4.1 Exponential Functions and 4.2 Applications

Properties of Exponential Functions: What is an exponential function?

Function where the **variable** is the exponent and the **base** is a positive constant. The simplest of these are of the form: $f(x) = a^x$, where $a > 0$

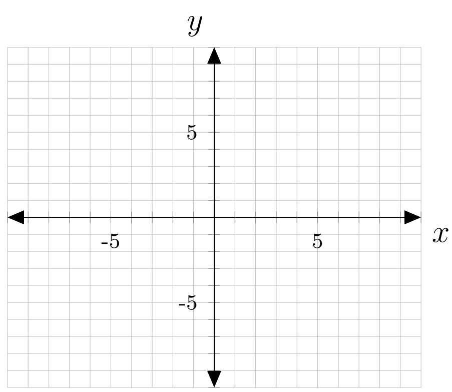
We will also consider what is arguably the most useful exponential function:

$$f(x) = e^x$$

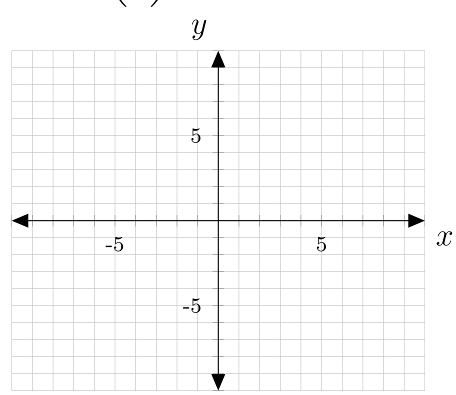
*Note: The **number *e*** is approximately 2.718281828459 (we will look at e in chapter 5)

Example: Graph each function:

i) $f(x) = 2^x$



ii) $f(x) = \left(\dfrac{3}{2}\right)^x$



Definition Exponential Growth: These are both examples of exponential growth. When $a > 1$, the function $f(x) = a^x$ has the set of all real numbers as its domain, the graph has the general shape above, and the following properties:

i) The graph is above the x-axis

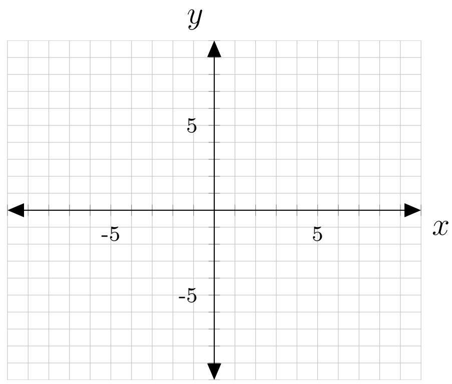
ii) The y-intercept is 1

iii) The graph climbs steeply to the right

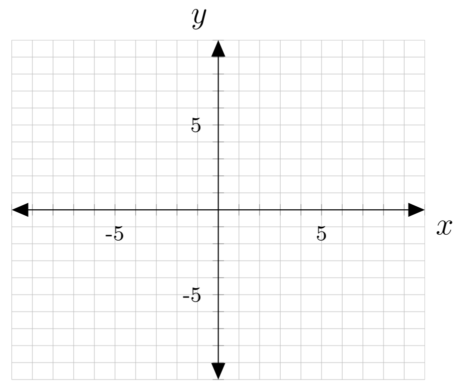
iv) The negative x-axis is a horizontal asymptote

v) The larger the base a, the more steeply the graph rises to the right

iii) $f(x) = 2^{-x}$



iv) $f(x) = \left(\dfrac{1}{10}\right)^x$



Definition Exponential Decay: These are both examples of exponential decay. When $0 < a < 1$, the function $f(x) = a^x$ has the set of all real numbers as its domain, the graph has the general shape above, and the following properties:

i) The graph is above the x-axis

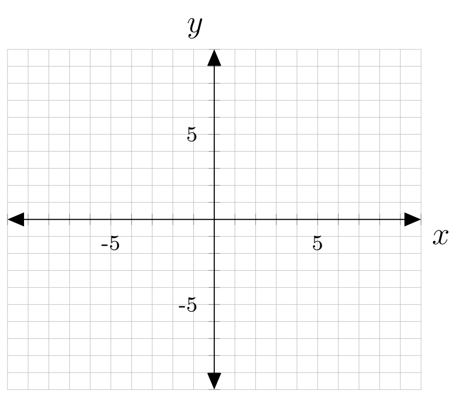
ii) The y-intercept is 1

iii) The graph falls sharply to the right

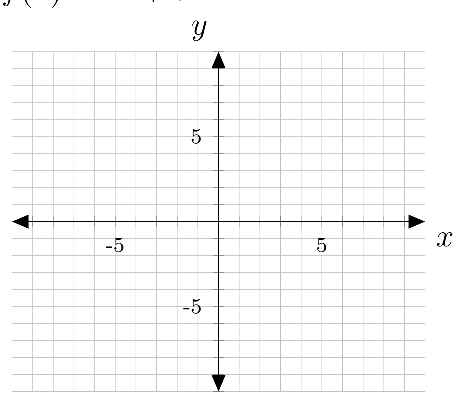
iv) The positive x-axis is a horizontal asymptote

v) The smaller the base a, the more sharply the graph falls to the right

v) $f(x) = 2^{x+6}$



vi) $f(x) = 2^x + 6$



Example: Give a rule of the form $f(x) = a^x$ for the exponential function whose graph contains the point $(-2, 121)$

Example: Give a rule of the form $f(x) = a^x$ for the exponential function whose graph contains the point $(4, 16)$

Types of Exponential Models:

$$f(t) = y_0 e^{kt} \qquad \text{or} \qquad f(t) = y_0 b^t$$

Where $f(t)$ is the amount present at time t, y_0 is the amount present at time $t = 0$, and k, b are constants that depend on growth.

Example: According to the Wall Street Journal OLED panels are increasingly being used for display screens on cell phones, digital cameras, and handheld TVs because they are thinner, provide crisper images, and use less energy than regular LED panels. The sales of OLED panels (in billions of dollars) are approximated by $f(x) = 0.07363e^{0.30447x}$ where $x = 6$ corresponds to the year 2006.

i) What were sales in 2000, the first year modeled?

ii) What were sales in 2006?

iii) When were sales approximately 3.86 billion dollars?

Example: When money is placed in a bank account that pays compound interest, the amount in the account grows exponentially. Suppose such an account grows from \$1000 to \$1316 in 7 years.

i) Find a growth function of the form $f(t) = y_0 b^t$ that gives the amount in the account at time t years.

ii) How much is in the account after 12 years?

Example: Sales of a new product often grow rapidly at first and then begin to level off with time. Suppose the annual sales of an inexpensive can opener are given by $S(x) = 10,000(1 - e^{-0.5x})$ where $x = 0$ corresponds to the time it went on the market.

i) What are the initial sales? Does this make sense?

ii) What are the sales in each of the first 3 years? What were the sales in the tenth year?

iii) What is the maximum the sales could possibly reach?

iv) What is the first full year when sales exceed 10,000?

4.1 Problems: 2, 3, 5, 10, 11, 17, 21-27, 29, 30, 35, 36, 37, 39, 41, 42, 43, 46, 53

4.2 Problems: 1, 3, 7, 8, 11*, 12*, 13, 22 (*without regression)

4.3 Logarithmic Functions

The Logarithm Function

If $x = 10^y$, then we say that y is the logarithm of x. So

$$y = \log(x) \text{ if and only if } x = 10^y$$

If $x = e^y$, then we say that y is the natural logarithm of x. So

$$y = \ln(x) \text{ if and only if } x = e^y$$

If $x = a^y$, then we say that y is the logarithm base a of x. So

$$y = \log_a(x) \text{ if and only if } x = a^y$$

Notice that since $a^y > 0$ for all y, we have that $x > 0$ with $x = e^y$.

This means that the logarithm functions are only defined for $x > 0$.

Exponential Form	Logarithmic Form

Properties of Logarithms

Let x and a be positive real numbers, $a \neq 1$, and r any real number. Then we have:

i) $\log_a(1) = 0$ and $\ln(1) = 0$ since $a^0 = 1$

ii) $\log_a(a) = 1$ and $\ln(e) = 1$ since $a^1 = a$

iii) $\log_a(a^r) = r$ and $\ln(e^r) = r$ since $(a)^r = (a^r)$

iv) $a^{\log_a(x)} = x$ and $e^{\ln(x)} = x$

v) $\log_a(xy) = \log_a(x) + \log_a(y)$

vi) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

vii $\log_a(x^r) = r \cdot \log_a(x)$

Example: Evaluate the logarithmic expression without using a calculator:

i) $\log_2 16$

ii) $\log_3 1$

iii) $\log 1$

iv) $\log_2 \sqrt{2}$

iii) $\log_e \frac{1}{e^2}$

iv) $\log_{1/2} 8$

v) $\ln e^{14.2}$

vi) $\log 0.01$

Example: Convert to a logarithmic equation:

i) $3^3 = 27$

ii) $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

Example: Convert to an exponential equation:

i) $\log_6 36 = 2$

ii) $\log_{\sqrt{3}} 81 = 8$

Example: Write the expression as a single logarithm with coefficient of 1:

i) $\log_a x + \log_a y - \log_a m$

ii) $2\log_m a - 3\log_m b^2$

iii) $3\ln 3 - \dfrac{1}{2}\ln 36$

iv) $\dfrac{1}{3}\log 8 + \log 5 - 3\log 2$

Example: Write each expression as a sum and/or difference of logarithms, with all variables to the first degree:

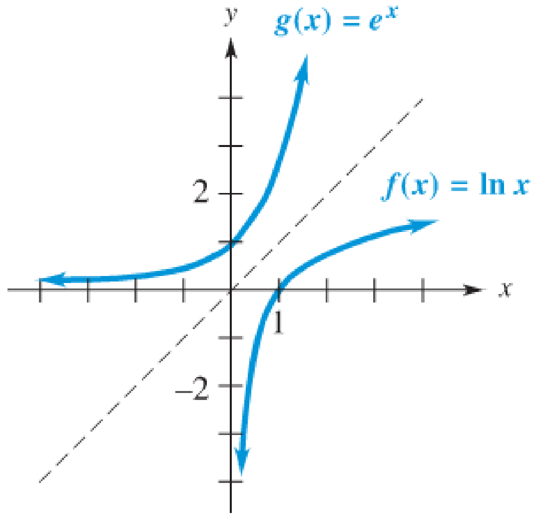
i) $\log_2 \dfrac{6x}{y}$

ii) $\log_5 \dfrac{5\sqrt{7}}{3}$

iii) $\log_m \sqrt{\dfrac{5r^3}{z^5}}$

iv) $\log_2 \dfrac{ab}{cd}$

*Note: Exponential and Logarithmic functions are actually inverses and their graphs are reflected over the line $y = x$. This means that we know basic points for logarithmic graphs.



Example: The life expectancy at birth of a person born in year x is approximated by the function $f(x) = 17.6 + 12.8 \ln x$, where $x = 10$ corresponds to 1910.

i) Find the life expectancy of a person born in 1910, 1960, and 2010.

ii) Graph f(x) on a graphing calculator with viewing window [-10, 140] x [-10, 100] and sketch.

iii) What does the shape of the graph suggest about a persons life expectancy at birth?

4.3 Problems: 1, 2, 6, 7, 9, 10, 13, 14, 17, 20, 21, 22, 31, 32, 58, 59, 60, 61, 63

4.4 Logarithmic and Exponential Equations

*Note: If $\log_a u = \log_a v$ then $u = v$, and if $a^u = a^v$ then $u = v$

Example: Solve each logarithmic equation. Express irrational solutions correct to the nearest thousandth:

i) $\log_2(x+9) - \log_2 x = log_2(x+1)$

ii) $\log_5(4x) = \log_5(x+3) + log_5(x-1)$

iii) $\log_6(2x+4) = 2$

iv) $\log(m+25) = 1 + \log(2m-7)$

iii) $\ln(x+2) - 4 = -\ln(x-2)$

iv) $\ln(e^y) - 2\ln(e) = -\ln(e^4)$

Example: Solve each exponential equation. Express irrational solutions correct to the nearest thousandth:

i) $2^x = 7$

ii) $10^{3y-9} = 7$

iii) $3e^{x^2} = 1200$

iv) $5^{6x-3} = 2^{4x+1}$

Example: According to projections by the US Census Bureau, the world population (in billions) is approximated by the function $f(x) = 4.834(1.011)^x$, where $x = 4$ corresponds to the year 1984. When will the population reach 7 billion?

Example: In the central Sierra Nevada mountains of California, the percent of moisture that falls as snow rather than rain is approximated reasonably well by $p(h) = 86.3 \ln h - 680$, where p is the percent of moisture as snow at an altitude of h feet (with $3000 \leq h < 8500$). At what altitude is 50 percent of the moisture snow?

4.4 Problems: 1, 6-9, 12, 14, 23, 28, 32, 33, 40, 66, 73