5.1: Antidifferentiation and the Indefinite Integral

Definition [Antiderivative or Integral]: If $f'(x)$ is the derivative of $f(x)$, then $f(x)$ is the **indefinite integral** (or **antiderivative**) of $f'(x)$.

Example: If $f'(x) = 2x$, what could $f(x)$ be? Is there more than one possible $f(x)$ function?

Claim: If both $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then they differ by a constant.

Proof:

Definition [Indefinite Integral]: Suppose that $F(x)$ is any one of the many members of the family of antiderivatives of $f(x)$. Then,

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant (**constant of integration**). Here $f(x)$ is called the **integrand** and dx is the **differential**.

Example: Find the following indefinite integrals

i) $\int 8\ dx$

ii) $\int 4x\ dx$

iii) $\int 4x + 8\ dx$

iv) $\int 3x^2\ dx$

Properties of Integrals:

i) If k is a constant,

$$\int kf(x)\ dx = k \int f(x)\ dx$$

ii) If $f(x)$ and $g(x)$ are two funtions,

$$\int [f(x) \pm g(x)]\ \ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

Remark [Some Common Integrals]:

i) Integral of a Constant: If k is a constant,

$$\int k\ dx = kx + c$$

ii) Integral of a Power Function: For $n \neq 1$,

$$\int x^n\ dx = \frac{1}{n+1} x^{n+1} + c$$

iii) Integral of $\frac{1}{x}$:

$$\int \frac{1}{x}\ dx = \ln |x| + c$$

iv) Integral of the Exponential Function:

$$\int e^{ax}\ dx = \frac{1}{a} e^{ax} + c$$

Example: Find the indefinite integrals

i) $\int 3x\ dx$

ii) $\int 4x^6\ dx$

iii) $\int 12x^2 - 3x + 6\ dx$

iv) $\int \frac{9}{x}\ dx$

v) $\int 5e^{4x}\ dx$

vi) $\int 3e^{2x} + 5x^2\ dx$

vii) $\int 21x^2 + 68x - 14\ dx$

viii) $\int \frac{4}{x^2} - 3e^x\ dx$

ix) $\int \frac{5}{\sqrt{x}} + 3\ dx$

x) $\int 5e^{4x} + 2x^3 - \frac{5}{x^{1/3}}\ dx$

Remark : Sometimes an “initial condition” is given along with an indefinite integral, and this allows us to solve for the constant of integration, as seen below.

Example:

i) Find $f(x)$ if $f'(x) = 2x + 3$ and $f(1) = -2$.

ii) Find $f(x)$ if $f'(x) = 3e^{3x}$ and $f(2) = e^2 + 6$.

Example:

a) Consider the marginal cost function

$$C'(x) = -.04x + 26$$

where x represents the level of production. Find the cost function, $C(x)$, if at a production level of 200 units, the cost is known to be \$9,000.

b) Find and interpret $C'(100)$ and $C(100)$.

5.1 Problems: 1, 2, 3, 4, 5, 8, 12, 14, 16, 17, 18, 24, 25, 43, 48, 49

5.2: Integration by Substitution

Motivation: How would we integrate $\int \frac{x}{1+x^2}\, dx$?

Remark [Suggestions for Substitutions]: If the integrand involves

i) a rational function, $f(x) = \frac{g(x)}{h(x)}$, let $u = h(x)$.

ii) a quotient with a natural logarithm in numerator, $f(x) = \frac{\ln(h(x))}{g(x)}$, let $u = \ln(h(x))$.

iii) an exponential equation, $e^{f(x)}$, let $u = f(x)$.

iv) a power or radical, $(f(x))^3$ or $\sqrt{f(x)}$, let $u = f(x)$.

Example: Find the following integrals

i) $\int 3x^2(x^3+1)^{10}\, dx$

ii) $\int 2xe^{x^2-1}\, dx$

iii) $\int \frac{6x-2}{3x^2-2x}\, dx$

iv) $\int \frac{x}{1+x^2}\, dx$

v) $\int (4x+3)\sqrt{2x^2+3x}\ dx$

vi) $\int \frac{\ln(3x)+5}{x}\ dx$

vii) $\int (x+2)(x-2)^5\ dx$

viii) $\int 10(9x+1)(9x^2+2x+7)^8\ dx$

Example: Suppose that Max has been learning to ride a bike. The number of yards that Max can ride before crashing, N, depends on the number of days he has been practicing, t. The number of yards he can ride changes according to the function

$$N'(t) = 24.5e^{-.35t}$$

After practicing for 10 days, Max can ride 50 yards before he crashes.

i) Find the function, $N(t)$, that describes how many yards Max can ride in terms of t.

ii) Find the number of yards that Max could ride per day after practicing for 5 days.

5.2 Problems: 1, 4, 5, 6, 9, 12, 14, 17, 18, 20, 36, 39, 42, 45

5.3: The Definite Integral

So far: Going from "rate of change" function to "total" function
Now: Accumulation of totals

The Fundamental Theorem of Calculus: For a function $f(x)$, if $f'(x)$ is continuous on $[a, b]$, then the definite integral of $f'(x)$ from a to b is

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

and represents the total amount accumulated by $f(x)$ as x increases from the **lower limit of integration**, a, to the **upper limit of integration**, b.

Example: If $A'(t)$ represents the rate at which the amount of money in an account will change after t months, then $A(t)$ would represent the amount of money in the account after t months, so

$$\int_a^b A'(t)\ dt = A(b) - A(a)$$

represents the amount of money accumulated in the account from month a to month b.

i) If $P'(t)$ represents the rate at which a population of a city is changing, interpret

$$\int_a^b P'(t)\ dt.$$

ii) If $C'(x)$ is the marginal cost of producing x items, interpret

$$\int_a^b C'(x)\ dx.$$

Example: John's Jerky Stand finds that the marginal revenue, in dollars, from the sale of x pounds of jerky per week is given by $R'(x) = 150 + 2x$. Assume that when zero pounds of jerky are sold, the revenue is zero.

a) If John makes 300 lbs. of jerky per week, what is his revenue?

b) By how much will the revenue increase if John increases production to 500 pounds per week?

Example: Find the following definite integrals.

i) $\int_1^2 2x \, dx$

ii) $\int_1^2 9x^3 \, dx$

Properties of Definite Integrals:

i) $\displaystyle\int_a^b kf(x)\ dx = k\int_a^b f(x)dx$

ii) $\displaystyle\int_a^a f(x)\ dx = 0$

iii) $\displaystyle\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

iv) $\displaystyle\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

v) If $a \leq c \leq b$, $\displaystyle\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

Example: Find the following definite integrals.

i) $\displaystyle\int_1^2 x^3 + 3\ dx$

ii) $\displaystyle\int_1^2 e^{2x}\ dx$

iii) $\displaystyle\int_0^5 \frac{6x}{x^2+4}\ dx$

iv) $\displaystyle\int_1^4 \sqrt{3x} - \frac{1}{x}\ dx$

Example: Find the following definite integrals.

i) $\int_{2}^{2} 7x^3 - 2x + e^x \; dx$

ii) $\int_{0}^{3} 2e^{5x} \; dx$

iii) $\int_{3}^{0} 2e^{5x} \; dx$

5.3 Problems: 3, 8, 13, 16, 17, 18, 20, 23, 26, 28, 29, 33, 34, 38, 56, 57, 60, 62, 65, 66

5.4: The Definite Integral and Area

Definite Integral as Area: The definite integral

$$\int_a^b f(x)\;dx$$

gives us the "area under the curve" (the area between $f(x)$ and the x-axis) from the vertical line $x = a$ to the vertical line $x = b$.

Example: The graph of $f(x)$ is given in Figure 1. Use a definite integral to express the area of the shaded region.

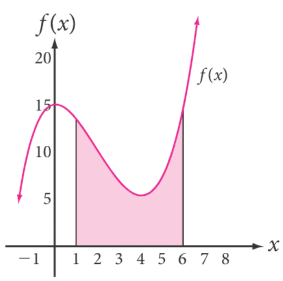


Figure 1

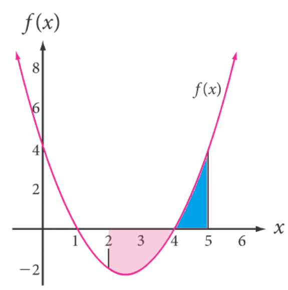


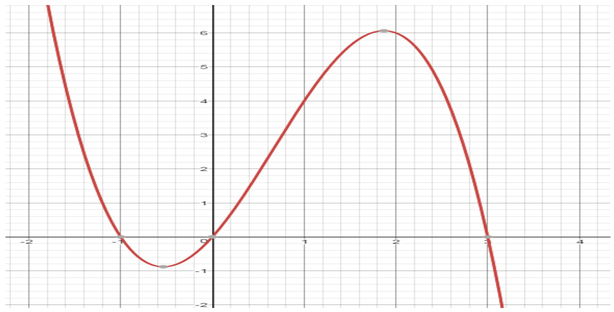
Figure 2

Example: The graph of $f(x)$ is given in Figure 2.

i) Find an expression for the area of the light/pink region.

ii) Find an expression for the area of the dark/blue region.

iii) Find an expression for the combined area of the two regions.



Example: The graph of $f(x) = -x^3 + 2x^2 + 3x$ is given above. Find

i) $\int f(x)\,dx$

ii) $\int_{-1}^{0} f(x)\,dx$

iii) $\int_{0}^{3} f(x)\,dx$

iv) $\int_{-1}^{3} f(x)\,dx$

Example: Determine the area of the region bounded by $f(x) = x^2 - 10$, $x = 4$, $x = 6$, and the x-axis.

Example: Determine the area of the region bounded by $f(x) = e^{-x+2}$, $x = 1$, $x = 5$, and the x-axis.

Revenue Streams: If you are continuously making money from some source, then we can calculate the accumulation of this revenue over time using a definite integral. This continuous revenue is called a **revenue stream** if the revenue is modeled by a function $R(t)$, then the accumulated value of the revenue stream from time $t = a$ to $t = b$ is given by the definite integral

$$\int_a^b R(t)\ dt.$$

Example: Sully invested money in stocks which he now expects to produce a revenue stream over the next 10 years modeled by

$$R(t) = 100\sqrt{2t + 5}.$$

Find the total revenue generated by this revenue stream over a 10 year period.

5.4 Problems: 1, 2, 3, 4, 5, 8, 9, 10, 11, 15, 17, 21, 22, 23, 24, 26, 27, 30, 31

5.5: Improper Integrals

Definition [Improper Integrals]: One version of an improper integral involves integrating a definite integral over an infinite interval. They are defined as follows

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$$
$$\int_{-\infty}^b f(x)\ dx = \lim_{a\to-\infty} \int_a^b f(x)\ dx$$

Take the integral first, then plug in a and b, and if the limit of that as a or b approaches $\pm\infty$ exists, then the integral converges and has a solution. If the limit does not exist, then we say the integral diverges.

Example: Evaluate $\displaystyle\int_1^\infty \frac{1}{x^2}\ dx$ or state that the integral is divergent.

Example: Evaluate $\int_1^\infty \frac{1}{x}\,dx$ or state that the integral is divergent.

Example: Evaluate $\int_{-1}^{1} \frac{1}{x^{6/7}}\, dx$ or state that the integral is divergent.

Example: Bill claims that he can make pies at a rate of $N(t) = 100e^{-.25t}$ pies per year, where t is time in years from now. Assuming Bill is correct, how many pies will he eventually be able to produce?

5.5 Problems: 1, 2, 5, 6, 33, 34, 37, 41