 \\
\sum_{s=0}^{n-j} \binom{n}{s} P^{(\alpha,-1)} S_{n-s}^{(j)} k^s & \text{if } k > 0
\end{cases} ,
\]
where the Jacobi–Stirling number $P^{(\alpha,-1)} S_{n}^{(j)}$ of order $(n, j)$ is defined by
\[
P^{(\alpha,-1)} S_{n}^{(j)} := \sum_{r=0}^{j} (-1)^{r+j} \frac{\Gamma(\alpha+r)\Gamma(\alpha+2r+1)[r(r+\alpha)]^n}{r!(j-r)!\Gamma(\alpha+2r)\Gamma(\alpha+j+r+1)},
\]
for $(n, j \in \mathbb{N}; j \leq n)$. We extend this definition to include
\[
P^{(\alpha,-1)} S_{n}^{(0)} := 1, \quad P^{(\alpha,-1)} S_{n}^{(j)} := 0 \quad \text{if } j \in \mathbb{N} \text{ and } 0 \leq n \leq j-1, \quad P^{(\alpha,-1)} S_{n}^{(n)} := 0 \quad \text{for } n \in \mathbb{N}.
\]
In [9], the authors prove that $P^{(\alpha,-1)} S_{n}^{(j)} > 0$ for $n, j \in \mathbb{N}, j \leq n$. Let
\[
W^{(\alpha,-1)}_n := \left( V^{(\alpha,-1)}_n, (\cdot, \cdot)^{(\alpha,-1)}_n \right).
\]
We will show that the vector space $W^{(\alpha,-1)}_n$ is the $n^{th}$ left-definite space associated with the pair $(L^{2}_{\alpha,n-1}(-1,1), A^{(\alpha,-1)})$, where $A^{(\alpha,-1)}$ is the self-adjoint Jacobi operator defined in (4.2) and (4.3). Using the results from [9] mutatis mutandis, Theorem 6 follows; for a proof see [5].

**Theorem 6.** Let $k > 0$. For each $n \in \mathbb{N}, W^{(\alpha,-1)}_n$ is a Hilbert space.

**Theorem 7.** The Jacobi polynomials $\{P^{(\alpha,-1)}_m\}_{m=1}^{\infty}$ form a complete orthogonal set in each $W^{(\alpha,-1)}_n$. Equivalently, the set of polynomials $\mathcal{P}$ is dense in $W^{(\alpha,-1)}_n$.

**Proof.** Fix $n \in \mathbb{N}$, and let $f \in W^{(\alpha,-1)}_n$, so
\[
f^{(n)} \in L^{2}_{\alpha+n,n-1}(-1,1).
\]
Since \( \{ P_{m}^{(\alpha+n,n-1)} \}_{m=0}^{\infty} \) is a complete orthonormal set in \( L_{\alpha+n,n-1}^{2}(-1,1) \), we know
\[
\sum_{m=0}^{r} c_{m,n}^{(\alpha-1)} P_{m}^{(\alpha+n,n-1)} \to f^{(n)} \quad \text{as} \quad r \to \infty \quad \text{in} \quad L_{\alpha+n,n-1}^{2}(-1,1) \quad (6.1)
\]
where \( c_{m,n}^{(\alpha-1)} \) are the Fourier coefficients given by
\[
c_{m,n}^{(\alpha-1)} = \int_{-1}^{1} f^{(n)}(t) P_{m}^{(\alpha+n,n-1)}(t)(1-t)^{\alpha+n}(1+t)^{n-1} dt
\]
for \( m \in \mathbb{N}_{0} \). For \( r \geq n \) define the polynomials
\[
p_{r}(t) := \sum_{m=\max\{2,n\}}^{r} c_{m-n,n}^{(\alpha-1)} \frac{((m-n)!)^{1/2} (\Gamma(\alpha+m))^{1/2}}{(m!)^{1/2} (\Gamma(\alpha+m+n))^{1/2}} P_{m}^{(\alpha-1)}(t).
\]
From
\[
\frac{d^{j}}{dt^{j}} P_{m}^{(\alpha-1)}(t) = \frac{(m!)^{1/2} (\Gamma(\alpha+m+j))^{1/2}}{((m-j)!)^{1/2} (\Gamma(\alpha+m))^{1/2}} P_{m-j}^{(\alpha+j,j-1)}(t),
\]
we see that, for \( j = 0, 1, \ldots, n \),
\[
p_{r}^{(j)}(t) = \sum_{m=\max\{2,n\}}^{r} c_{m-n,n}^{(\alpha-1)} \frac{((m-n)!)^{1/2} (\Gamma(\alpha+m+j))^{1/2}}{(\Gamma(\alpha+m+n))^{1/2} ((m-j)!)^{1/2}} P_{m-j}^{(\alpha+j,j-1)}(t).
\]
In particular, by (6.1),
\[
p_{r}^{(n)}(t) = \sum_{m=\max\{2,n\}}^{r} c_{m-n,n}^{(\alpha-1)} P_{m-n}^{(\alpha+n,n-1)} = \sum_{l=0}^{r-\max\{2,n\}} c_{l,n}^{(\alpha-1)} P_{l}^{(\alpha+n,n-1)}
\]
\[
= \sum_{m=0}^{s} c_{m,n}^{(\alpha-1)} P_{m}^{(\alpha+n,n-1)} \to f^{(n)}
\]
as \( r \to \infty \) in \( L_{\alpha+n,n-1}^{2}(-1,1) \). Furthermore, by Riesz-Fischer, there exists a subsequence \( \{ p_{r_{j}}^{(n)} \} \) of \( \{ p_{r}^{(n)} \} \) such that
\[
p_{r_{j}}^{(n)} \to f^{(n)} \quad \text{for a.e.} \quad t \in (-1,1).
\]
By Dirichlet’s test, the sequence
\[
\left\{ \frac{c_{m-n,n}^{(\alpha-1)} ((m-n)!)^{1/2} (\Gamma(\alpha+m+j))^{1/2}}{(\Gamma(\alpha+m+n))^{1/2} ((m-j)!)^{1/2}} \right\} \in \ell^{2},
\]
so there exists a \( g_{j} \in L_{\alpha+j,j-1}^{2}(-1,1) \) such that
\[
p_{r_{j}} \to g_{j} \quad \text{in} \quad L_{\alpha+j,j-1}^{2}(-1,1).
\]
(6.2)
For a.e. \( a, t \in (-1, 1) \),

\[
\int_a^t p_{r_j}^{(n)}(u)du \to \int_a^t f^{(n)}(u)du.
\]

Integrate both sides and obtain

\[
p_{r_j}^{(n-1)}(t) \to f^{(n-1)}(t) + c_1 \quad \text{for a.e. } t \in (-1, 1) \tag{6.3}
\]

for some constant \( c_1 \). Passing through the subsequence implies

\[
g_{n-1}(t) = f^{(n-1)}(t) + c_1 \quad \text{for a.e. } t \in (-1, 1).
\]

From (6.3), we see that

\[
\int_a^t p_{r_j}^{(n-1)}(u)du \to \int_a^t f^{(n-1)}(u)du + c_1 \int_a^t du,
\]

that is,

\[
p_{r_j}^{(n-2)}(t) \to f^{(n-2)}(t) + c_1 t + c_2 \quad \text{for a.e. } t \in (-1, 1)
\]
or

\[
g_{n-2}(t) = f^{(n-2)}(t) + c_1 t + c_2 \quad \text{for a.e. } t \in (-1, 1).
\]

Continue this process to see that for \( j \in \{0, 1, \ldots, n-1\} \),

\[
g_j(t) = f^{(j)}(t) + q_{n-j+1} \quad \text{for a.e. } t \in (-1, 1),
\]

where \( q_{n-j-1} \) is a polynomial of degree \( \leq n-j-1 \) and where \( q_{n-j-1}' = q_{n-j-2} \).

Hence, with (6.2),

\[
p_r^{(j)} \to f^{(j)} + q_{n-j-1} \quad \text{in } L^2_{\alpha+j,j-1}(-1, 1) \tag{6.4}
\]

For \( r \geq n \), define \( \pi_r(t) := p_r(t) - q_{n-1}(t) \). Note that, with (6.4),

\[
\pi_r^{(j)}(t) = p_r^{(j)}(t) - q_{n-1}^{(j)}(t) = p_r^{(j)}(t) - q_{n-j-1}(t) \to f^{(j)}(t).
\]

Hence, as \( r \to \infty \),

\[
\left( \|f - \pi_r\|_{\alpha,-1}^{(n,k)} \right)^2 = \sum_{j=0}^n c_j^{(\alpha,-1)}(n,k) \\
\times \int_{-1}^{-1} \left| f^{(j)}(t) - \pi_r^{(j)}(t) \right|^2 (1-t)^{\alpha+j}(1+t)^{j-1} dt \to 0.
\]

\[\square\]

**Lemma 8.** For \( p, q \in \mathcal{P} \), the vector space of all polynomials \( p : [-1, 1] \to \mathbb{C} \), and for \( n \geq 1 \),

\[
(p, q)_n^{(\alpha,-1)} = \left( \left( A^{(\alpha,-1)} \right)^n p, q \right)_{\alpha,-1}.
\]
Nonclassical Jacobi Polynomials and Sobolev Orthogonality

Proof. First we note that this may be restated as

\[
\left(\ln_{\alpha,-1}[p], q\right)_{\alpha,-1} = \int_{-1}^{1} \ln_{\alpha,-1}[p](x)q(x)w_{\alpha,-1}(x)dx
= \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k)p(j)(x)\overline{q(j)}(x)(1-x)^{j+\alpha}(1+x)^{j-1}dx.
\]

(6.5)

Since the Jacobi polynomials form a basis for \( P \), it suffices to prove (6.5) for \( p = P_m^{(\alpha,-1)} \) and \( q = P_r^{(\alpha,-1)} \) for arbitrary \( m, r \in \mathbb{N}_0 \). From

\[
\ln_{\alpha,-1}[P_m^{(\alpha,-1)}](x) = (m(m-1) + k)nP_m^{(\alpha,-1)}(x) \quad (m \in \mathbb{N}_0)
\]

and

\[
\left( P_r^{(\alpha,-1)}, P_m^{(\alpha,-1)} \right)_{\alpha,-1} = \delta_{r,m} \quad (r, m \in \mathbb{N}_0),
\]

the left-hand side of (6.5) becomes

\[
\left( \ln_{\alpha,-1}[P_m^{(\alpha,-1)}], P_r^{(\alpha,-1)} \right)_{\alpha,-1} = \int_{-1}^{1} \ln_{\alpha,-1}[P_m^{(\alpha,-1)}](x)\overline{P_r^{(\alpha,-1)}}(x)w_{\alpha,-1}(x)dx
= (m(m-1) + k)n\delta_{r,m}.
\]

(6.6)

Upon using (2.3) for \( \alpha > -1, \beta = -1 \) and the recurrence relation for the \( c_j^{(\alpha,-1)}(n,k) \), that is,

\[
(m(m+\alpha) + k)n = \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k) \frac{m!(m+\alpha+j-1)!}{(m-j)!(m+\alpha-1)!}
\]

the right-hand side of (6.5) becomes

\[
\sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k) \times \int_{-1}^{1} \left( P_m^{(\alpha,-1)}(x) \right)^{(j)}(x)\left( \overline{P_r^{(\alpha,-1)}}(x) \right)^{(j)}(x)(1-x)^{j-1}(1+x)^{j-1}dx
= \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k) \frac{m!(m+\alpha+j-1)!}{(m-j)!(m+\alpha-1)!} \delta_{r,m}
= (m(m+\alpha) + k)n\delta_{r,m}.
\]

(6.8)

Comparing (6.6) and (6.8) completes the proof of the lemma. \( \Box \)
Theorem 8. For \( k > 0 \), let \( A^{(\alpha,-1)} \) be the Jacobi self-adjoint operator in \( L^2_{\alpha,-1}(-1,1) \), defined in (4.2) and (4.3), having the sequence of Jacobi polynomials \( \{P_m^{(\alpha,-1)}\}_{m=1}^{\infty} \) as eigenfunctions. For each \( n \in \mathbb{N} \), let

\[
V_n^{(\alpha,-1)} = \left\{ f : (-1,1) \to \mathbb{C} \mid f \in AC_{loc}^{(n-1)}; f^{(j)} \in L^2_{\alpha+j,j-1}(-1,1), j=0,\ldots,n \right\}
\]

and, for \( f, g \in V_n^{(\alpha,-1)} \),

\[
(f,g)_{n}^{(\alpha,-1)} = \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k) \int_{-1}^{1} f^{(j)}(t) g^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{j-1} dt.
\]

Then \( W_n^{(\alpha,-1)} = \left( V_n^{(\alpha,-1)}, \langle \cdot, \cdot \rangle_n^{(\alpha,-1)} \right) \) is the \( n \)th left-definite space associated with the pair \( (L^2_{\alpha,-1}(-1,1), A^{(\alpha,-1)}) \). Moreover, the Jacobi polynomials \( \{P_m^{(\alpha,-1)}\}_{m=1}^{\infty} \) form a complete orthogonal set in each \( W_n^{(\alpha,-1)} \), and they satisfy the orthogonality relation

\[
(P_m^{(\alpha,-1)}, P_l^{(\alpha,-1)})_{n} = (m(m-1) + k)\delta_{m,l}.
\]

Furthermore, define

\[
B_n^{(\alpha,-1)} := D\left( B_n^{(\alpha,-1)} \right) \subset W_n^{(\alpha,-1)} \to W_n^{(\alpha,-1)}
\]

by

\[
B_n^{(\alpha,-1)} f := l_{\alpha,-1}[f] \quad (f \in D\left( B_n^{(\alpha,-1)} \right) := V_{n+2}^{(\alpha,-1)}).
\]

Then \( B_n^{(\alpha,-1)} \) is the \( n \)th left-definite operator associated with \( (L^2_{\alpha,-1}(-1,1), A^{(\alpha,-1)}) \). Lastly, the spectrum of \( B_n^{(\alpha,-1)} \) is given by

\[
\sigma\left( B_n^{(\alpha,-1)} \right) = \{m(m-1) + k \mid m \in \mathbb{N}_0 \} = \sigma\{A^{(\alpha,-1)}\},
\]

with the Jacobi polynomials \( \{P_m^{(\alpha,-1)}\}_{m=1}^{\infty} \) forming a complete set of eigenfunctions of each \( B_n^{(\alpha,-1)} \).

Proof. Let \( n \in \mathbb{N} \). We need to show that \( W_n^{(\alpha,-1)} \) satisfies the five properties in Definition 1 in Sect. 5. (i) \( W_n^{(\alpha,-1)} \) is a Hilbert space (see Theorem 6 and Ref. [5]). (ii) We need to show

\[
D\left( (A^{(\alpha,-1)})^n \right) \subset W_n^{(\alpha,-1)} \subset L^2_{\alpha,-1}(-1,1).
\]

Let \( f \in D\left( (A^{(\alpha,-1)})^n \right) \). Since the Jacobi polynomials \( \{P_m^{(\alpha,-1)}\}_{m=1}^{\infty} \) form a complete orthonormal set in \( L^2_{\alpha,-1}(-1,1) \), we see that

\[
p_j \to f \quad \text{in} \ L^2_{\alpha,-1}(-1,1) \quad \text{as} \ j \to \infty \quad (6.9)
\]
where
\[ p_j(t) := \sum_{m=0}^{j} c_m^{(\alpha,-1)} P_m^{(\alpha,-1)}(t) \quad (t \in (-1, 1)), \]
and, for \( m \in \mathbb{N}_0 \),
\[ c_m^{(\alpha,-1)} := \left( f, P_m^{(\alpha,-1)} \right)_{\alpha,-1} = \int_{-1}^{1} f(t) P_m^{(\alpha,-1)}(t) (1 - t)^\alpha (1 + t)^{-1} dt. \]
Since \((A^{(\alpha,-1)})^n f \in L^2_{\alpha,-1}(-1, 1)\), we see that as \( j \to \infty \),
\[ \sum_{m=0}^{j} \tilde{c}_m^{(\alpha,-1)} P_m^{(\alpha,-1)} \to (A^{(\alpha,-1)})^n f \quad \text{in} \quad L^2_{\alpha,-1}(-1, 1) \]
where
\[ \tilde{c}_m^{(\alpha,-1)} := \left( (A^{(\alpha,-1)})^n f, P_m^{(\alpha,-1)} \right)_{\alpha,-1} = (f, (A^{(\alpha,-1)})^n P_m^{(\alpha,-1)})_{\alpha,-1} \]
\[ = (m(m + \alpha) + k)^n \left( f, P_m^{(\alpha,-1)} \right)_{\alpha,-1} \]
\[ = (m(m + \alpha) + k)^n c_m^{(\alpha,-1)}; \]
consequently,
\[ (A^{(\alpha,-1)})^n p_j \to (A^{(\alpha,-1)})^n f \]
in \( L^2_{\alpha,-1}(-1, 1) \) as \( j \to \infty \). Moreover, by Lemma 8,
\[ \left( \|p_j - p_r\|^{(\alpha,-1)}_n \right)^2 = \left( (A^{(\alpha,-1)})^n [p_j - p_r], p_j - p_r \right)_{\alpha,-1} \]
\[ \to 0 \quad \text{as} \quad j, r \to \infty \]
so \( \{p_j\}^{\infty}_{j=0} \) is Cauchy in \( W_n^{(\alpha,-1)} \). Since \( W_n^{(\alpha,-1)} \) is a Hilbert space (Theorem 6), there exists
\[ g \in W_n^{(\alpha,-1)} \subset L^2_{\alpha,-1}(-1, 1) \]
such that
\[ p_j \to g \quad \text{in} \quad W_n^{(\alpha,-1)} \quad \text{as} \quad j \to \infty. \]
Furthermore, since
\[ (f, f)_n^{(\alpha,-1)} = \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n, k) \left\| f^{(j)} \right\|_{j+\alpha,j-1}^2 \]
\[ \geq c_0^{(\alpha,-1)}(n, k) \left\| f^{(j)} \right\|_{\alpha,-1}^2 = k^n (f, f)_{\alpha,-1}, \]
we see that
\[ \|p_j - g\|_{\alpha,-1} \leq k^{-n/2} \|p_j - g\|_n^{(\alpha,-1)}, \]
and hence,

\[ p_j \to g \quad \text{in } L^2_{\alpha,-1}(-1,1). \]  

(6.10)

Comparing (6.9) and (6.10), \( f = g \in W^{(\alpha,-1)}_n \).

(iii) We need to show: \( \mathcal{D}(\{A^{(\alpha,-1)}_n\}) \) is dense in \( W^{(\alpha,-1)}_n \). Since the set of polynomials is contained in \( \mathcal{D}(\{A^{(\alpha,-1)}_n\}) \) and is dense in \( W^{(\alpha,-1)}_n \) (by Theorem 7), \( \mathcal{D}(\{A^{(\alpha,-1)}_n\}) \) is dense in \( W^{(\alpha,-1)}_n \). Furthermore, from Theorem 7, the Jacobi polynomials \( \{P^{(\alpha,-1)}_m\}_{m=1}^{\infty} \) form a complete orthonormal set in \( W^{(\alpha,-1)}_n \).

(iv) We need to show that \( \langle f, f \rangle^{(\alpha,-1)}_n \geq k^n \langle f, f \rangle^{(\alpha,-1)}_1 \) for \( f \in V^{(\alpha,-1)}_n \). This is clear from the definition of \( \langle \cdot, \cdot \rangle^{(\alpha,-1)}_n \).

(v) We need to show: \( \langle f, g \rangle^{(\alpha,-1)}_n = \langle (A^{(\alpha,-1)}_n)f, g \rangle^{(\alpha,-1)}_1 \) for \( f \in \mathcal{D}(\{A^{(\alpha,-1)}_n\}) \) and \( g \in V^{(\alpha,-1)}_n \). This is true for any \( f, g \in \mathcal{P} \) by Lemma 8. Let \( f \in \mathcal{D}(\{A^{(\alpha,-1)}_n\}) \subset W^{(\alpha,-1)}_n, g \in W^{(\alpha,-1)}_n \). Since the set of polynomials \( \mathcal{P} \) is dense in both \( W^{(\alpha,-1)}_n \) and \( L^2_{\alpha,-1}(-1,1) \), and since convergence in the space \( W^{(\alpha,-1)}_n \) implies convergence in \( L^2_{\alpha,-1}(-1,1) \) (by (iv)), there exist sequences \( \{p_j\}_{j=0}^{\infty} \) and \( \{q_j\}_{j=0}^{\infty} \) such that

\[ p_j \to f \quad \text{in } W^{(\alpha,-1)}_n \text{ as } j \to \infty \]

\[ \left( A^{(\alpha,-1)} \right)^n p_j \to \left( A^{(\alpha,-1)} \right)^n f \]

in \( L^2_{\alpha,-1}(-1,1) \) as \( j \to \infty \) and

\[ q_j \to g \]

in \( W^{(\alpha,-1)}_n \) and \( L^2_{\alpha,-1}(-1,1) \) as \( j \to \infty \). Hence, by Lemma 8,

\[ \left( \left( A^{(\alpha,-1)} \right)^n f, g \right)_{-1,-1} = \lim_{j \to \infty} \left( \left( A^{(\alpha,-1)} \right)^n p_j, q_j \right)_{\alpha,-1} = \lim_{j \to \infty} \left( p_j, q_j \right)_n = \langle f, f \rangle^{(\alpha,-1)}_n. \]

The results listed in the theorem on \( B^{(\alpha,-1)}_n \) and the spectrum of \( B^{(\alpha,-1)}_n \) follow immediately from the general left-definite theory. \( \square \)

7. The Sobolev Orthogonality of the Jacobi Polynomials

If we renormalize the Jacobi polynomials as follows:

\[ \widetilde{P}_0^{(\alpha,-1)}(x) := 1, \quad \widetilde{P}_1^{(\alpha,-1)}(x) := \left( \frac{\alpha + 2}{2^{\alpha+2}} \right)^{1/2} (x + 1), \]

and, for \( n \geq 2 \),
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\[ \tilde{P}_n^{(\alpha,-1)}(x) := \frac{(2n + \alpha)^{\frac{3}{2}}}{2^{n+\alpha/2} (n + \alpha)} \sum_{j=0}^{n} \binom{n + \alpha}{n - j} \binom{n - 1}{j} \left( \frac{x - 1}{2} \right)^j \left( \frac{x + 1}{2} \right)^{n-j}, \]

we obtain the following theorem; a proof can be found in [12]. A key result in establishing this result is in the following important identity

\[ \tilde{P}_n^{(\alpha,-1)}(x) = (n + \alpha)!(n - 1)! \frac{(x + 1)\tilde{P}_{n-1}^{(\alpha,1)}(x)}{2n!(n + \alpha - 1)!} \quad (n \geq 2) \]

whose proof can easily be checked.

**Theorem 9.** The Jacobi polynomials \( \{\tilde{P}_n^{(\alpha,-1)}(x)\}_{n=0}^{\infty} \) are orthonormal with respect to the Sobolev inner product

\[ \phi(f, g) := f(-1)\overline{g}(-1) + \int_{-1}^{1} (1-x)^{\alpha+1} f'(x)\overline{g'}(x)dx, \]

that is,

\[ \phi(\tilde{P}_n^{(\alpha,-1)}, \tilde{P}_m^{(\alpha,-1)}) = \delta_{nm} \quad (n, m \in \mathbb{N}_0). \]

8. Jacobi Polynomials and a Self-Adjoint Operator in a Sobolev Space

**Definition 3.** Define

\[ W_\alpha := \{ f : [-1, 1) \to \mathbb{C} \mid f \in AC[-1, 1) ; f' \in L^2_{\alpha+1,0}(-1,1) \} \]

\[ \phi(f, g) := f(-1)\overline{g}(-1) + \int_{-1}^{1} f'(x)\overline{g'}(x)(1-x)^{\alpha+1}dx \quad (f, g \in W_\alpha). \]

Write \( \|f\|_\phi^2 = \phi(f, f) \) for \( f \in W_\alpha \).

**Theorem 10.** \((W_\alpha, \phi(\cdot, \cdot))\) is a Hilbert space.

**Proof.** Let \( \{f_n\} \subset W_1 \) be a Cauchy sequence. Hence

\[ \|f_n - f_m\|_\phi^2 = |f_n(-1) - f_m(-1)|^2 + \int_{-1}^{1} |f'_n(x) - f'_m(x)|^2 (1-x)^{\alpha+1}dx \]

\[ \to 0 \quad \text{as} \ n, m \to \infty. \]

In particular, since

\[ \int_{-1}^{1} |f'_n(x) - f'_m(x)|^2 (1-x)^{\alpha+1}dx \leq \|f_n - f_m\|_\phi^2, \]

we see that \( \{f'_n\} \) is Cauchy in \( L^2((-1,1);(1-x)^{\alpha+1}). \)
Since $L^2((-1,1); (1-x)^{\alpha+1})$ is complete, there exists $g \in L^2_{\alpha+1,0}(-1,1)$ such that
\[ f'_n \to g \quad \text{as } n \to \infty \quad \text{in } L^2_{\alpha+1,0}(-1,1). \tag{8.1} \]
Also, since
\[ |f_n(-1) - f_m(-1)|^2 \leq \|f_n - f_m\|_\phi^2 \]
we see that the sequence $\{f_n(-1)\}$ is Cauchy in $\mathbb{C}$ and, hence, there exists $A \in \mathbb{C}$ such that
\[ f_n(-1) \to A. \tag{8.2} \]
Furthermore, since $f_n \in AC([-1,1), n \in \mathbb{N})$, we see that
\[ \frac{1}{1} \int_{-1}^{1} f'_n(t)(1-t)^{\alpha+1} dt \to \frac{1}{1} \int_{-1}^{1} g(t)(1-t)^{\alpha+1} dt, \]
Since $g \in AC([-1,1)$, we may define $f : [-1,1) \to \mathbb{C}$ by
\[ f(x) = A + \int_{-1}^{x} g(t) dt. \]
It is clear that $f \in AC[-1,1)$ and $f'(x) = g(x) \in L^2_{\alpha+1,0}(-1,1)$ for a.e. $x \in [-1,1)$, so $f \in W_\alpha$. Furthermore, $f(-1) = A$. Now
\[ \|f_n - f\|_\phi^2 = |f_n(-1) - f(-1)|^2 + \int_{-1}^{1} |f'_n(t) - f'(t)|^2 (1-t)^{\alpha+1} dt \]
\[ = |f_n(-1) - A|^2 + \int_{-1}^{1} |f'_n(t) - g(t)|^2 (1-t)^{\alpha+1} dt \to 0 \]
as $n \to \infty$ by (8.1) and (8.2). Thus, $(W_\alpha, \phi(\cdot, \cdot))$ is complete. \qed

**Theorem 11.** Let $W_\alpha$ and $\phi(\cdot, \cdot)$ be as before, and
\[ W_{\alpha,1} := \{ f \in W_\alpha \mid f(-1) = 0 \} \]
\[ W_{\alpha,2} := \{ f \in W_\alpha \mid f'(x) = 0 \}. \]
Then $W_{\alpha,1}$ and $W_{\alpha,2}$ are closed, orthogonal subspaces of $W_\alpha$ and $W_\alpha = W_{\alpha,1} \oplus W_{\alpha,2}$.

**Proof.** Since $W_{\alpha,2}$ is one-dimensional, it is a closed subspace of $W_\alpha$. The orthogonal complement of $W_{\alpha,2}$ is given by
\[ W_{\alpha,2}^\perp := \{ f \in W_\alpha \mid \phi(f, g) = 0 \ (g \in W_{\alpha,2}) \}. \]
To see that $W_{\alpha,1} \subset W_{\alpha,2}^{\perp}$, let $f \in W_{\alpha,1}, g \in W_{\alpha,2}$ and consider
\[
\phi(f, g) = f(-1)g(-1) + \int_{-1}^{1} f'(x)g'(x)(1 - x)^{\alpha+1}dx = 0.
\]
The first summand vanishes because $f \in W_{\alpha,1}$, and the integral is 0 because $g \in W_{\alpha,2}$. Now let $f \in W_{\alpha}$. We need to find $f_1 \in W_{\alpha,1}$ and $f_2 \in W_{\alpha,2}$ such that $f = f_1 + f_2$. Let $f_1(x) = f(x) - f(-1)$ and $f_2(x) = f(-1)$; clearly, $f_i \in W_{\alpha,i}$ for $i = 1, 2$. \hfill $\square$

The next result shows that, surprisingly, the space $W_{\alpha,1}$ is precisely the first left-definite vector space $V_{1}^{(\alpha,-1)}$. It is this theorem that allows us to construct a self-adjoint operator $T_{\alpha}$ in $W_{\alpha}$ having the entire sequence of Jacobi polynomials $\{P_n^{(\alpha,-1)}\}_{n=0}^{\infty}$ as eigenfunctions. For $n \geq 2$,
\[
\tilde{P}_n^{(\alpha,-1)}(x) = \frac{(n + \alpha)!(n - 1)!}{2n!(n + \alpha - 1)!}(x + 1)\tilde{P}_{n-1}^{(\alpha,1)}(x).
\]

**Theorem 12.** $W_{\alpha,1} = V_{1}^{(\alpha,-1)}$.

**Proof.** Notice that functions in $W_{\alpha,1}$ are defined on $[-1, 1)$ while functions in $V_{1}^{(\alpha,-1)}$ are defined on $(-1, 1)$ so the connection between these two spaces is not immediately obvious.

$V_{1}^{(\alpha,-1)} \subseteq W_{\alpha,1}$: Let $f \in V_{1}^{(\alpha,-1)}$. Since $V_{1}^{(\alpha,-1)} \subset \Delta_{\alpha,-1}$ we see, by Lemma 2, that $f \in AC[-1, 1)$ and $f(-1) = 0$ so $f \in W_{\alpha,1}$.

$W_{\alpha,1} \subseteq V_{1}^{(\alpha,-1)}$: Let $f \in W_{\alpha,1}$. From the definition of both spaces, it clearly suffices to show that $f \in L_{2,\alpha,-1}^{2}(-1, 1)$. For $-1 < x < 0$,
\[
(1 - x)^{\alpha/2}(1 + x)^{-1/2} \int_{-1}^{x} f'(t)dt = (1 - x)^{\alpha/2}(1 + x)^{-1/2} f(x)
\]
since $f(-1) = 0$. We use Theorem 1 on $(-1, 0)$ with $\psi(x) = (1 - x)^{\alpha/2}(1 + x)^{-1/2}$ and $\varphi(x) = 1$. Clearly, $\psi$ and $\varphi$ satisfy the conditions of Theorem 1. Moreover
\[
\int_{-1}^{x} dt \int_{x}^{0} (1 - t)^{\alpha}(1 + t)^{-1} dt \leq c \int_{-1}^{x} dt \int_{x}^{0} \frac{dt}{1 + t} = -c(x + 1) \ln(1 + x),
\]
and, since this is a bounded function on $(-1, 0)$, we see that $f \in L_{2,\alpha,-1}^{2}(-1, 0)$. A similar argument shows that $f \in L_{2,\alpha,-1}^{2}(0, 1)$ and this completes the proof. \hfill $\square$

**Theorem 13.** The inner products $\phi(\cdot, \cdot)$ and $(\cdot, \cdot)_{1}^{(\alpha,-1)}$ are equivalent on $W_{\alpha,1} = V_{1}^{(\alpha,-1)}$. 
Proof. First of all, \((W_{\alpha,1}, \phi(\cdot, \cdot))\) is a Hilbert space, and, by definition, \((V_1^{(\alpha,-1), \alpha,-1})\) is a Hilbert space. Let \(f, W_{\alpha,1} = V_1^{(\alpha,-1)}\). Then

\[
\|f\|_\phi^2 = \int_{-1}^1 |f'|^2 (1-x)^{\alpha+1} dx \leq \int_{-1}^1 [ |f'|^2 (1-x)^{\alpha+1} + k(1-x)^\alpha (1+x)^{-1} |f|^2 ] dx = (\|f\|_1)^2.
\]

By the Open Mapping Theorem, these inner products are equivalent.

We now construct a self-adjoint operator \(T_{\alpha,1}\) in the space \(W_{\alpha,1}\), generated by the Jacobi expression \(l_{\alpha,-1}\), having the sequence of Jacobi polynomials \(\{P_n^{(\alpha,-1)}\}_{n=1}^\infty\) as eigenfunctions. Recall that by Theorem 12, \(V_1^{(\alpha,-1)} = W_{\alpha,1}\).

We also know that the operator

\[
B_1^{(\alpha,-1)} : D(B_1^{(\alpha,-1)}) := V_3^{(\alpha,-1)} \subset V_1^{(\alpha,-1)} \rightarrow V_1^{(\alpha,-1)}
\]

namely, the first left-definite operator associated with \((L_{\alpha,-1}^2(-1,1), A^{(\alpha,-1)})\), is self-adjoint and given specifically by

\[
B_1^{(\alpha,-1)}[f](x) = l_{\alpha,-1}[f](x) \text{ (a.e. } x \in (-1,1))
\]

\[
f \in D(B_1^{(\alpha,-1)}) = V_3^{(\alpha,-1)},
\]

where

\[
f \in D(B_1^{(\alpha,-1)}) = V_3^{(\alpha,-1)} = \{ f : (-1,1) \rightarrow \mathbb{C} \mid f, f', f'' \in AC_{loc}(-1,1); (1-x)^{(\alpha+3)/2}(1+x)f''', (1-x)^{(\alpha+2)/2}(1+x)^{1/2}f'', (1-x)^{(\alpha+1)/2}f', (1-x)^{\alpha/2}(1+x)^{-1/2}f \in L^2(-1,1) \}.
\]

More specifically, \(B_1^{(\alpha,-1)}\) is self-adjoint with respect to the first left-definite inner product \((\cdot, \cdot)_{V_3^{(\alpha,-1)}}\) which is equivalent to the inner product \(\phi(\cdot, \cdot)\). We shall prove that the operator \(T_{\alpha,1} : D(T_{\alpha,1}) \subset W_{\alpha,1} \rightarrow W_{\alpha,1}\) given by

\[
T_{\alpha,1} f = B_1^{(\alpha,-1)} f = l_{\alpha,-1}[f]
\]

\[f \in D(T_{\alpha,1}) := V_3^{(\alpha,-1)}
\]
is self-adjoint in \((W_{\alpha,1}, \phi(\cdot, \cdot))\).

Lemma 9. \(T_{\alpha,1}\) in \((W_{\alpha,1}, \phi(\cdot, \cdot))\) is densely defined.

Proof. The Jacobi polynomials \(\{P_n^{(\alpha,-1)}\}_{n=1}^\infty\) are eigenfunctions of \(T_{\alpha,1}\) and they are complete in \(V_3^{(\alpha,-1)}\).

Theorem 14. \(T_{\alpha,1}\) is symmetric in \((W_{\alpha,1}, \phi(\cdot, \cdot))\).
Proof. From the previous lemma, it suffices to show that $T_{\alpha,1}$ is Hermitian. Let $f, g \in \mathcal{D}(T_{\alpha,1}) = V_3^{(\alpha,-1)}$. Since $V_3^{(\alpha,-1)} \subset V_1^{(\alpha,-1)}$ and $T_{\alpha,1}f, T_{\alpha,1}g \in V_1^{(\alpha,-1)}$, we know that

$$f(-1) = g(-1) = 0 = T_{\alpha,1}f(-1) = T_{\alpha,1}g(-1).$$

Integration by parts shows that $\phi(T_{\alpha,1}f, g) = \phi(f, T_{\alpha,1}g)$.

\[ \square \]

**Theorem 15.** The operator $T_{\alpha,1}$ has the following properties:

1. $T_{\alpha,1}$ is self-adjoint in $(W_{\alpha,1}, \phi(\cdot, \cdot))$.
2. $\sigma(T_{\alpha,1}) = \{n(n + \alpha) + k \mid n \geq 2\}$.
3. $\{P_n^{(\alpha,-1)}\}_{n \geq 1}$ is a complete orthonormal set of eigenfunctions of $T_{\alpha,1}$ in the space $(W_{\alpha,1}, \phi(\cdot, \cdot))$.
4. $T_{\alpha,1}$ is bounded below by $kI$ in $(W_{\alpha,1}, \phi(\cdot, \cdot))$.

**Proof.** For (iii): We know that $\{P_n^{(\alpha,-1)}\}_{n \geq 0}$ is a complete orthonormal set in $(W_{\alpha}, \phi(\cdot, \cdot))$ and we know that $W_{\alpha} = W_{\alpha,1} \oplus W_{\alpha,2}$. Also, we have $W_{\alpha,2} = \text{span}\{P_0^{(\alpha,-1)}\}$ and so $W_{\alpha,1} = W_{\alpha,2}^\perp = \text{span}\{P_n^{(\alpha,-1)}\}_{n \geq 1}$. We next prove that $T_{\alpha,1}$ is closed in $(W_{\alpha,1}, \phi(\cdot, \cdot))$. Take a sequence $\{f_n\} \subseteq \mathcal{D}(T_{\alpha,1}) = V_3^{(\alpha,-1)}$ such that

$$f_n \rightarrow f \quad \text{in} \quad (W_{\alpha,1}, \phi(\cdot, \cdot))$$

$$T_1f_n \rightarrow g \quad \text{in} \quad (W_{\alpha,1}, \phi(\cdot, \cdot)).$$

We show that $f \in \mathcal{D}(T_{\alpha,1})$ and $T_{\alpha,1}f = g$. We know that $B_1^{(\alpha,-1)}$ is self-adjoint and hence closed in $(W_{\alpha,1}, (\cdot, \cdot)_1^{(\alpha,-1)})$, and we know, since $\phi(\cdot, \cdot)$ and $(\cdot, \cdot)_1^{(\alpha,-1)}$ are equivalent, there exist constants $c_1$ and $c_2$ such that

$$c_1 \|f\|_\phi \leq \|f\|_1 \leq c_2 \|f\|_\phi \quad (f \in W_{\alpha,1} = V_1).$$

Hence,

$$\|f_n - f\|_1 \leq c_2 \|f_n - f\|_\phi \rightarrow 0;$$

that is,

$$f_n \rightarrow f \quad \text{in} \quad (W_{\alpha,1}, (\cdot, \cdot)_1^{(\alpha,-1)})$$

and

$$\|T_{\alpha,1}f_n - g\|_1 \leq c_2 \|T_{\alpha,1}f_n - g\|_\phi \rightarrow 0;$$

hence

$$T_{\alpha,1}f_n \rightarrow g \quad \text{in} \quad (W_{\alpha,1}, (\cdot, \cdot)_1^{(\alpha,-1)})$$
and since $T_{\alpha,1}$ is closed in $(W_{\alpha,1}, (\cdot, \cdot)_{1}^{(\alpha,-1)})$, we see that $f \in \mathcal{D}(T_{\alpha,1})$ and $T_{\alpha,1}f = g$. Also, we know that, for $n \geq 2$,

$$
(T_{\alpha,1}P_{n}^{(\alpha,-1)})(x) = l_{\alpha,-1}[P_{n}^{(\alpha,-1)}](x)
= (n(n + \alpha) + k)P_{n}^{(\alpha,-1)}(x).
$$

This implies

$$
\{n(n + \alpha) + k \mid n \geq 2\} \subseteq \sigma(T_{\alpha,1}).
$$

Since $\{P_{n}^{(\alpha,-1)}\}_{n \geq 1}$ is complete and $\lambda_n := n(n + \alpha) + k \to \infty$, we know that

$$
\sigma(T_{\alpha,1}) = \{n(n + \alpha) + k \mid n \geq 2\}
$$

which proves (ii) and (iii). To summarize: $T_{\alpha,1}$ is a closed, symmetric operator with a complete set of eigenfunctions. From [14], $T_{\alpha,1}$ is self-adjoint. This proves (i). To prove (iv), let $f \in \mathcal{D}(T_{\alpha,1})$. Then, since $T_{\alpha,1} : V_{3}^{(\alpha,-1)} \subset V_{1}^{(\alpha,-1)} \to V_{1}^{(\alpha,-1)}$,

$$
\phi(T_{\alpha,1}f, f) = (T_{\alpha,1}f)(-1)\bar{f}(-1) + \int_{-1}^{1} (T_{\alpha,1}f)'(x)\bar{f}'(x)(1 - x)^{\alpha+1} dx
= \int_{-1}^{1} (T_{\alpha,1}f)'(x)\bar{f}'(x)(1 - x)^{\alpha+1} dx
= \int_{-1}^{1} \left[ \left| (1 - x)^{\alpha+1}f'(x) \right|^2 + k |f'(x)|^2 (1 - x)^{\alpha+1} \right] dx
\geq k \int_{-1}^{1} |f'(x)|^2 (1 - x)^{\alpha+1} dx = k\phi(f, f).
$$

Next, we define the operator $T_{\alpha,2} : \mathcal{D}(T_{\alpha,2}) \subset W_{\alpha,2} \to W_{\alpha,2}$ by

$$
(T_{\alpha,2}f)(x) = l_{\alpha,-1}[f](x)
$$

$$
\mathcal{D}(T_{\alpha,2}) := W_{\alpha,2}.
$$

It is straightforward to check that $T_{\alpha,2}$ is symmetric in $W_{\alpha,2}$ and since the domain of $T_{\alpha,2}$ is the entire space, it follows that $T_{\alpha,2}$ is self-adjoint.

We now construct the self-adjoint operator $T_{\alpha}$ in $(W_{\alpha}, \phi(\cdot, \cdot))$ that is generated by the Jacobi differential expression $l_{\alpha,-1}[\cdot]$, having the entire set of Jacobi polynomials $\{P_{n}^{(\alpha,-1)}\}_{n \geq 0}$ as eigenfunctions and having spectrum $\sigma(T_{\alpha}) = \{n(n + \alpha) + k \mid n \in \mathbb{N}_0\}$. For $f \in W_{\alpha}$, write

$$
f = f_1 + f_2
$$
where \( f_i \in W_{\alpha,i}, (i = 1, 2) \). Define \( T_\alpha : \mathcal{D}(T_\alpha) \subset W_\alpha \rightarrow W_\alpha \) by

\[
T_\alpha f = T_{\alpha,1}f_1 + T_{\alpha,2}f_2 = l_{\alpha,-1}[f_1] + l_{\alpha,-1}[f_2] = l_{\alpha,-1}[f],
\]

\[
\mathcal{D}(T_\alpha) = \mathcal{D}(T_{\alpha,1}) \oplus \mathcal{D}(T_{\alpha,2}).
\]

A proof that operators of this form are self-adjoint can be found in [10, Theorem 11.1]. Furthermore, since we know explicitly the domains of \( T_{\alpha,1} \) and \( T_{\alpha,2} \), we can specifically determine the domain \( \mathcal{D}(T_\alpha) \) of \( T_\alpha \).

**Theorem 16.** \( T_\alpha \) is self-adjoint in \((W_\alpha, \phi(\cdot, \cdot))\) and

\[
\mathcal{D}(T_\alpha) = \{ f : [-1, 1) \rightarrow \mathbb{C} \mid f \in AC[-1, 1); f', f'' \in AC_{loc}(-1, 1); (1 - x)^{\alpha+3}/2(1 + x)f''', (1 - x)^{\alpha+2}/2(1 + x)^{1/2}f'', (1 - x)^{\alpha+1}/2 f' \in L^2(-1, 1) \}
\]

\[
= \{ f : [-1, 1) \rightarrow \mathbb{C} \mid f \in AC[-1, 1); f', f'' \in AC_{loc}(-1, 1); f(-1) = 0 \}
\]

\[
= W_{\alpha,1}
\]

\[
V_1^{(\alpha,-1)} = \mathcal{D}(T_{\alpha,1}) = \{ f : (-1, 1) \rightarrow \mathbb{C} \mid f \in AC_{loc}(-1, 1); f', f'' \in AC_{loc}(-1, 1); (1 - x)^{\alpha+3}/2(1 + x)f''', (1 - x)^{\alpha+2}/2(1 + x)^{1/2}f'', (1 - x)^{\alpha+1}/2 f' \in L^2(-1, 1) \}
\]

\[
= \{ f : [-1, 1) \rightarrow \mathbb{C} \mid f \in AC[-1, 1); f', f'' \in AC_{loc}(-1, 1); f(-1) = 0; (1 - x)^{\alpha+3}/2(1 + x)f''', (1 - x)^{\alpha+2}/2(1 + x)^{1/2}f'', (1 - x)^{\alpha+1}/2 f' \in L^2(-1, 1) \}.
\]

**Theorem 17.** Let

\[
\mathcal{D} := \{ f : [-1, 1) \rightarrow \mathbb{C} \mid f \in AC[-1, 1); f', f'' \in AC_{loc}(-1, 1); (1 - x)^{\alpha+3}/2(1 + x)f''', (1 - x)^{\alpha+2}/2(1 + x)^{1/2}f'', (1 - x)^{\alpha+1}/2 f' \in L^2(-1, 1) \}.
\]

Then \( \mathcal{D}(T_\alpha) = \mathcal{D} \).
Proof. We first show $\mathcal{D}(T_\alpha) \subseteq \mathcal{D}$. Let $f \in \mathcal{D}(T_\alpha) = \mathcal{D}(T_{\alpha,1}) \oplus \mathcal{D}(T_{\alpha,2})$. Write

$$f = f_1 + f_2$$

where $f_1 \in \mathcal{D}(T_{\alpha,1}) = V_3^{(\alpha,-1)} \subseteq \mathcal{D}$, $f_2 \in \mathcal{D}(T_{\alpha,2}) \subseteq \mathcal{D}$. Then $f \in \mathcal{D}$. To show that $\mathcal{D} \subseteq \mathcal{D}(T_\alpha)$, let $f \in \mathcal{D}$. Write

$$f(x) = f_1(x) + f_2(x)$$

where

$$f_1(x) := f(x) + f(-1)$$

$$f_2(x) := -f(-1).$$

Then $f_1 \in \mathcal{D}$ and $f_1(-1) = 0$; that is, $f_1 \in V_3^{(\alpha,-1)} = \mathcal{D}(T_{\alpha,1})$. Also, $f_2''(x) = 0$ so $f_2 \in \mathcal{D}(T_{\alpha,2})$. Together, $f \in \mathcal{D}(T_\alpha)$.

\[\square\]

References


Nonclassical Jacobi Polynomials and Sobolev Orthogonality


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