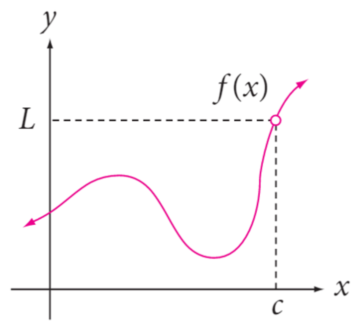
Introduction to Limits

Definition [Limit Notation]: The limit of $f(x)$, as x approaches c, is L, can be written as

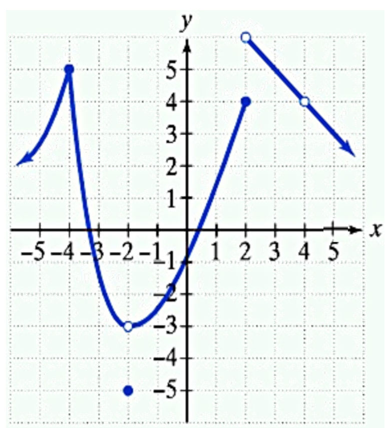
$$\lim_{x \to c} f(x) = L,$$

meaning $f(x)$ gets really close to L as x gets sufficiently close to c.



Theorem [Limits]: $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$. That is, $f(x)$ approaches L from **both** the left side ($x \to c^-$) and the right side ($x \to c^+$).

Example: Use the graph of $f(x)$ below to find the following



i) $\lim_{x \to -4^-} f(x) =$ $\quad \lim_{x \to -4^+} f(x) =$ $\quad \lim_{x \to -4} f(x) =$ $\quad f(-4) =$

ii) $\lim_{x \to -2^-} f(x) =$ $\quad \lim_{x \to -2^+} f(x) =$ $\quad \lim_{x \to -2} f(x) =$ $\quad f(-2) =$

iii) $\lim_{x \to 2^-} f(x) =$ $\quad \lim_{x \to 2^+} f(x) =$ $\quad \lim_{x \to 2} f(x) =$ $\quad f(2) =$

iv) $\lim_{x \to 4^-} f(x) =$ $\quad \lim_{x \to 4^+} f(x) =$ $\quad \lim_{x \to 4} f(x) =$ $\quad f(4) =$

Properties of Limits:

i) If k is a constant, then $\lim_{x\to c} k = k$.

ii) If $P(x)$ is a polynomial function, then $\lim_{x\to c} P(x) = P(c)$.

iii) If $R(x) = \frac{N(x)}{D(x)}$ is a rational function, then $\lim_{x\to c} R(x) = \frac{N(c)}{D(c)}$, ** as long as $D(c) \neq 0$.

iv) $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$.

v) If k is a constant, then $\lim_{x\to c} k \cdot f(x) = k \lim_{x\to c} f(x)$.

vi) $\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$.

vii) $\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$

viii) $\lim_{x\to c} \sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)}$.

Example: Evaluate the following limits.

i) $\lim_{x\to 2} -7$

ii) $\lim_{x\to -1} 3x^2 - x + 4$

Definition [Undefined form]: When evaluating $\lim_{x\to c} f(x)$, if substituting c into $f(x)$ gives you

$$f(c) = \frac{k}{0},$$

where k is some nonzero number, **then we say the limit does not exist**.

Example: Evaluate the limit or state that it does not exist.

i) $\lim_{x\to 3} \frac{x+1}{(x-1)^2}$

ii) $\lim_{x\to 1} \frac{x+1}{(x-1)^2}$

Definition [Indeterminate form]: When evaluating $\lim_{x\to c} f(x)$, if substituting c into $f(x)$ gives you

$$f(c) = \frac{0}{0},$$

then this is an **indeterminate form**, and you must find a common factor in the numerator and denominator, and cancel them out, then try substituting c into the simplified version of the function.

Example: Evaluate the limit or state that it does not exist.

i) $\lim_{x\to 4} \frac{(x+2)(x-4)}{x-4}$

ii) $\lim_{x\to 6} \frac{(x)(x-6)(x+1)}{x^2-6x}$

iii) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

iv) $\lim_{x\to 1} \frac{x^2-1}{(x-1)(x+2)}$

Example: Find the limits

i) $\lim_{x\to\infty} x =$ $\qquad$ $\lim_{x\to\infty} x^3 =$ $\qquad$ $\lim_{x\to\infty} \frac{x^4}{x^2} =$

ii) $\lim_{x\to\infty} \frac{1}{x} =$ $\qquad$ $\lim_{x\to\infty} \frac{1}{x^3} =$ $\qquad$ $\lim_{x\to\infty} \frac{x^2}{x^7} =$

*Notice a pattern? Good!

> **Theorem :** If $\lim_{x\to\infty} \frac{N(x)}{D(x)} = \frac{\infty}{\infty}$, then there are three possible cases.
>
> i) If degree $N(x) <$ degree $D(x)$, then the limit equals 0.
>
> ii) If degree $N(x) =$ degree $D(x)$, then the limit equals the ratio of the coefficients of the dominant terms.*
>
> iii) If degree $N(x) >$ degree $D(x)$, then the limit equals infinity.
>
> *the dominant term of a polynomial is the term with the largest power (i.e., the coefficient of the dominant term in $f(x) = 2x^2 + 3x + 1$ is 2).

Example: Find the limit as x approaches infinity.

i) $\lim_{x\to\infty} \frac{4x^4+3x-7}{9x^3-x^2+4x+1}$

ii) $\lim_{x\to\infty} \frac{x^2+4x^3-1}{5x^3+x-9}$

iii) $\lim_{x\to\infty} \frac{2x^2+3x-1}{4x^3-2x^2+12}$

Problems: Homework Quiz 11

Functions and Continuity

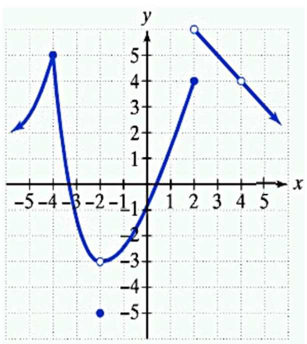
Definition [Continuity Conditions]: If $f(x)$ is a function, then $f(x)$ is continuous at $x = a$ if all three of the following conditions are satisfied.

i) $f(a)$ is defined.

ii) $\lim_{x \to a} f(x)$ exists.

iii) $\lim_{x \to a} f(x) = f(a)$.

Example: Is the graphed function continuous at $x = -4$? $x = -2$? $x = 2$? $x = 3$? $x = 4$? If not, explain why.



Definition [Continuity of Polynomial and Rational Functions]:

i) Polynomial functions,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

are continuous for all $x \in \mathbb{R}$.

ii) Rational functions

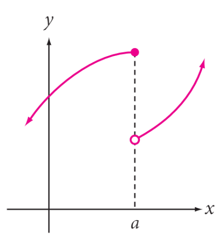
$$f(x) = \frac{h(x)}{g(x)}$$

are continuous everywhere except x values that make the denominator zero (i.e. for every x value in the function's domain).

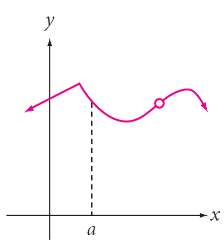
Example: For which values of x is $f(x) = 3x^2 + x - 1$ continuous?

Example: For which values of x is $f(x) = \frac{x^2+1}{(x-2)(x+1)}$ continuous?

Example: Is the graphed function continuous at $x = a$?



Example: Is the graphed function continuous at $x = a$?



Example: Determine if $f(x)$ is continuous at $x = 5$ where

$$f(x) = \begin{cases} 2x - 1, & x < 5 \\ 14 - x, & x \geq 5 \end{cases}$$

Example: Determine if $f(x)$ is continuous at $x = -1$ where

$$f(x) = \begin{cases} -x^2 - 2, & x < -1 \\ 3x - 1, & x \geq -1 \end{cases}$$

Example: Determine if $f(x)$ is continuous at $x = 3$ where

$$f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3}, & x \neq 3 \\ 7, & x = 3 \end{cases}$$

Problems: Homework Quiz 12

Instantaneous Rates of Change

Remember, we have the following two definitions that we will use to build the next idea.

Definition [Average Rate of Change of a Function]: $f(x)$ over the interval $[x_1, x_2]$.

$$\text{Average rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Definition [Difference Quotient]: Consider the average rate of change of $f(x)$ over the interval $[x, x+h]$. Then

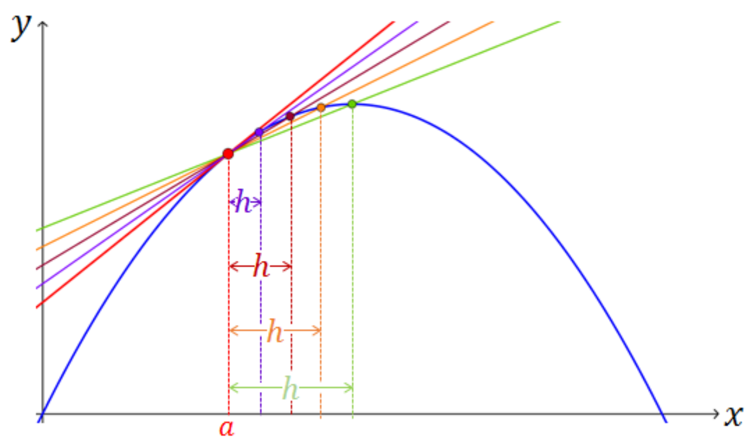
$$\text{Average rate of change} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$

Definition [Instantaneous Rate of Change or Derivative]: Consider taking the average rate of change of $f(x)$ over the interval $[x_1, x_2]$ as x_2 moves extremely close to x_1, (i.e. $h \to 0$). Then the limit of the difference quotient gives the slope at any point x.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*Note that direct substitution leads to $\frac{0}{0}$, so we need to simplify first. Further, the difference between this and average rate of change is this asks rate of change at one particular point.

The original average rate of change formula is much easier to look at, but the reason we have the difference quotient is so that we can let h become arbitrarily small, and we can find the **"instantaneous rate of change"** or the **slope of the tangent line at x**.



Example: Find the derivative of $f(x) = 5$

Example: Find the derivative of $f(x) = 3x - 4$

Example: Find the derivative of $f(x) = x^2 - 2x + 1$

Example: Find the derivative of $f(x) = \frac{3}{x}$ at $x = 1$ and $x = 4$.

Example: Find the derivative of $f(x) = 3x^2 - 1$ at $x = -1$ and $x = 3$.

Problems: Homework Quiz 12