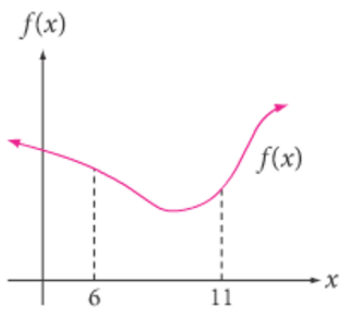
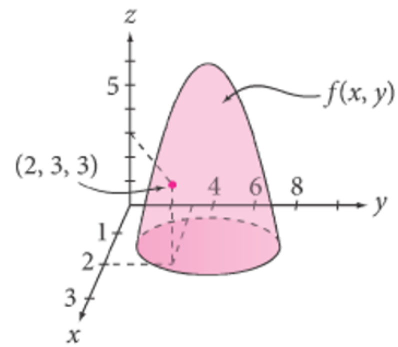
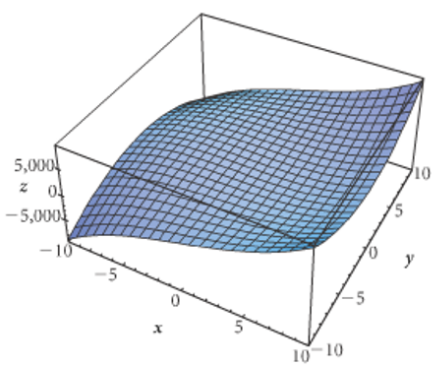
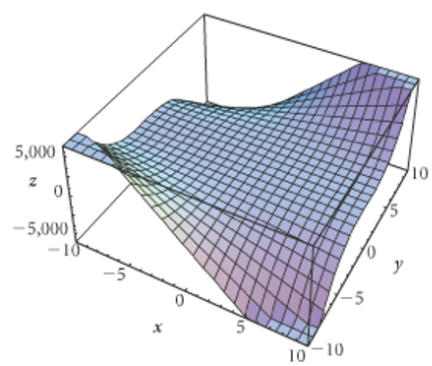
7.1: Functions of Several Variables

Definition [Functions of Several Variables]:
A <u>function of two variables</u> assigns each input pair (x, y) exactly one output number $f(x, y)$ (often plotted on the z axis in three dimensions).
A <u>function of three variables</u> assigns each input triplet (x, y, z) exactly one output number $f(x, y, z)$.









Example: Find $f(0, 0)$ and $f(1, 2)$ if $f(x, y) = e^x y^3 + 3y$.

Example: Find $f(0,0,0)$ and $f(1,-2,3)$ if $f(x,y,z) = 3^z + 2xy - 4z^2$.

Example: Find $\lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$ if $f(x,y) = 2x^2 + y$.

Example: Find $\lim_{h \to 0} \frac{f(x,y+h)-f(x,y)}{h}$ if $f(x,y) = x - 3x^2 + y^2 - 4y$

Example: Find $\lim\limits_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$ for $f(x,y) = 2xy + x$

7.1 Problems: 1, 2, 8, 11, 13, 14, 15, 30

7.2: Partial Derivatives

Notation: Partial Derivatives
For a function $f(x,y)$, the partial derivative with respect to x can be written as:

$$\frac{\partial f}{\partial x},\ f_x,\ f_x(x,y)$$

For a function $f(x,y)$, the partial derivative with respect to x can be written as:

$$\frac{\partial f}{\partial y},\ f_y,\ f_y(x,y)$$

Partial Derivatives: For a function $f(x,y)$, the partial derivative with respect to x is defined as:

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

For a function $f(x,y)$, the partial derivative with respect to y can be written as:

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

How Do I Do This?
Given a function $f(x,y)$, the partial derivative with respect to x of $f(x,y)$ is found by taking the derivative of $f(x,y)$ w.r.t. x and treating y as a constant.

Given a function $f(x,y)$, the partial derivative with respect to y of $f(x,y)$ is found by taking the derivative of $f(x,y)$ w.r.t. y and treating x as a constant.

Example: For the function $f(x,y) = 2x^2 + 3y$, find

i) $\frac{\partial f}{\partial x}(x,y)$

ii) $\frac{\partial f}{\partial y}(x,y)$

iii) $\left.\frac{\partial f}{\partial x}\right|_{(2,1)}$

iv) $\left.\frac{\partial f}{\partial y}\right|_{(-2,4)}$

Example: For the function $f(x,y) = \frac{x+3y^2}{x^2+4y}$, find

i) $f_x(x,y)$

ii) $f_y(x,y)$

iii) $f_x(1,-1)$

iv) $f_y(3,2)$

Second-Order Partial Derivatives: Just like with regular derivatives, we can find higher order partial derivatives. We will be interested in the second-order partial derivatives, of which there are four:

$$f_{xx},\ f_{yy},\ f_{xy},\ f_{yx}$$

Example: For the function $f(x,y) = 3xy + 2x - y^2$, find

i) $f_x(x,y)$

ii) $f_y(x,y)$

iii) $f_{xx}(x,y)$

iv) $f_{yy}(x,y)$

v) $f_{xy}(x,y)$

vi) $f_{yx}(x,y)$

Example: For the function $f(x, y) = \ln(xy + 2x^3)$, find

i) $f_x(x, y)$

ii) $f_y(x, y)$

iii) $f_{xy}(x, y)$

iv) $f_{xx}(x, y)$

Example: For the function $f(x, y, \lambda) = x + 3xy^2 - x^2y^3 + 4y\lambda$, find

i) $f_x(x, y, \lambda)$

ii) $f_y(x, y, \lambda)$

iii) $f_\lambda(x, y, \lambda)$

iv) $f_{\lambda y}(x, y, \lambda)$

Example: For the function $f(x, y) = x^2 + 7y^2 + 3x + 4y + 10$, find the values of x and y so that both $f_x(x, y) = 0$ and $f_y(x, y) = 0$.

Example: A company selling cat collars has a revenue in dollars given by $R(x, y) = 20x + 15y + x^2 + 2y^2$, where x is the number of green collars sold, and y is the number of gold collars sold.

i) What is the revenue if 10 green and 20 gold collars are sold?

ii) How can we expect the revenue to change if one more *green* collar is sold?

iii) How can we expect the revenue to change if one more *gold* collar is sold?

7.2 Problems: 1, 2, 3, 4, 6, 7, 9, 14, 15, 19, 22, 25, 43

7.3: Optimization of Functions of Two Variables

Definition [Relative Extrema]:

i) $f(x, y)$ has a **relative maxima** at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) "close" to (a, b), as seen in Figure 1.

ii) $f(x, y)$ has a **relative minima** at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) "close" to (a, b), as seen in Figure 2.

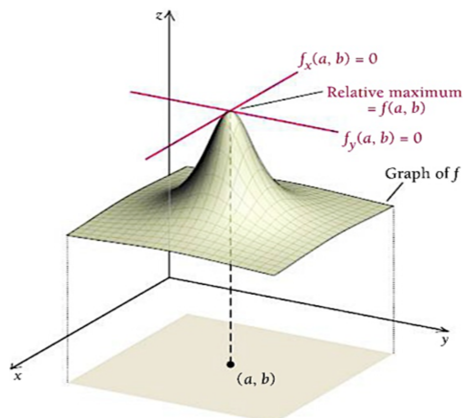


Figure 1: Relative Maximum

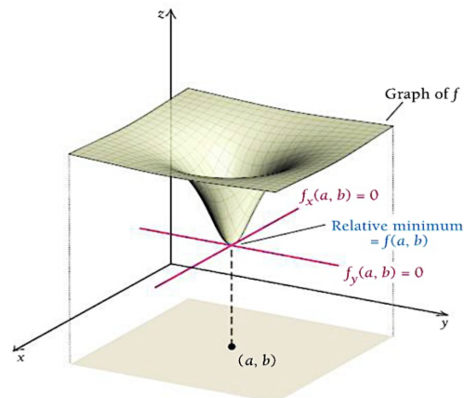
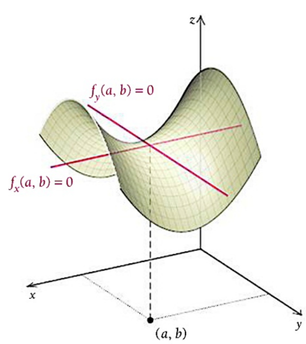


Figure 2: Relative Minimum

Theorem [Location of critical points]: The **critical points** of $f(x, y)$ occur wherever

$$f_x(x, y) = 0 \quad \text{AND} \quad f_y(x, y) = 0$$

These point are usually relative extrema, however, there is one exception called a saddle point, as seen below.



Theorem [D-Test]: To find the relative minimum/maximum values of $f(x, y)$:

i) Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.

ii) Solve the system of equations, $f_x(x, y) = 0$ and $f_y(x, y) = 0$, and let (a, b) be the solution.

iii) Evaluate $D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$

iv) Then,

  a) if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) produces a relative minimum.

  b) if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) produces a relative maximum.

  c) if $D(a, b) < 0$, then (a, b) is a saddle point.

  d) if $D(a, b) = 0$, then the test provides no information.

Example: Find and classify, if possible, all relative extrema and saddle points for

$$f(x, y) = \frac{3}{2}x^2 + y^2 + 6x - 8y + 9.$$

Example: Find and classify, if possible, all relative extrema and saddle points for

$$f(x,y) = x^2 + xy + y^2 - 3x.$$

Example: Find and classify, if possible, all relative extrema and saddle points for

$$f(x, y) = 4xy - x^3 - 2y^2.$$

Example: A manufacturer markets a product in two states, Texas and Alaska, and wants to price them differently in each state, The manufacturer wishes to sell x units of the product in Texas and y units in Alaska. To do so, he must set the price in Texas at $p_T = 86 - \frac{x}{18}$ dollars, and he must set the price in Alaska at $p_A = 122 - \frac{y}{18}$ dollars. The cost of producing all $x + y$ items is $45,000 + 4(x + y)$ dollars. How many items should be produced for Texas and Alaska, respectively, to maximize the manufacturer's profit, and what is the maximum profit?

7.3 Problems: 1, 2, 3, 4, 5, 12, 21, 22, 23, 24, 25

7.4: Constrained Maxima and Minima

What are constrained maxima and minima?
So far we have looked at optimizing functions of two variables that have the largest domain possible. But often, there are many conditions on how the variables must be related in order to get a practical maximum or minimum.

Below we see an example where c_1 is the maximum value of $f(x, y)$. However, if we only look at input values that are related to each other by $g(x, y)$ (so they sit on the line in Figure 2), the maximum value then becomes c_2.

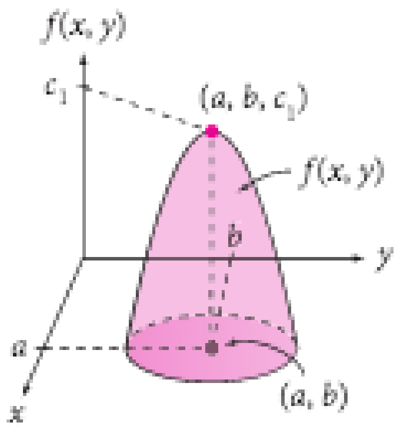


Figure 1

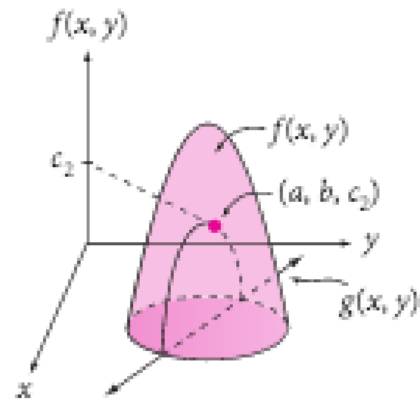


Figure 2

Example: Find and classify the relative extreme points of $f(x, y) = 3xy$ subject to the constraint $2x + y = 8$ by using substituion.

Theorem [The Method of Lagrange Multipliers]: To locate potential relative extreme points of a function $f(x, y)$ subject to the constraint $g(x, y) = 0$,

i) Set the constraint equal to zero so that it is in the form $g(x, y) = 0$.

ii) Construct the Lagrangian function

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

iii) Find the first-order partial derivatives F_x, F_y, and F_λ.

iv) Locate the critical points by solving the system of equations

$$\begin{cases} F_x &= 0 \\ F_y &= 0 \\ F_\lambda &= 0 \end{cases}$$

for x, y, and λ. Each order pair solution (x, y) is a critical point. The relative extreme points are among these critical points.

Note: You do also need to evaluate $f(x, y)$ in order to talk about the maximum of the function, as opposed to the pair (x, y) where the maximum is located.

Note: It is possible to determine whether the critical values are maxima or minima by plugging in a test point that *also happens to be a solution of the constraint equation*, but in this class, for any problem involving Lagrange Multipliers, you will be told whether to maximize or minimize the function. Then we may assume that there is a maximum or minimum at the critical value, as expected.

What is λ?

If we let M be the constrained maximum or minimum value of $f(x, y)$, subject to the constraint $g(x, y) = c$ for c a constant, then

$$\frac{dM}{dc} = \lambda$$

i.e. λ is the rate at which the constrained maximum or minimum value of the function $f(x, y)$ changes with respect to c.

Example: Using Lagrange multipliers, maximize the function subject to the given constraint.

$$f(x, y) = 2y - 6x^2, \quad \text{subject to} \quad 2x + y = 4$$

Example: Using Lagrange multipliers, minimize the function subject to the given constraint.

$$f(x, y) = y^2 + 6x, \quad \text{subject to} \quad y - 2x = 0$$

Example: Using Lagrange multipliers, maximize the function subject to the given constraint.

$$f(x, y) = 25 - x^2 - y^2, \quad \text{subject to} \quad x + y = -1$$

Example: The production of a manufacturer is given by the Cobb-Douglas production function

$$f(x, y) = 4x^{5/8}y^{3/8}$$

where x represents the number of units of labor (in hours) and y represents the number of units of capital (in dollars) invested. Labor costs \$9.50 per hour, there are 8 hours in a working day, and 255 working days in a year. The manufacturer has allocated \$2,232,576 this year for labor and capital.

i) How should the money be allocated to labor and capital to maximize productivity this year?

ii) Find and interpret the marginal productivity of money.

7.4 Problems: 1, 2, 3, 6, 8, 9, 17, 21, 22