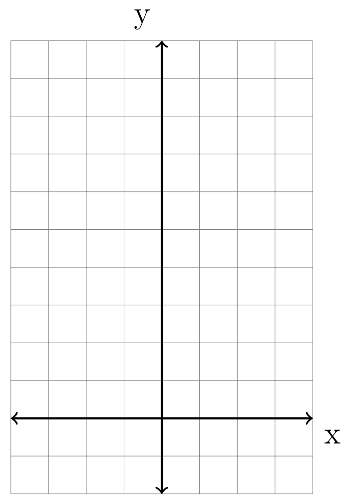
4.1: The Exponential Functions

Definition [Exponential Functions]: Exponential functions have the variable, x, in the exponent:
$f(x) = b^x$
The number that is raised to a power is called the **base**. In our class, the base will always be positive, and not equal to one (because $b = 1$ isn't very interesting!!!).

Let's look at $f(x) = b^x$ where $b > 1$. When $b > 1$, this is an exponential *growth* function.

Graph $f(x) = 2^x$.

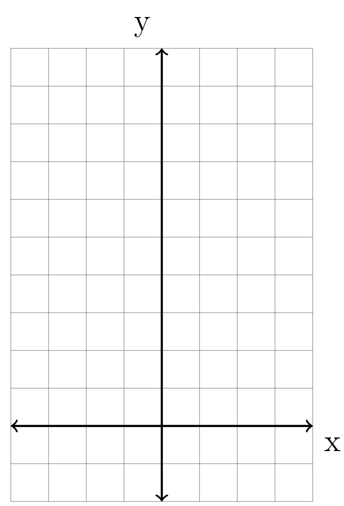


x	y
-2	
-1	
0	
1	
2	
3	

Find $\lim_{x\to\infty} 2^x =$ and $\lim_{x\to-\infty} 2^x =$.

Now let's look at $f(x) = b^x$ where $0 < b < 1$. When $0 < b < 1$, this is an exponential *decay* function.

Graph $f(x) = \left(\frac{1}{2}\right)^x$.



x	y
-2	
-1	
0	
1	
2	
3	

Find $\lim_{x\to\infty} \left(\frac{1}{2}\right)^x =$ and $\lim_{x\to-\infty} \left(\frac{1}{2}\right)^x =$.

Compound Interest:

If you invest P dollars in an account with rate r that is compounded n times per year, after t years the amount in the account, $A(t)$, is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example: Say you have \$1000 to invest in a savings account. If the annual interest rate is 5%, find the amount in your account after four years if the interest is compounded

a) annually.

b) quarterly.

c) monthly.

d) daily.

Question: What if we compounded the interest as often as possible? In other words, what if your interest was *continuously* compounding?

Remember from last semester: This was answered by defining the number e and writing a formula for continuous compound interest using e.

Definition [The number e]: The number e is an irrational number defined by

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \approx 2.718281828459045...$$

Natural Exponential Function

A lot of growth or decay in business and in life/social sciences can be represented by exponential growth with base e to represent *continuous* growth, A_0 the initial amount, k the growth constant, t the time, and $f(t)$ the amount after some time:

$$f(t) = A_0 e^{kt} \text{ or } f(t) = A_0 e^{-kt}$$

You may have seen this equation for continuous growth in the business/financial setting, written as

$$A = Pe^{rt}$$

where P represents the principal (initial) amount of money invested, r is the interest rate, t is time, and A is the amount obtained by the investment.

Example: Say you have $1000 to invest in a savings account. If the annual interest rate is 5%, find the amount in your account after four years if the interest is compounded continuously.

Other examples of usefulness:

Exponential Function Properties

Exponential functions have the following important properties:

1) The domains are all real numbers, i.e. can be calculated for any value of x.

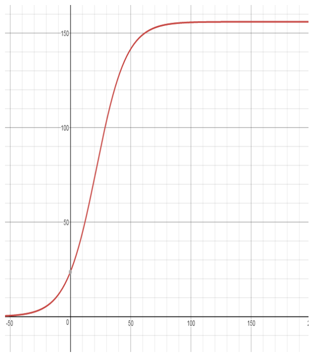
2) The range of each is all positive real numbers, i.e. never 0 or negative.

3) They are continuous.

4) Model either growth or decay.

Logistic Growth Function
A variation on this natural growth function is the **logistic growth function**,

$$f(t) = \frac{M}{1 + Ae^{-kt}}$$



This function models *restricted growth*; in other words, the growth is continuous as before, but there is some upper limit to how much it can grow. This is found by taking the limit of f as t goes to infinity.

Example: Demand for a product depends on the price by

$$D(p) = 500 + \frac{1000}{1 + e^{-0.05p}}$$

where D is the quantity demanded and p is the price in dollars.

a) How many units of the product will be sold when the price is \$50?

b) As the price raises, what level of sales will the demand stabilize at?

4.1 Problems: 17, 18, 33, 38, 39, 42, 45, 49

4.2: The Natural Logarithm Function

The Natural Logarithm Function
If $a = e^t$, then we say that t is the natural logarithm of a. So

$$t = \ln(a) \text{ if and only if } a = e^t$$

Notice that since $e^t > 0$ for all t, we have that $a > 0$ with $a = e^t$.
This means that $\ln(a)$ is only defined for $a > 0$.

Example: Convert each logarithmic equation to exponential form.

a) $2 = \ln(6x)$

b) $x + 1 = \ln(y - 3)$

c) $-3 = \ln(x + 5)$

Example: Convert each exponential equation to logarithmic form.

a) $e^{3x} = 2$

b) $e^{x+y} = 2y$

c) $e^{-3+x} = 2x - 1$

Example: Find $\ln(e)$ and $\ln(1)$.

Properties of Logarithms

1. $\ln(e) = 1$ and $\ln(1) = 0$
2. $\ln(e^x) = x$
3. $\ln(xy) = \ln(x) + \ln(y)$
4. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
5. $\ln(x^r) = r \cdot \ln(x)$

Example: Expand the logarithmic expression so that each variable has an exponent of 1.

$$\ln \frac{2x^3}{y^2 z}$$

Example: Write the logarithmic expression as an expression with a single logarithm.

$$\ln(x) - 2\ln(y) + 10\ln(z)$$

Example: Solve the exponential equation.

$$4e^{3x-1} = 12$$

What is $\lim_{x\to\infty} \ln(x) =$?

Example: Find $\lim_{x\to\infty} \ln(3x + 4)$

Natural Logarithm Function Properties

Natural logarithm functions have the following important properties:

1) The domain is the set of all positive numbers. (Think inverse of exponential)

2) The range is all real numbers.

3) For $0 < x < 1$, $\ln(x) < 0$ and for $x > 1$, $\ln(x) > 0$

4) $\ln(1) = 0$

5) As x increases, $f(x) = \ln(x)$ increases but at a decreasing rate.

4.2 Problems: 3, 4, 5, 6, 7, 9, 10, 12, 14, 17, 21, 24, 25, 26, 27

4.3: Differentiating the Natural Logarithm Function

The Derivative of the Natural Logarithm Function
If $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$.

Chain Rule for Natural Logarithm Function
If $f(x) = \ln[g(x)]$, then $f'(x) = \frac{1}{g(x)} \cdot g'(x)$.

Example: Find the derivative of $y = \ln(4x)$.

Example: Find the derivative of $y = \ln(x^2 + 3)$.

Example: Find the derivative of $y = \ln((4x^2 + 10x)^7)$.

Example: Find the derivative of $y = 4x^2 \ln(3x - 1)$

Example: The daily cost (in hundreds of dollars) to produce x thousand pounds of cat food is given by

$$C(x) = 100 \ln \sqrt{4x^2 - 300}$$

If the manufacturer is currently producing 50 thousand pounds of cat food per day, how much can the cost be expected to change if production is increased to 51 thousand pounds per day?

4.3 Problems: 2, 5, 6, 9, 11, 12, 15, 43, 46, 47*, 48*

*Use what we have learned about first and second derivatives of functions to answer these questions.

4.4: Differentiating the Natural Exponential Function

The Derivative of the Exponential Function
If $f(x) = e^x$, then $f'(x) = e^x$. **Chain Rule for Exponential Function**

If $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)} \cdot g'(x)$.

Example: Find the derivative of $y = e^{10x+1}$.

Example: Find the second derivative of $y = e^{x^3+2x-1}$.

Example: Find the derivative of $y = e^{\sqrt{x^2+10}}$.

Example: Find the derivative of $y = x^4 e^{x^2}$.

Example: Find the derivative of $f(x) = \frac{e^x}{2e^x + e^{2x}}$.

Example: Find the derivative of $e^{e^{e^{x^2+1}}}$.

4.4 Problems: 1, 6, 9, 12, 13, 14, 15, 16, 19, 20, 25, 45, 46