1.2 Algebra and Composition with Functions

Definition [Domain]: For a function $f(x)$, the domain is all x values s.t. $f(x)$ exists. In simple terms, there are only two ways that an x value **is not** in the domain of $f(x)$, that is,

i) if it makes the denominator zero (you can never divide by zero).

Ex. $f(x) = \frac{1}{x-1}$ has a domain of all real numbers $\mathbb{R}$, except $x = 1$.

ii) if it makes the inside of a square root negative (you can never take the square root of a negative number).

Ex. $f(x) = \sqrt{x+5}$ has a domain of $[-5, \infty)$ (solved $x + 5 \geq 0$).

If we are given two functions $f(x)$ and $g(x)$ we can define four new functions as follows.

Properties of Functions:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example: Let $f(x) = 3x - 4$ and $g(x) = 5x + 1$. Write a formula for each of the following functions.

i) $(f+g)(x)$

ii) $(f-g)(x)$

iii) $(fg)(x)$

iv) $\left(\frac{f}{g}\right)(x)$

Example: Let $f(x) = x + 1$, $g(x) = 2x - 3$, and $h(x) = x^2$. Find the following.

i) $(f - g)(4)$

ii) $(f + g - h)(-2)$

iii) $(gh - f)(1)$

iv) $\left(\frac{f}{g}\right)(3)$

Definition [Composite Function]: Given two functions $f(x)$ and $g(x)$, we define

$$(f \circ g)(x) = f(g(x))$$

Example: Let $f(x) = x - 3$, $g(x) = (x + 2)^2$. Find the following.

i) $(f \circ g)(x)$

ii) $(f \circ g)(1)$

iii) $(g \circ f)(x)$

iv) $(g \circ f)(3)$

Definition [Revenue, Cost, Profit]:

The **revenue function** is the price per unit, p, multiplied by the number of units sold, x. It can be modeled by

$$\text{Revenue} = (\text{Number of items sold})(\text{Price of each item})$$
$$R(x) = x \cdot p$$

The **cost function** is the total cost of producing x units, and it can be modeled by

$$C(x) = (\text{variable cost}) + (\text{fixed cost})$$

The **profit** can be modeled by

$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$P(x) = R(x) - C(x)$$

Example: A office supplies manufacturer knows that the number of binders he can sell each week is related to the price of binders by the equation $x = 1200 - 50p$, where x is the number of binders and p is the price per binder. What price should he charge if he wants a weekly revenue of $\$3,000$?

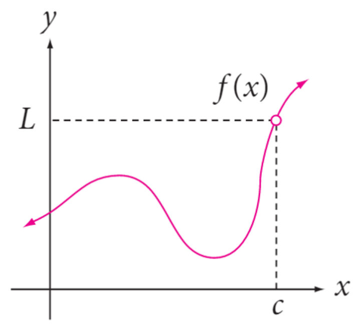
1.2 Problems: 1, 2, 19, 20, 23, 26, 27, 30, 39, 42, 43, 44, 48

1.4 Introduction to Limits

Definition [Limit Notation]: The limit of $f(x)$, as x approaches c, is L, can be written as

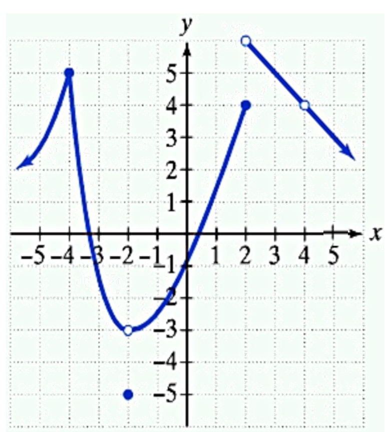
$$\lim_{x \to c} f(x) = L,$$

meaning $f(x)$ gets really close to L as x gets sufficiently close to c.



Theorem [Limits]: $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$. That is, $f(x)$ approaches L from **both** the left side ($x \to c^-$) and the right side ($x \to c^+$).

Example: Use the graph to find the following



i) $\lim_{x \to -4^-} f(x) =$ $\quad\lim_{x \to -4^+} f(x) =$ $\quad\lim_{x \to -4} f(x) =$ $\quad f(-4) =$

ii) $\lim_{x \to -2^-} f(x) =$ $\quad\lim_{x \to -2^+} f(x) =$ $\quad\lim_{x \to -2} f(x) =$ $\quad f(-2) =$

iii) $\lim_{x \to 2^-} f(x) =$ $\quad\lim_{x \to 2^+} f(x) =$ $\quad\lim_{x \to 2} f(x) =$ $\quad f(2) =$

iv) $\lim_{x \to 4^-} f(x) =$ $\quad\lim_{x \to 4^+} f(x) =$ $\quad\lim_{x \to 4} f(x) =$ $\quad f(4) =$

Properties of Limits:

i) If k is a constant, then $\lim_{x \to c} k = k$.

ii) If $P(x)$ is a polynomial function, then $\lim_{x \to c} P(x) = P(c)$.

iii) If $R(x) = \frac{N(x)}{D(x)}$ is a rational function, then $\lim_{x \to c} R(x) = \frac{N(c)}{D(c)}$, ** as long as $D(c) \neq 0$.

iv) $\lim_{x \to c}[f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$.

v) If k is a constant, then $\lim_{x \to c} k \cdot f(x) = k \lim_{x \to c} f(x)$.

vi) $\lim_{x \to c}[f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$.

vii) $\lim_{x \to c}[f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$

viii) $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)}$.

Example: Evaluate the following limits.

i) $\lim_{x \to 2} -7$

ii) $\lim_{x \to -1} 3x^2 - x + 4$

Definition [Undefined form]: When evaluating $\lim_{x \to c} f(x)$, if substituting c into $f(x)$ gives you

$$f(c) = \frac{k}{0},$$

where k is some nonzero number, **then we say the limit does not exist**.

Example: Evaluate the limit or state that it does not exist.

i) $\lim_{x \to 3} \frac{x+1}{(x-1)^2}$

ii) $\lim_{x \to 1} \frac{x+1}{(x-1)^2}$

Definition [Indeterminate form]: When evaluating $\lim_{x \to c} f(x)$, if substituting c into $f(x)$ gives you

$$f(c) = \frac{0}{0},$$

then this is an **indeterminate form**, and you must find a common factor in the numerator and denominator, and cancel them out, then try substituting c into the simplified version of the function.

Example: Evaluate the limit or state that it does not exist.

i) $\lim_{x \to 4} \frac{(x+2)(x-4)}{x-4}$

ii) $\lim_{x \to 6} \frac{(x)(x-6)(x+1)}{x^2-6x}$

iii) $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$

iv) $\lim_{x \to 1} \frac{x^2-1}{(x-1)(x+2)}$

Example: Find the limits

i) $\lim_{x\to\infty} x =$ $\qquad$ $\lim_{x\to\infty} x^3 =$ $\qquad$ $\lim_{x\to\infty} \frac{x^4}{x^2} =$

ii) $\lim_{x\to\infty} \frac{1}{x} =$ $\qquad$ $\lim_{x\to\infty} \frac{1}{x^3} =$ $\qquad$ $\lim_{x\to\infty} \frac{x^2}{x^7} =$

*Notice a pattern? Good!

Theorem : If $\lim_{x\to\infty} \frac{N(x)}{D(x)} = \frac{\infty}{\infty}$, then there are three possible cases.

i) If degree $N(x) <$ degree $D(x)$, then the limit equals 0.

ii) If degree $N(x) =$ degree $D(x)$, then the limit equals the ratio of the coefficients of the dominant terms.*

iii) If degree $N(x) >$ degree $D(x)$, then the limit equals infinity.**

*the dominant term of a polynomial is the term with the largest power (i.e., the coefficient of the dominant term in $f(x) = 2x^2 + 3x + 1$ is 2). **∞ suffices in business applications, however more general applications could result in -∞

Example: Find the limit as x approaches infinity.

i) $\lim_{x\to\infty} \frac{4x^4+3x-7}{9x^3-x^2+4x+1}$

ii) $\lim_{x\to\infty} \frac{x^2+4x^3-1}{5x^3+x-9}$

iii) $\lim_{x\to\infty} \frac{2x^2+3x-1}{4x^3-2x^2+12}$

1.4 Problems: 3, 4, 5, 7, 8, 9, 23, 24, 30, 31, 34, 38, 39, 44, 48, 51

1.5 Functions and Continuity

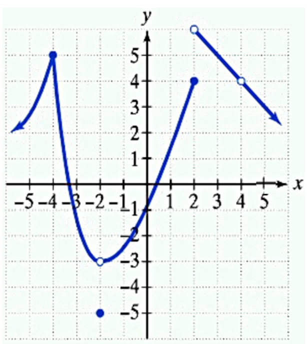
Definition [Continuity Conditions]: If $f(x)$ is a function, then $f(x)$ is continuous at $x = a$ if all three of the following conditions are satisfied.

i) $f(a)$ is defined.

ii) $\lim_{x \to a} f(x)$ exists.

iii) $\lim_{x \to a} f(x) = f(a)$.

Example: Is the graphed function continuous at $x = -4$? $x = -2$? $x = 2$? $x = 3$? $x = 4$? If not, explain why.



Definition [Continuity of Polynomial and Rational Functions]:

i) Polynomial functions,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

are continuous for all $x \in \mathbb{R}$.

ii) Rational functions

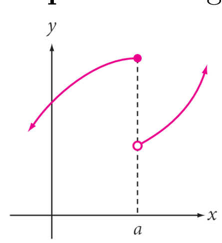
$$f(x) = \frac{h(x)}{g(x)}$$

are continuous everywhere except x values that make the denominator zero (i.e. for every x value in the function's domain).

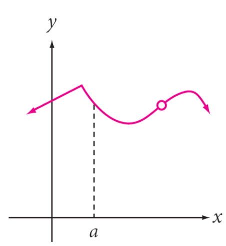
Example: For which values of x is $f(x) = 3x^2 + x - 1$ continuous?

Example: For which values of x is $f(x) = \frac{x^2+1}{(x-2)(x+1)}$ continuous?

Example: Is the graphed function continuous at $x = a$?



Example: Is the graphed function continuous at $x = a$?



Example: Determine if $f(x)$ is continuous at $x = 5$ where

$$f(x) = \begin{cases} 2x - 1, & x < 5 \\ 14 - x, & x \geq 5 \end{cases}$$

Example: Determine if $f(x)$ is continuous at $x = -1$ where

$$f(x) = \begin{cases} -x^2 - 2, & x < -1 \\ 3x - 1, & x \geq -1 \end{cases}$$

Example: Determine if $f(x)$ is continuous at $x = 3$ where

$$f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3}, & x \neq 3 \\ 7, & x = 3 \end{cases}$$

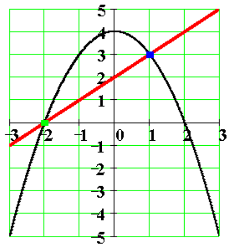
1.5 Problems: 2, 3, 5, 8, 11, 12, 16, 19, 23, 24, 28, 29

1.6 Average and Instantaneous Rates of Change

Definition [Average Rate of Change of a Function]: $f(x)$ over the interval $[x_1, x_2]$.

$$\text{Average rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example: Find the average rate of $f(x) = -x^2 + 4$ over the interval $[-2, 1]$



Example: If Dallas is 110 miles away and it took me 90 minutes to get there, what is the average rate of change of my position? What is another name for the rate of change of your position?

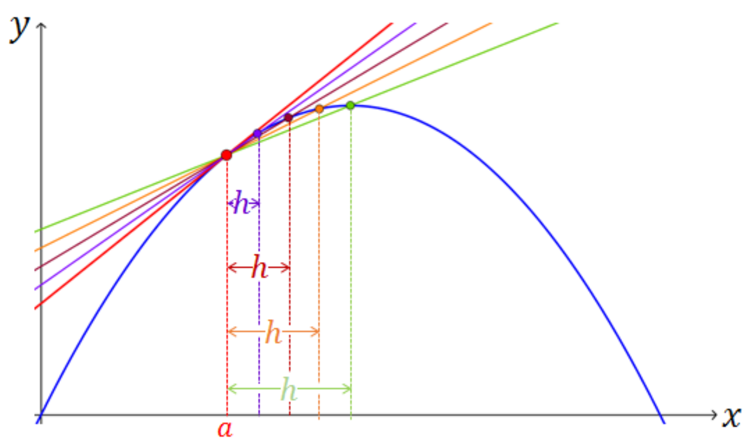
Definition [Difference Quotient]: Consider the average rate of change of $f(x)$ over the interval $[x, x + h]$. Then

$$\text{Average rate of change} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$

Example: Find and simplify the difference quotient for $f(x) = 7x + 2$.

Example: Find and simplify the difference quotient for $f(x) = \frac{3}{x}$.

The original average rate of change formula is much easier to look at, but the reason we have the difference quotient is so that we can let h become arbitrarily small, and we can find the **"instantaneous rate of change"** or the **slope of $f(x)$ at any point** x.



Definition [Instantaneous Rate of Change]: Consider taking the average rate of change of $f(x)$ over the interval $[x_1, x_2]$ as x_2 moves extremely close to x_1, (i.e. $h \to 0$). Then the limit of the difference quotient gives the slope at any point x.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: Find the instantaneous rate of change of $f(x) = \frac{3}{x}$ at $x = 1$ and $x = 4$.

Example: Find the instantaneous rate of change of $f(x) = 3x^2 - 1$ at $x = -1$ and $x = 3$.

1.6 Problems: 4, 5, 6, 7, 10, 11, 12, 13, 18, 19, 23, 24, 33, 38