6.1 Graphs of Sine and Cosine Functions

Properties of Sine and Cosine:

Each real number t determines a point $P(t) = (x, y)$ on the unit circle. Since $\cos t = x$ and $\sin t = y$, this implies that the domain of the sine and cosine functions are $(-\infty, \infty)$.

If (x, y) is a point on the unit circle, then $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

Therefore, the range of the sine and cosine functions is $[-1, 1]$.

If $P(t) = (x, y)$ is the terminal point associated with a real number t, then $Q(-t) = (x, -y)$ is the terminal point associated with $-t$. By definition, this means that

$$\cos(-t) = x = \cos t$$

and

$$\sin(-t) = -y = -\sin t.$$

Therefore, the sine function is an odd function and the cosine function is an even function.

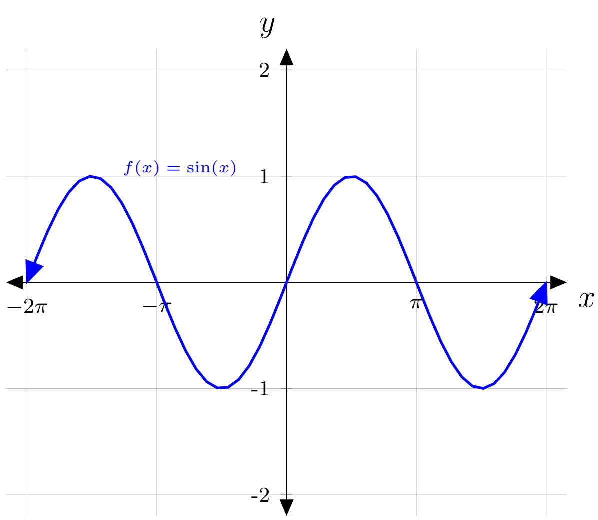
A function f is said to be **periodic** if there is a positive number p such that

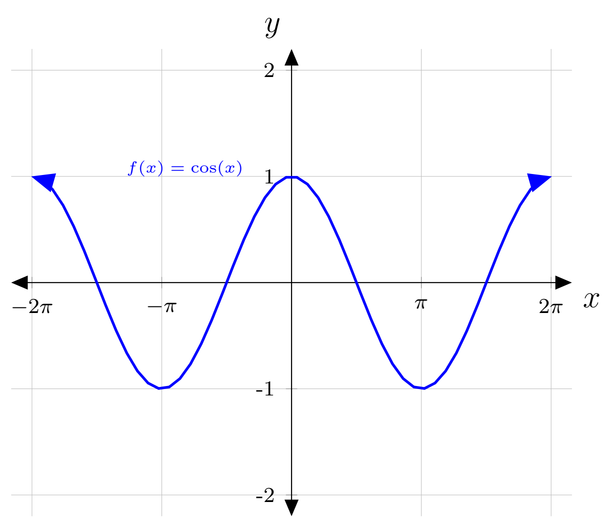
$$f(x + p) = f(x)$$

for every x in the domain of f. The smallest such value is called the **period** of f.

Periodic functions are nice in that once you have graphed one period length (or **cycle**), you know what the graph looks like for all real numbers.

Since the terminal point associated with a real number t is the same as the terminal point associated with $t + 2\pi$, the sine and cosine functions are periodic with period 2π.





Sinusoidal Graphs

The graphs of the sine and cosine functions and their transformations are called **sinusoidal graphs** or **sinusoidal curves**. In this section, we study sinusoidal curves of the form

$$S(x) = a\sin(b(x-c)) + d$$

and

$$C(x) = a\cos(b(x-c)) + d,$$

where $a \neq 0$ and $b \neq 0$.

The **amplitude** is given by $|a|$.

The **period** is given by $2\pi/b$.

The **phase shift** is given by c.

The **vertical shift** is given by d.

Since S and C are just transformations of the sine and cosine functions, respectively, we expect them to have the same shape and behavior as the sine and cosine functions. Therefore, we find "special" values of x such that $S(x)$ and $C(x)$ are easy to compute.

When we graphed the sine and cosine functions, we evaluated them at

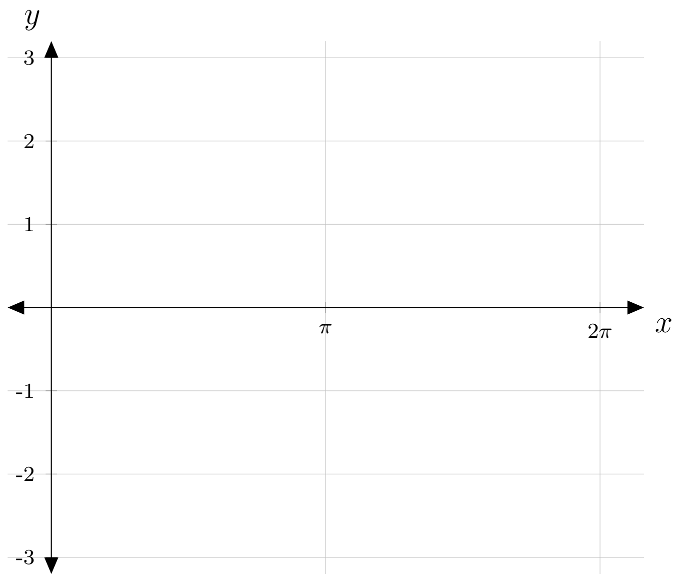
$$x = 0, \quad x = \pi/2, \quad x = \pi, \quad x = 3\pi/2, \quad \text{and} \quad x = 2\pi.$$

Since we are applying a horizontal shift of c units and a horizontal stretch factor of b, it makes sense to evaluate $S(x)$ and $C(x)$ at

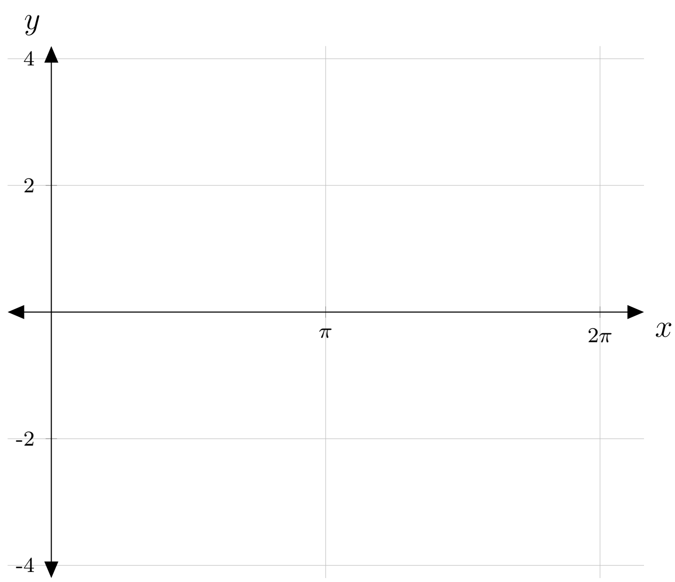
$$x = c, \quad x = c + \frac{\pi}{2b}, \quad x = c + \frac{\pi}{b}, \quad x = c + \frac{3\pi}{2b}, \quad \text{and} \quad x = c + \frac{2\pi}{b}.$$

x	$S(x) = a\sin(b(x-c)) + d$	$C(x) = a\cos(b(x-c)) + d$
c		
$c + \frac{\pi}{2b}$		
$c + \frac{\pi}{b}$		
$c + \frac{3\pi}{2b}$		
$c + \frac{2\pi}{b}$		

Example: Graph one period of the function $S(x) = 2\sin(x - \pi/4) + 1$.



Example: Graph one period of the function $C(x) = -3\cos(2x + \pi) - 1$.



*Some problems ask you to graph on larger intervals. To do this, graph one period as above and then continue the pattern in both directions until you reach the desired interval.

6.1 Problems: 6, 11, 13, 18, 19, 23, 25-27, 30-32, 38, 39

6.2 Graphs of Other Trigonometric Functions

Properties of the Tangent Function:

Since $\tan x = \sin x / \cos x$, we have that $\tan x$ is defined for all real numbers x except where $\cos x = 0$. By looking at the unit circle and periodicity of cosine, we have that

$$\cos x = 0 \quad \text{when} \quad x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$$

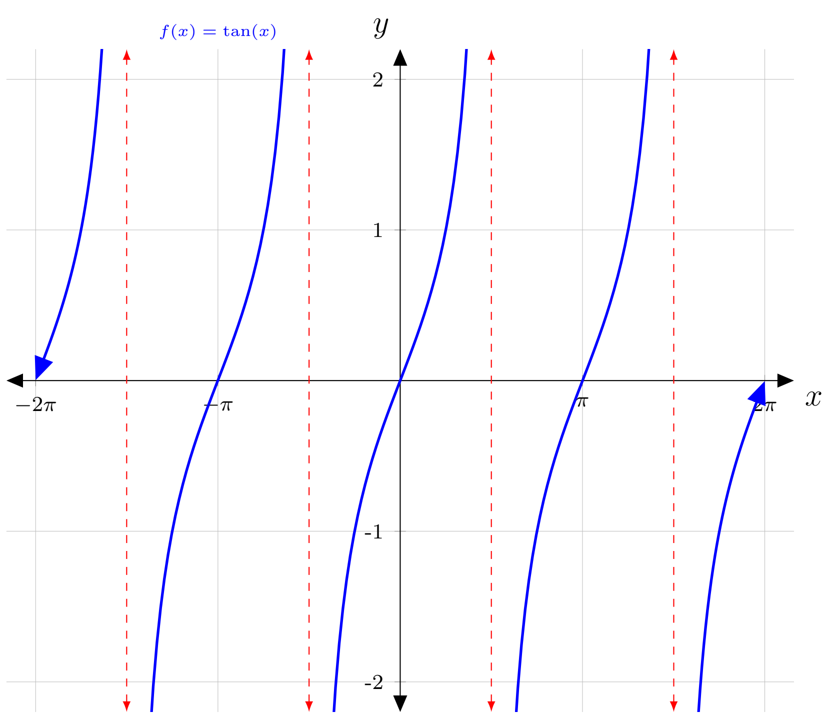
Therefore, the domain of the tangent function is all real numbers x such that x is not an odd multiple of $\pi/2$. It follows that the tangent function has a vertical asymptote at each odd multiple of $\pi/2$.

Given a real number k, we can find real numbers $x \neq 0$ and y such that $y/x = k$. The point (x, y) lies on the terminal side of an angle θ in standard position. Therefore, for each real number k, there is an angle θ such that $\tan\theta = k$, and so the range of the tangent function is $(-\infty, \infty)$.

If $P(t) = (x, y)$ is the terminal point associated with the real number t, then $P(t + \pi) = (-x, -y)$ is the terminal point associated with $t + \pi$. Therefore,

$$\tan(t + \pi) = \frac{-y}{-x} = \frac{y}{x} = \tan t,$$

and so the tangent function is a periodic function with period π.



We now want to look at graphs of functions of the form

$$T(x) = a\tan(b(x-c)) + d,$$

where $a \neq 0$ and $b \neq 0$.

The **vertical stretch factor** is given by $|a|$.

The **period** is given by π/b.

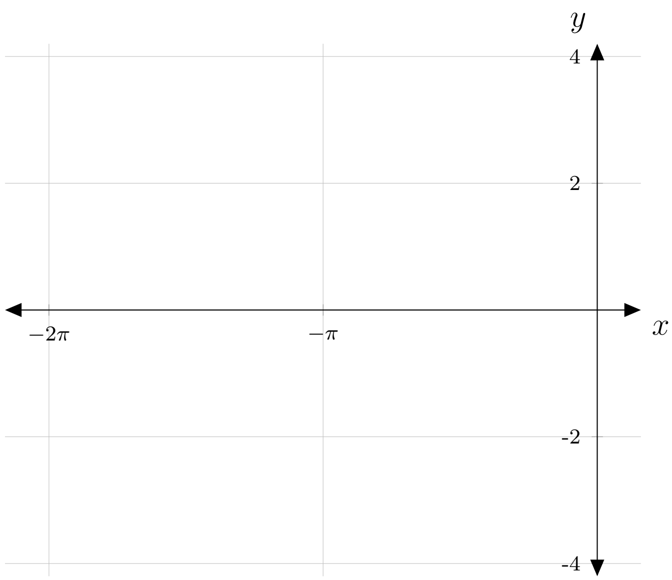
The **phase shift** is given by c.

The **vertical shift** is given by d.

As with the graphs of sinusoidal functions, we want to apply the transformations to the values of x we used above and evaluate T at the transformed x values.

x	$T(x) = a\tan(b(x-c)) + d$
$c - \frac{\pi}{2b}$	
$c - \frac{\pi}{4b}$	
c	
$c + \frac{\pi}{4b}$	
$c + \frac{\pi}{2b}$	

Example: Graph one period of the function $T(x) = -3\tan\left[\frac{1}{2}(x+\pi)\right] + 1$.



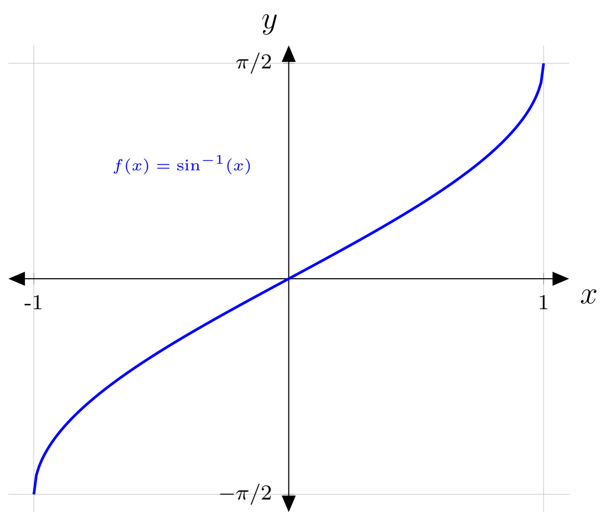
6.2 Problems: 22-24, 57 (Wait until after we cover Section 5.4 to work 57)

6.3 Inverse Trigonometric Functions

The Inverse Sine Function

Recall that a function f has an inverse if it passes the horizontal line test. Notice, since the sine function is periodic, it does not pass the horizontal line test. However, if we restrict the domain of the sine function to $[-\pi/2, \pi/2]$, then the resulting function is one-to-one and has the same range as the sine function, and so it has an inverse. This inverse function is called the **inverse sine** (or **arcsine**) function and is denoted by $\sin^{-1} x$ (or $\arcsin x$)

The equation $y = \sin^{-1} x$ means $\sin y = x$, where $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$.



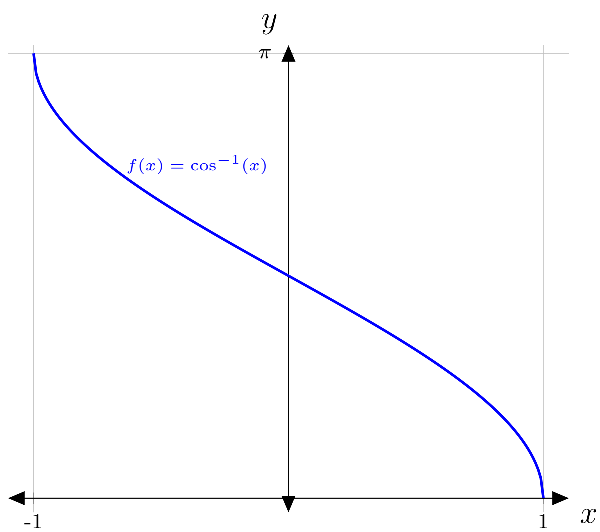
Example: Find the exact value of y, where $y = \sin^{-1}(\sqrt{3}/2)$.

Example: Find the exact value of y, where $y = \sin^{-1}(-1/2)$.

The Inverse Cosine Function

Similarly, if we restrict the domain of the cosine function to $[0, \pi]$, then we obtain a function that has an inverse. The inverse of this function is called the **inverse cosine** (or **arccosine**) function and is denoted by $\cos^{-1} x$ (or $\arccos x$).

The equation $y = \cos^{-1} x$ means $\cos y = x$, where $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$.



Example: Find the exact value of y, where $y = \cos^{-1}(\sqrt{2}/2)$.

Example: Find the exact value of y, where $y = \cos^{-1}(-1/2)$.

The Inverse Tangent Function

The **inverse tangent** (or **arctangent**) function, denoted by $\tan^{-1} x$ (or $\arctan x$), results from restricting the domain of the tangent function to $(-\pi/2, \pi/2)$.

The equation $y = \tan^{-1} x$ means $\tan y = x$, where $-\infty < x < \infty$ and $-\pi/2 < y < \pi/2$.

Example: Find the exact value of y, where $y = \tan^{-1} 0$.

Example: Find the exact value of y, where $y = \tan^{-1}(-1)$.

Other Inverse Trigonometric Functions

Inverse Cotangent: The equation $y = \cot^{-1} x$ means $\cot y = x$, where $-\infty < x < \infty$ and $0 < y < \pi$.

Inverse Cosecant: The equation $y = \csc^{-1} x$ means $\csc y = x$, where $|x| \geq 1$ and $-\pi/2 \leq y \leq \pi/2$, $y \neq 0$.

Inverse Secant: The equaiton $y = \sec^{-1} x$ means $\sec y = x$, where $|x| \geq 1$ and $0 \leq y \leq \pi$, $y \neq \pi/2$.

Composition of Trigonometric and Inverse Trigonometric Functions

Using properties of inverse functions, we have the following:

If $-\pi/2 \leq x \leq \pi/2$, then $\sin^{-1}(\sin x) = x$.

If $0 \leq x \leq \pi$, then $\cos^{-1}(\cos x) = x$.

If $-\pi/2 < x < \pi/2$, then $\tan^{-1}(\tan x) = x$.

Example: Find the exact value of $\sin^{-1}(\sin(-\pi/8))$.

Example: Find the exact value of $\cos^{-1}(\cos(5\pi/4))$.

*Remember Pythagorean Theorem Result: $\sin^2\theta + \cos^2\theta = 1$

Example: Find the exact value of $\cos(\sin^{-1}(x/3))$.

6.3 Problems: 2, 8-11, 14, 15, 22-26, 30, 32, 34-39