2.1A: The Derivative of a Function

Definition [Limit Definition of the Derivative]: The derivative of the function $f(x)$, denoted $f'(x)$ (and read as "f prime of x"), is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists. If the limit exists, we also might say the function is "differentiable." The derivative is "derived" from the function $f(x)$ and describes:

- the instantaneous rate of change of $f(x)$
- the slope of the tangent line to $f(x)$

Derivative Notations For a function $f(x) = y$, the following are equivalent ways of representing the derivative of $f(x)$:

$$f'(x) \qquad y' \qquad \frac{df}{dx} \qquad \frac{dy}{dx} \qquad \frac{d}{dx}f \qquad \frac{d}{dx}y$$

To indicate the value of the derivative at $x = a$, we can write

$$f'(a) \qquad y'|_{x=a} \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

Example: Use the limit definition to find the derivative of each function. Then interpret the derivative.

i) $f(x) = 3x - 7$

ii) Find $f(2)$ and $f'(2)$ from part i). Interpret the meaning of each.

iii) $f(x) = 4x^2 - 5x + 1$

iv) Find $f(3)$ and $f'(3)$ from part iii). Interpret the meaning of each.

v) $f(x) = \frac{9}{x}$

vi) Find $f(1)$ and $f'(1)$ from part v). Interpret the meaning of each.

2.1A Problems: 1, 2, 3, 4, 5, 6

2.1B: The Derivative of a Function

Fundamental Rules of Differentiation

1) **Derivative of a Constant Function**
   If $y = c$ where c is any real number, then $\frac{dy}{dx} = 0$.
2) **Derivative of a Simple Power**
   If $y = x^n$, where n is any real number, then $\frac{dy}{dx} = n \cdot x^{n-1}$.
3) **Derivative of y=x**
   If $y = x$, then $y' = 1$.
4) **Derivative of a Constant Times a Function**
   If $f(x)$ is a differentiable function and c is a constant, then $\frac{d}{dx}cf(x) = c \cdot \frac{d}{dx}f(x)$.
5) **Derivative of y=cx**
   If c is a constant and $y = cx$, then $y' = c$.
6) **Derivative of a Sum or a Difference**
   If $f(x)$ and $g(x)$ are both differentiable functions, then

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

Example: Find the derivative of each function.

i) $y = 1239742$

ii) $f(x) = x^6$

iii) $f(x) = \sqrt[3]{x^4}$

iv) $g(x) = x^{-3}$

v) $y = \frac{1}{x^7}$

vi) $f(x) = \frac{2}{x^2}$

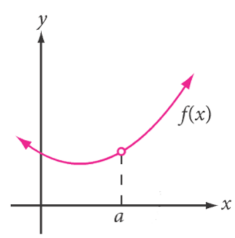
vii) $g(x) = 3x^5$

viii) $f(x) = \frac{1}{x} + 3x^2 - 14x + 5$

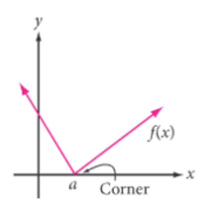
Example: Let $f(x) = x^2 - 3x + 5$. Find the x-values for which $f'(x) = 0$. What can you say about the graph of $f(x)$ at the points where $f'(x) = 0$?

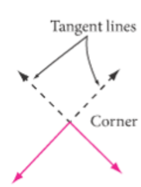
When is a function NOT differentiable?

A function $f(x)$ will not be differentiable at $x = a$ if:

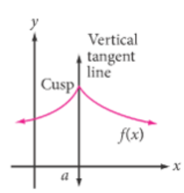


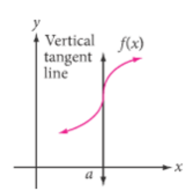
1) $f(x)$ is not continuous at $x = a$





2) $f(x)$ has a corner at $x = a$





3) $f(x)$ has a vertical tangent line at $x = a$.

Definition [Marginal Change]: For a revenue, cost, or profit function, we often want to know how these quantities will change with the increase or decrease of one item. When these functions are linear, we know exactly the rate of change: it is called the slope and it tells us how much the function increases/decreases per unit. When functions are nonlinear, we could find the actual difference to increase from, say x to $x+1$ items, by $f(x+1)-f(x)$. But we can also approximate the difference by finding the instantaneous rate of change, $f'(x)$. This approximation is the marginal change in revenue/cost/profit. **Note:** The closer the function is to a straight line, the better the approximation will be.

Example: Your lemonade stand can predict its profit (in cents) by the function $P(x) = x^2 - 15$, where x is the number of cups. If you have already sold 10 cups of lemonade, approximately how much should your profit change by selling one more cup? What is the actual change in profit?

Example: Biologists have determined that the relationship between heart rate h (in beats per minute) and body weight w (in pounds) is approximated by the function

$$h(w) = \frac{250}{x^{1/4}}.$$

Find and interpret $h'(w)$ for an 81-pound child.

2.1B Problems: 7, 8, 11, 14, 16, 17, 18, 19, 20, 22, 23, 25, 26, 36, 38, 41

2.2: Differentiating Products and Quotients

The Product Rule
If $u(x)$ and $v(x)$ are differentiable functions, then

$$\frac{d}{dx}[u(x) \cdot v(x)] = \frac{d}{dx}[u(x)] \cdot v(x) + u(x) \cdot \frac{d}{dx}[v(x)]$$

or

$$[u(x) \cdot v(x)]' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Example: Find the derivative of $y = x^3 \cdot x^6$ two different ways.

Example: Find the derivative of $f(x) = (2x - 1)(3x^2 + 5)$.

Example: Find and interpret the derivative of the cost function $C(x) = (5x^3 + 2x)(4x - 1)$ at $x = 10$ units sold.

Example: For the function $g(x) = (x^2 - 9)(x^2 + 4)$, find

i) The values of x for which $g(x) = 0$

ii) The values of x for which $g'(x) = 0$

iii) What do we expect to see on the graph of $g(x)$ at each of these values?

The Quotient Rule
If $u(x)$ and $v(x)$ are differentiable functions and $v(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x) \cdot \frac{d}{dx}[u(x)] - u(x) \cdot \frac{d}{dx}[v(x)]}{[v(x)]^2}$$

or

$$\left[\frac{u(x)}{v(x)}\right]' = \frac{v(x) \cdot [u(x)]' - u(x) \cdot [v(x)]'}{[v(x)]^2}$$

Note: $\left[\frac{u(x)}{v(x)}\right]' \neq \frac{u'(x)}{v'(x)}$!

Example: Find the derivative of $f(x) = \frac{x^2}{x^4}$ two ways.

Example: Find the derivative of $h(x) = \frac{x^3 - 4x + 1}{x^2 + 2}$.

Example: Find the derivative of $y = \frac{2x-5}{6x^3+x-7}$.

Example: The formula $N(t) = \frac{80t}{105t-80}$ was developed by psychologist L.L. Thurstone and relates the number N of facts that a person can remember t hours after memorizing them. Find and interpret $N(1)$, $N'(1)$, $N(3)$, and $N'(3)$.

Example: Economists have established that the demand D for an item decreases as the price x increases. The daily number D of tubes of roofing tar that people are willing to buy at x cents each is given by

$$D(x) = \frac{85,000}{\sqrt{x} + 10}, \quad 200 \leq x \leq 700$$

If the current price per tube is \$2.50, find and interpret the marginal demand for tubes of roofing tar.

2.2 Problems: 2, 3, 5, 6, 9, 16, 17, 20, 22, 23, 27, 31, 32, 34, 36, 42, 43

2.3: Higher Order Derivatives

Higher Order Derivatives:
Given a differentiable function f, we can find its derivative f'. Since f' is also a function, we may be able to find its derivative, f'' (read "f double prime"). This is the second derivative of f. As long as the derivatives exist, we can continue finding the third derivative of f, the fourth derivative of f, etc.

Notation for Higher Order Derivatives:

	1st Derivative	2nd Derivative	3rd Derivative	4th Derivative	...	n-th Derivative
f	f'	f''	f'''	$f^{(4)}$	...	$f^{(n)}$
f	$\frac{df}{dx}$	$\frac{d^2f}{dx^2}$	$\frac{d^3f}{dx^3}$	$\frac{d^4f}{dx^4}$	...	$\frac{d^nf}{dx^n}$
y	y'	y''	y'''	$y^{(4)}$	...	$y^{(n)}$
y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^4y}{dx^4}$	...	$\frac{d^ny}{dx^n}$

Example: Find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$ for $f(x) = 3x^4 - x^2 + 5x - 1$.

Example: Find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$ for $f(x) = x^7 - 5x^3 + 3\sqrt{x}$.

Example: Find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$ for $f(x) = \frac{4x^2+5x-10}{x}$.

Example: Let's say you are driving your car and the function $f(x)$ tells us the distance you are from your house after x minutes.

- What does the derivative of f represent?

- What does it mean if this is positive/negative?

- What does the second derivative of f represent?

- What does it mean if this is positive/negative?

Interpreting Higher Order Derivatives:
Just as the first derivative f' describes the rate of change of f, the second derivative f'' describes the rate of change of f'.

Sign of 1st Derivative	Sign of 2nd Derivative	Interpretation
$f' > 0$	$f'' > 0$	f is increasing at an increasing rate
$f' < 0$	$f'' < 0$	f is decreasing at an increasing rate
$f' > 0$	$f'' < 0$	f is increasing at a decreasing rate
$f' < 0$	$f'' > 0$	f is decreasing at a decreasing rate

Example: Suppose that $S(r)$ represents a company's monthly sale volume (in millions of dollars) when r (in millions of dollars) is spent on research and development. Interpret the information provided by $S'(1.6) = 5.8$ and $S''(1.6) > 0$.

Example: Suppose that $N(t)$ represents the number of people (in thousands) in a city who have heard of a new product being advertised on radio and television t days after the beginning of the advertisement. Interpret the information provided by $N'(21) = 0.6$ and $N''(21) < 0$.

Example: The function

$$P(x) = \frac{180x}{110 + x^2}$$

relates the weekly profit P (in thousands of dollars) a company makes to the number x (in thousands) of automobile seat covers it sells each week. In this case,

$$P'(x) = \frac{180(110 - x^2)}{(110 + x^2)^2} \text{ and } P''(x) = \frac{-360x(330 - x^2)}{(110 + x^2)^3}$$

What is the trend when the company sells 10,000 seat covers?

2.3 Problems: 1, 4, 5, 6, 9, 10, 13, 14, 15, 16, 20, 23, 28, 29, 31

2.4: The Chain Rule and General Power Rule

The Chain Rule

Suppose y is a function of u and u is a function of x. Then a change in x produces a change in u, which produces a change in y. We find these individual changes and multiply to determine how quickly y is changing relative to x, i.e.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Another way to say this is that if $y = (f \circ g)(x)$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Example: Suppose $y = u^3$ and $u = 3x - 1$. Find $\frac{dy}{dx}$.

The General Power Rule

If y is a differentiable function of u, and u is a differentiable function of x with $y = [u(x)]^n$, then

$$\frac{dy}{dx} = n[u(x)]^{n-1} \cdot \frac{du}{dx}$$

Note: This is a special case of the chain rule, where $f(u) = u^n$.

Example: Find the derivative of $y = (4x^3 + 5x + 2)^6$.

Example: Find the derivative of $f(x) = \sqrt[5]{(4x+5)^3}$.

Example: Find the derivative of $g(x) = 3x^2(6x - 8)^{1/6}$.

Bypassing the Quotient Rule
If $f(x) = \frac{N(x)}{D(x)}$, we can use the quotient rule to find the derivative, or we can find the derivative by rewriting $f(x) = N(x) \cdot (D(x))^{-1}$ and taking the derivative using the product, power, and chain rules combined.

Example: For $f(x) = \frac{(x^2-6)^4}{x^2-3}$, find $f'(x)$ without using the quotient rule.

Example: For $g(x) = \frac{(3x-1)^5}{x+8}$, find $g'(x)$ without using the quotient rule.

Example: Demand for a product depends on the price of the product by

$$x(p) = 420p^{-2}$$

where x is the number of units demanded and p is the price of the product in dollars. The cost per unit sold depends on the number produced as

$$c(x) = 20 + 10x^{-3}$$

where c is the cost per unit in dollars.

i) Write a symbolic description that shows how the cost of producing a unit changes with the sales price.

ii) How does the cost of production change as the price increases from \$50 to \$51 per unit?

2.4 Problems: 5, 8, 9, 10, 11, 14, 17, 19, 21, 22, 25, 27, 28, 43, 46, 48, 49

2.5: Implicit Differentiation

Implicit vs. Explicit Functions
All of the functions we have looked at so far are *explicit functions*, where the output variable (often y) is isolated on one side of the equation, e.g. $y = x^3 - x$. Here, the relationship between the variables x and y is stated directly (explicitly).
An *implicit function* is one in which the relationship between the variables is stated, but less directly. This makes it more work (and sometimes impossible!) to isolate the output variable, e.g. $y^3 - x^2y^4 + x = 17$.

Implicit Power Rule
Let $y = f(x)$ be an implicit function of x. Then for $y^n = [f(x)]^n$, by the general power rule, we have

$$\frac{d}{dx}y^n = ny^{n-1}y' \quad \text{or} \quad \frac{d}{dx}[f(x)]^n = n \cdot [f(x)]^{n-1} \cdot f'(x)$$

Implicit Differentation
To differentiate implicitly:

1. Differentiate both sides of the equation with respect to the appropriate variable (often x)
2. Apply the rules for differentiation (Power, Product, Quotient, and Chain) as necessary. Any time an expression involving y is differentiated, y' should be a factor in the result.
3. Solve for y'. (This will often involve rearranging terms and factoring out y' to isolate it.)

Example: Find y' given that $y^2 - 3x^4 = 7$.

Example: Find $\frac{dy}{dx}$ given that $xy^2 + 3x^3y^3 = y - 1$.

Example: Find y' for $3x + xy^4 + y = 131$ at the point $(1, 2)$.

Example: Oil is leaking in a circular shape from a tanker in such a way that the radius of the circular spill is increasing at the rate of 2 feet per hour. To manage the spill, the oil company needs to know the rate at which the area covered by oil is increasing. Find the rate at which the area is increasing when the radius of the spill is 300 feet. (Hint: Use the formula for the area of a circle, $A = \pi r^2$.)

Example: Say the price p of donuts is related to the demand x by

$$4x + xp - 9p = 36$$

If donuts are currently selling for \$2 each and the number of sales is 12 per day, find the approximate change in sales if the price is raised by \$1.

Example: The monthly revenue R (in dollars) of a telephone polling service is related to the number x of completed responses by the function

$$R(x) = -12,000 + 25\sqrt{3.5x^2 + 25x} \quad 0 \leq x \leq 1,500$$

If the number of completed responses is increasing at the rate of 10 forms per month, find the rate at which the monthly revenue is changing per month when 750 responses have been completed.

2.5 Problems: 1, 5, 6, 11, 15, 18, 19, 21, 27, 28, 29, 32