5.1 Angles

Definition [Ray and Angle]: A **ray** is a part of a line made up of a point, called the **endpoint**, and all the points on one side of the endpoint.

An **angle** is formed by rotating a ray about its endpoint.

Properties of Angles:

The angle's **initial side** is the ray's original position, while the angle's **terminal side** is the ray's position after the rotation. The endpoint is called the **vertex** of the angle. A curved arrow drawn near the angle's vertex indicates both the direction and amount of rotation from the initial side to the terminal side.

The Greek letters α (alpha), β (beta), γ (gamma), and θ (theta) are often used to name angles. If the rotation is counterclockwise, the result is a **positive angle**; if the rotation is clockwise, the result is a **negative angle**.

*Note- Rotation in both directions is unrestricted.

Angles that have the same initial and terminal side are called **coterminal angles**.

Properties of Angles in Cartesian Coordinate System:

An angle in the Cartesian coordinate system is said to be in **standard position** if its vertex is at the origin and its initial side lies on the positive x-axis.

An angle in standard position is a **quadrantal** angle if its terminal side lies on a coordinate axis; it is said to **lie in a quadrant** if its terminal side lies in that quadrant.

We measure angles by determining the amount of rotation from the initial side to the terminal side. Two units of measurement for angles are **degrees** and **radians**.

Measuring Angles Using Degrees

A measure of one **degree** (denoted by 1°) is assigned to the angle resulting from a rotation that is 1/360 of a complete revolution counterclockwise about the vertex.

An angle formed by rotating the initial side counterclockwise one full rotation so that the terminal and initial sides coincide has measure 360°.

An **acute angle** has measure between 0° and 90°; that is, an acute angle in standard position lies in the first quadrant.

A **right angle** has measure 90°, or 1/4 of a revolution; that is, a right angle in standard position is a quadrantal angle whose terminal side lies on the positive y-axis.

An **obtuse angle** has measure between 90° and 180°; that is, an obtuse angle in standard position lies in the second quadrant.

A **straight angle** has measure 180°, or 1/2 of a revolution; that is, a straight angle in standard position is a quadrantal angle whose terminal side lies on the negative x-axis.

Measuring Angles Using Radians

An angle whose vertex is at the center of a circle is called a **central angle**. A central angle intercepts the arc of the circle from the initial side to the terminal side.

The smallest positive central angle that intercepts an arc of length equal to the radius of the circle is said to have measure 1 **radian**.

The radian measure θ of a central angle that intercepts an arc of length s on a circle of radius r is given by

$$\theta = \frac{s}{r} \text{ radians.}$$

Notice that if $\theta = 180°$, then $s = \frac{1}{2}C = \frac{1}{2}(2\pi r) = \pi r$, and hence

$$180° = \frac{\pi r}{r} = \pi \text{ radians} \Rightarrow \mathbf{1° = \frac{\pi}{180} \text{ radians}} \text{ and } \mathbf{1 \text{ radian}} = \mathbf{\frac{180°}{\pi}}$$

Therefore, we can convert from degrees to radians by multiplying by $\pi/180$, and we can convert from radians to degrees by multiplying by $180/\pi$.

Example: Convert each angle from degrees to radians.

i) $60°$ ii) $90°$ iii) $225°$

Example: Convert each angle from radians to degrees.

i) $\frac{\pi}{6}$ ii) $\frac{3\pi}{2}$ iii) $\frac{7\pi}{4}$

Definition [Arc Length]:

The length s of the arc intercepted by a central angle with **radian** measure θ in a circle of radius r is given by

$$s = r\theta$$

Example: A circle has a radius of 9 inches. Find the length of the arc intercepted by a central angle with measure $210°$.

Example: Find the diameter of the circle whose central angle of $15°$ intercepts an arc of length $\pi/6$ inches.

5.1 Problems: 7, 10, 12, 21, 22, 26, 28, 30, 32, 35, 38, 39, 41

5.2 Unit Circle: Sine, Cosine, 5.3 Other Trigonometric Functions

Definition [The Unit Circle]:

The **unit circle** is the circle of radius 1 with its center at the origin. In the xy-plane, it is the set of points (x, y) that satisfy the equation

$$x^2 + y^2 = 1.$$

Notice that the circumference of the unit circle is 2π; the unit circle is symmetric with respect to the x-axis, the y-axis, the origin, and the line $y = x$; the x-intercepts are ± 1; and the y-intercepts are ± 1.

Example

Example:

Find all points on the unit circle whose y-coordinate is $3/5$.

Definition [Trigonometric Functions of Real Numbers]:

If t is a real number, then we can associate with t a point $P(t) = (x, y)$ on the unit circle by finding the point where the terminal side of the angle in standard position with measure t radians intersects the unit circle. The point P is called the **terminal point** associated with the real number t.

The correspondence between real numbers and terminal points on the unit circle is used to define six important functions, called the **trigonometric functions**. The full names of these functions are **sine, cosine, tangent, cosecant, secant,** and **cotangent**, which are usually abbreviated by $\sin$, $\cos$, $\tan$, $\csc$, $\sec$, and $\cot$, respectively.

Unit Circle Definitions of the Trigonometric Functions of Real Numbers

Let t be any real number and let $P(t) = (x, y)$ be the terminal point on the unit circle associated with t. Then

$$\sin t = y \qquad \csc t = \frac{1}{y} = \frac{1}{\sin t}$$

$$\cos t = x \qquad \sec t = \frac{1}{x} = \frac{1}{\cos t}$$

$$\tan t = \frac{y}{x} = \frac{\sin t}{\cos t} \qquad \cot t = \frac{x}{y} = \frac{\cos t}{\sin t} = \frac{1}{\tan t}$$

(Pythagorean Theorem) Result: $\sin^2 t + \cos^2 t = 1$

Notice that $\sec t$ and $\tan t$ are undefined when $x = 0$, and $\csc t$ and $\cot t$ are undefined when $y = 0$.

Example: Let $x = -3/7$. Find the value $y < 0$ for which (x, y) is on the unit circle, and then write the values of all six trigonometric functions at t, where t is the real number associated to the terminal point $P(t) = (x, y)$.

Example: Finding Exact Trigonometric Function Values

Find the values of the six trigonometric functions at $t = 3\pi/2$.

In general, it is difficult to find the terminal point associated with a real number t. Therefore, it is useful to have a table of common real numbers used in calculations and their associated terminal points. **You should memorize this table**.

Real number t	**Terminal Point** $P(t) = (x, y)$
0	$(1, 0)$
$\pi/6$	$(\sqrt{3}/2, 1/2)$
$\pi/4$	$(\sqrt{2}/2, \sqrt{2}/2)$
$\pi/3$	$(1/2, \sqrt{3}/2)$
$\pi/2$	$(0, 1)$
π	$(-1, 0)$
$3\pi/2$	$(0, -1)$
2π	$(1, 0)$

For other numbers, we can use the symmetries of the unit circle to find their terminal points.

Example:

i) Find $\sin t$, $\cos t$, and $\tan t$ for $t = -\pi/3$.

ii) Find $\sin t$, $\cos t$, and $\tan t$ for $t = 5\pi/6$.

iii) Find $\sin t$, $\cos t$, and $\tan t$ for $t = 5\pi/4$.

Trigonometric Function Values of an Angle θ

Let θ be an angle in standard position, and let $(x, y) \neq (0, 0)$ be a point on the terminal side of θ. Then

$$\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

is a point on the unit circle. Our previous definitions of the six trigonometric functions then imply the following definitions of the six trigonometric functions of θ.

Let θ be an angle in standard position, let $P(x, y) \neq (0, 0)$ be any point on the terminal side of θ, and let $r = \sqrt{x^2+y^2}$. Then $r > 0$ and

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y} = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x} = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

As before, we have that $\sec\theta$ and $\tan\theta$ are undefined when $x = 0$, and $\csc\theta$ and $\cot\theta$ are undefined when $y = 0$.

Example: Suppose θ is an angle whose terminal side contains the point $P(-1, 3)$. Find the exact values of the six trigonometric functions of θ.

Signs of Trigonometric Functions

Suppose an angle θ is not quadrantal and its terminal side contains the point $(x, y) \neq (0, 0)$. Since $r = \sqrt{x^2 + y^2} > 0$, it follows that the signs of x and y determine the signs of the six trigonometric functions of θ, as summarized in the following table:

	θ lies in Quadrant I	**θ lies in Quadrant II**	**θ lies in Quadrant III**	**θ lies in Quadrant IV**
$\sin\theta$	Positive	Positive	Negative	Negative
$\cos\theta$	Positive	Negative	Negative	Positive
$\tan\theta$	Positive	Negative	Positive	Negative
$\csc\theta$	Positive	Positive	Negative	Negative
$\sec\theta$	Positive	Negative	Negative	Positive
$\cot\theta$	Positive	Negative	Positive	Negative

Example: Given that $\tan\theta = 3/2$ and $\cos\theta < 0$, find the exact values of $\sin\theta$ and $\sec\theta$.

Definition [Reference Angles]:

Let θ be an angle in standard position that is not a quadrantal angle. The **reference angle** for θ is the **acute** angle θ_0 formed by the terminal side of θ and the x-axis.

It follows from the symmetries of the circle that the absolute value of any of the six trigonometric functions at θ is equal to its value at θ_0.

Example: Find the exact value of $\sin 945°$ and $\cot 945°$.

5.2 Problems: 6-9, 11, 13, 15, 17, 19, 21, 34, 35, 39, 41-44, 50, 53, 58, 59

5.3 Problems: 18-21, 38, 39, 41, 49, 51

5.4 Right Triangle Trigonometry

Trigonometric Ratios and Functions

Consider a right triangle with vertices A, B, and C, where $C = 90°$. Let θ be the angle formed at vertex A, and let a, b, and c represent the lengths of the sides opposite the vertices A, B, and C, respectively.

Now place the angle θ in standard position. The x-coordinate of the point corresponding to the vertex B is b, and its y-coordinate is a. Since the point (b, a) is on the terminal side of θ, we can use our previous definitions to define the trigonometric functions of θ as ratios of the sides of the triangle.

We use the words **opposite** for the length of the leg opposite θ, **adjacent** for the length of the leg adjacent to θ, and **hypotenuse** for the length of the hypotenuse.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{a}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}$$

We will often use the vertex itself instead of θ to denote the angle; for example, we will say $\sin A$ instead of $\sin\theta$.

Example: Let ABC be a triangle with $C = 90°$, $a = 3$, and $b = \sqrt{7}$. Find $\sin A$ and $\tan B$.

Solving Right Triangles

To **solve a triangle** means to find all unknown side lengths and angle measures. Unless otherwise stated, we assume that the right angle occurs at vertex C, and that a, b, and c denote the side lengths opposite vertices A, B, and C, respectively (in particular, c is always the length of the hypotenuse).

Example: Solve right triangle ABC if $A = 30°$ and $c = 6$.

Example: Solve right triangle ABC if $a = 3$ and $b = 3$.

Applications

Angles that are measured between a line of sight and a horizontal line occur in many applications. If the line of sight is **above** the horizontal line, the angle between these two lines is called the **angle of elevation**. If the line of sight is **below** the horizontal line, the angle between the two lines is called the **angle of depression**.

Example: You are standing on the roof of building A, which is 20 feet tall. You are an unknown distance away from building B, which is of unknown height. The angle of elevation from your friend at the bottom of building B is $45°$, and the angle of depression from your friend at the top of building B is $60°$. Find the height of the building and how far away from the building you are.

Example: You are walking towards a building of unknown height. From your current position, the angle of elevation is $30°$. After walking 100 feet towards the building, the angle of elevation is $45°$. Find the height of the building.

5.4 Problems: 17-22, 29-31, 43, 45, 47, 54, 56