Applications of

Spectral Functions

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Contents

Introduction

This document was created based on lecture notes from a graduate class given by Klaus Kirsten at Baylor University in the spring of 2019 as a preliminary to my research with him and Fritz Gesztesy. I have expanded on some of the ideas, included some of the basics concepts from complex analysis used, and included exercises with detailed solutions. The script is intended as a rough draft on lecture notes giving some definitions and interesting results for practice in Chapter 1 with the background of the research on spectral functions associated to Sturm-Liouville operators given in Chapter 2. This will continue to be updated and improved with the addition of our continued research into applications of the ζ-function associated to Sturm-Liouville operators.

I am indebted to Klaus for his work on this course and to both Klaus and Fritz in our research together.

Chapter 1

The Prime Number Theorem

Our goal in Chapter 1 is to provide an analytic proof of the Prime Number Theorem (see below) as well as give some useful properties of some special functions that are of particular interest in defining spectral functions and their uses in chapter 2. We will largely follow [Apo76], in particular chapters 2, 3, 4, 12, and 13.

Theorem (The Prime Number Theorem). *If $x > 0$, let $\pi(x)$ denote the number of primes not exceeding x. The Prime Number Theorem states that*

$$\lim_{x\to\infty} \frac{\pi(x)\ln x}{x} = 1.$$

Hence the number of primes acts like $\dfrac{x}{\ln x}$ as $x \to \infty$.

1.1 Big O Notation: Asymptotic Equality of Functions

Definition 1.1 (Big O Notation)**.** *If $g(x) > 0$ for all $x \geq a$, we write $f(x) = \mathcal{O}(g(x))$ to mean that the quotient $\dfrac{f(x)}{g(x)}$ is bounded for $x \geq a$; that is, there exists a constant $M > 0$ such that $|f(x)| \leq M \cdot g(x)$ for all $x \geq a$.*

Lemma 1.2. *We have that $f(t) = \mathcal{O}(g(t))$ for $t \geq a$ implies*

$$\int_a^x f(t)\ dt = \mathcal{O}\left(\int_a^x g(t)\ dt\right) \text{ for } x \geq a.$$

Proof. We have

$$\left|\int_a^x f(t)\ dt\right| \leq \int_a^x |f(t)|\ dt \leq \int_a^x M \cdot g(t)\ dt = M \int_a^x g(t)\ dt.$$

Thus the assertion holds. □

Definition 1.3. *If* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$*, we say* $f(x)$ *is asymptotical to* $g(x)$ *as* $x \to \infty$*, and write* $f(x) \sim g(x)$ *as* $x \to \infty$*.*

1.2 Summation Formula

Theorem 1.4 (Euler's summation formula)**.** *If* f *has continuous derivative,* f'*, on the interval* $[y,x]$ *where* $0 < y < x$*, then*

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\ dt + \int_y^x (t - \lfloor t \rfloor) f'(t)\ dt + f(x)(\lfloor x \rfloor - x) - f(y)(\lfloor y \rfloor - y).$$

Proof. For n and $n-1$ in $[y,x]$ we evaluate

$$\int_{n-1}^n \lfloor t \rfloor f'(t)\ dt = (n-1)\int_{n-1}^n f'(t)\ dt = (n-1)[f(n) - f(n-1)] = [nf(n) - (n-1)f(n-1)] - f(n).$$

Summing from $n = \lfloor y \rfloor + 2$ to $n = \lfloor x \rfloor$ we see

$$\int_{\lfloor y \rfloor+1}^{\lfloor x \rfloor} \lfloor t \rfloor f'(t)\ dt = \lfloor x \rfloor f(\lfloor x \rfloor) - (\lfloor y \rfloor + 1) f(\lfloor y \rfloor + 1) - \sum_{n=\lfloor y \rfloor+2}^{\lfloor x \rfloor} f(n)$$

$$= \lfloor x \rfloor f(\lfloor x \rfloor) - \lfloor y \rfloor f(\lfloor y \rfloor + 1) - \sum_{n=\lfloor y \rfloor+1}^{\lfloor x \rfloor} f(n)$$

$$= -\int_{\lfloor x \rfloor}^{x} \lfloor t \rfloor f'(t)\ dt + \lfloor x \rfloor f(x) - \int_y^{\lfloor y \rfloor+1} \lfloor t \rfloor f'(t)\ dt - \lfloor y \rfloor f(y) - \sum_{n=\lfloor y \rfloor+2}^{\lfloor x \rfloor} f(n).$$

Now add the following to see the assertion,

$$0 = \int_y^x f(t)\ dt + \int_y^x t f'(t)\ dt - x f(x) + y f(y).$$

□

Theorem 1.5 (Abel's identity)**.** *For any aritheoremetic function (defined on* $\mathbb{Z}^+$ *mapped to* $\mathbb{C}$*)* $a(n)$*, let*

$$A(x) = \sum_{n\le x} a(n),\ \text{where } A(x) = 0 \text{ if } x < 1.$$

Assume f *has a continuous derivative on the interval* $[y,x]$*, where* $0 < y < x$*. Then we have*

$$\sum_{y<n\le x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)\ dt.$$

Proof. Let $k = \lfloor x \rfloor$ and $m = \lfloor y \rfloor$. Note that $A(\cdot)$ is piecewise constant, in particular $A(x) = A(k)$ and $A(y) = A(m)$. We calculate

$$\sum_{y<n\leq x} a(n)f(n) = \sum_{n=m+1}^{k} a(n)f(n) = \sum_{n=m+1}^{k} [A(n) - A(n-1)]f(n)$$

$$= \sum_{n=m+1}^{k} A(n)f(n) - \sum_{n=m}^{k-1} A(n)f(n+1) = \sum_{n=m}^{k-1} A(n)[f(n) - f(n+1)] + A(k)f(k) - A(m)f(m+1)$$

$$= -\sum_{n=m+1}^{k-1} A(n) \int_{n}^{n+1} f'(t)\ dt + A(k)f(k) - A(m)f(m+1)$$

$$= -\sum_{n=m+1}^{k-1} \int_{n}^{n+1} A(t)f'(t)\ dt + A(k)f(k) - A(m)f(m+1)$$

$$= -\int_{m+1}^{k} A(t)f'(t)\ dt - \int_{k}^{x} A(t)f'(t)\ dt + A(x)f(x) - \int_{y}^{m+1} A(t)f'(t)\ dt - A(y)f(y)$$

and the result holds. □

Lemma 1.6. *For any aritheoremetic function $a(n)$, let*

$$A(x) = \sum_{n\leq x} a(n), \text{ where } A(x) = 0 \text{ if } x < 1.$$

Then

$$\sum_{n\leq x} (x-n)a(n) = \int_{1}^{x} A(t)\ dt.$$

Proof. We use Abel's identity with $f(x) = x$, $y = 1$ to see

$$\sum_{a<n\leq x} a(n)f(n) = \sum_{2\leq n\leq x} a(n) \cdot n = A(x) \cdot x - a(1) \cdot 1 - \int_{1}^{x} A(t)\ dt$$

and so

$$\sum_{n\leq x} (x-n)a(n) = \int_{1}^{x} A(t)\ dt.$$

□

Remark 1.7. This allows us to reformulate the Prime Number Theorem and to use continuous functions instead of piecewise continuous functions. ⋄

1.3 Some Elementary Number Theoretic Functions and a Reformulation of the Prime Number Theorem

Definition 1.8. *For every integer $n \geq 1$, we define the Mangoldt function*

$$\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^m \text{ for some prime } p \text{ and } m \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For example, $\Lambda(1) = 0$, $\Lambda(2) = \ln 2$, $\Lambda(3) = \ln 3$, $\Lambda(4) = \ln 2$, $\Lambda(5) = \ln 5$, $\Lambda(6) = 0$, *etc.*

Definition 1.9. *For $x > 0$, we define Chebyshev's ψ-function by the formula*

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

Definition 1.10. *For $x > 0$, we define Chebyshev's θ-function by the formula*

$$\theta(x) = \sum_{p \leq x} \ln(p)$$

where p runs over all primes less than or equal to x.

Lemma 1.11. *Let $\log_2 x = \dfrac{\ln x}{\ln 2}$. Then*

$$\psi(x) = \sum_{m \leq \log_2 x} \theta(x^{1/m}).$$

Proof. We have by definition

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{\substack{p \\ p^m \leq x}} \Lambda(p^m) = \sum_{m=1}^{\infty} \sum_{p \leq x^{1/m}} \ln p.$$

The sum is empty for m big enough such that

$$x^{1/m} < 2 \Longleftrightarrow \frac{1}{m} \ln x < \ln 2 \Longleftrightarrow m > \frac{\ln x}{\ln 2} = \log_2 x$$

is satisfied, so

$$\psi(x) = \sum_{m \leq \log_2 x} \sum_{p \leq x^{1/m}} \ln p = \sum_{m \leq \log_2 x} \theta(x^{1/m}).$$

□

Theorem 1.12. *For $x > 0$, we have*

$$0 \leq \frac{\psi(x)}{x} - \frac{\theta(x)}{x} \leq \frac{\ln^2 x}{2\sqrt{x}\ln 2}.$$

Proof. We note that for $0 < x < 1$, the identity is trivially satisfied. We also note for $x \geq 1$ that

$$\theta(x) = \sum_{p \leq x} \ln p \leq \sum_{p \leq x} \ln x \leq x \ln x,$$

therefore by lemma 1.6

$$0 \leq \psi(x) - \theta(x) = \sum_{2 \leq m \leq \log_2 x} \theta(x^{1/m}) \leq \sum_{2 \leq m \leq \log_2 x} x^{1/m} \ln(x^{1/m}) \leq \log_2 x \sqrt{x} \ln \sqrt{x} = \frac{\sqrt{x}\ln x}{2\ln 2} \ln x$$

which is the assertion. □

Theorem 1.13. *For $x \geq 2$, we have*

$$\theta(x) = \pi(x)\ln x - \int_2^x \frac{\pi(t)}{t}\, dt$$

and

$$\pi(x) = \frac{\theta(x)}{\ln x} + \int_2^x \frac{\theta(x)}{\ln^2 t}\, dt.$$

Proof. We will use Abel's identity, therefore we introduce the characteristic function $a(n)$ of the primes defined by

$$a(n) = \begin{cases} 1 & \text{if } n \text{ is a prime} \\ 0 & \text{otherwise.} \end{cases}$$

With this notation

$$\pi(x) = \sum_{p \leq x} 1 = \sum_{1 \leq n \leq x} a(n)$$

and

$$\theta(x) = \sum_{p \leq x} \ln p = \sum_{1 \leq n \leq x} a(n) \ln n.$$

We apply Abel's identity with $f(x) = \ln x$ to see

$$\theta(x) = \sum_{1 \leq n \leq x} a(n)\ln n = \pi(x)\ln x - \pi(1)\ln 1 - \int_1^x \pi(t)\frac{1}{t}\, dt = \pi(x)\ln x - \int_2^x \frac{\pi(t)}{t}\, dt$$

where we have used the fact that $\pi(t) = 0$ for $t \leq 2$.

"Inverting" the procedure, we now let $b(n) = a(n)\ln n$ so that

$$\theta(x) = \sum_{n \le x} b(n) \text{ and } \pi(x) = \sum_{\frac{3}{2} \le n \le x} \frac{b(n)}{\ln n}.$$

We apply Abel's identity once again with $f(x) = \dfrac{1}{\ln(x)}$ to see

$$\pi(x) = \sum_{\frac{3}{2} \le n \le x} \frac{b(n)}{\ln n} = \frac{\theta(x)}{\ln x} - \frac{\theta(\frac{3}{2})}{\ln \frac{3}{2}} - \int_{\frac{3}{2}}^{x} \theta(t) \left(-\frac{1}{t \ln^2 t} \right) \, dt = \frac{\theta(x)}{\ln x} + \int_{2}^{x} \frac{\theta(t)}{t \ln^2 t} \, dt$$

where we have used the fact that $\theta(t) = 0$ for $t \le 2$. □

Theorem 1.14. *The following are equivalent:*

i) $\displaystyle\lim_{x \to \infty} \frac{\pi(x)\ln x}{x} = 1,$

ii) $\displaystyle\lim_{x \to \infty} \frac{\theta(x)}{x} = 1,$

iii) $\displaystyle\lim_{x \to \infty} \frac{\psi(x)}{x} = 1.$

Proof. i) ⇒ ii): From theorem 1.13 we have i) implies ii) if

$$\lim_{x \to \infty} \frac{1}{x} \int_{2}^{x} \frac{\pi(t)}{t} \, dt = 0.$$

From i) we know that $\dfrac{\pi(t)}{t} = \mathcal{O}\left(\dfrac{1}{\ln t}\right)$ for $t \ge 2$, so

$$\frac{1}{x} \int_{2}^{x} \frac{\pi(t)}{t} \, dt = \mathcal{O}\left(\frac{1}{x} \int_{2}^{x} \frac{1}{\ln t} \, dt \right).$$

We estimate for $x \ge 4$,

$$\int_{2}^{x} \frac{1}{\ln t} \, dt = \int_{2}^{\sqrt{x}} \frac{1}{\ln t} \, dt + \int_{\sqrt{x}}^{x} \frac{1}{\ln t} \, dt \le \frac{\sqrt{x}}{\ln 2} \, dt + \frac{x - \sqrt{x}}{\ln \sqrt{x}},$$

so that as $x \to \infty$ we have

$$\frac{1}{x} \int_{2}^{x} \frac{\pi(t)}{t} \, dt \to 0.$$

ii) $\Rightarrow$ i): Again from theorem 1.13 we have that ii) implies i) if

$$\lim_{x\to\infty} \frac{\ln x}{x} \int_2^x \frac{\theta(t)}{t\ln^2 t}\, dt = 0.$$

From ii) we know that $\dfrac{\theta(t)}{t} = \mathcal{O}(1)$ for $t \geq 2$, so

$$\frac{\ln x}{x} \int_2^x \frac{\theta(t)}{t\ln^2 t}\, dt = \mathcal{O}\left(\frac{\ln x}{x} \int_2^x \frac{1}{\ln^2 t}\, dt\right).$$

We estimate for $x \geq 4$,

$$\int_2^x \frac{1}{\ln^2 t}\, dt = \int_2^{\sqrt{x}} \frac{1}{\ln^2 t}\, dt + \int_{\sqrt{x}}^x \frac{1}{\ln^2 t}\, dt \leq \frac{\sqrt{x}}{\ln^2 2} + \frac{x-\sqrt{x}}{\ln^2 \sqrt{x},}$$

so that as $x \to \infty$ we have

$$\frac{\ln x}{x} \int_2^x \frac{\theta(t)}{t\ln^2 t}\, dt \to 0.$$

ii) $\iff$ iii): Consequence of theorem 1.12. □

Remark 1.15. This shows that the Prime Number Theorem is equivalent to

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1 \text{ or } \theta(x) \sim x$$

and

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1 \text{ or } \psi(x) \sim x.$$

◇

Remark 1.16. Analyzing continuous functions is easier than analyzing functions with jumps, so we will next establish a relation between the asymptotic behavior of $\psi(x)$ and of $\psi_1(x) = \int_1^x \psi(t)\, dt$.
◇

Theorem 1.17. *Let $A(x) = \sum_{n\leq x} a(n)$ and let $A_1(x) = \int_1^x A(t)\, dt$. Assume further that $a(n) \geq 0$ for all n. If we have the asymptotic formula $A_1(x) \sim Lx^c$ as $x \to \infty$ for some $c > 0$ and $L > 0$, then we also have $A(x) \sim cLx^{c-1}$ as $x \to \infty$. That is to say, formal differentiation gives the correct result.*

Proof. We note first that $A(x)$ is monotone increasing since $a(n) \geq 0$. Choosing any $\beta > 1$, we see that

$$A_1(\beta x) - A_1(x) = \int_x^{\beta x} A(u)\, du \geq \int_x^{\beta x} A(x)\, du = (\beta - 1)xA(x).$$

This shows
$$xA(x) \le \frac{1}{\beta - 1}(A_1(\beta x) - A_1(x))$$
or in particular
$$\frac{A(x)}{x^{c-1}} \le \frac{1}{\beta - 1}\left(\frac{A_1(\beta x)}{(\beta x)^c}\beta^c - \frac{A_1(x)}{x^c}\right),$$
and furthermore
$$\limsup_{x\to\infty}\frac{A(x)}{x^{c-1}} \le \frac{1}{\beta - 1}\limsup_{x\to\infty}\left(\frac{A_1(\beta x)}{(\beta x)^c}\beta^c - \frac{A_1(x)}{x^c}\right) = \frac{1}{\beta-1}(L\beta^c - L) = \frac{\beta^c - 1}{\beta - 1}L.$$
Given that $\beta > 1$ was arbitrary, we send $\beta \to 1^+$ to see
$$\limsup_{x\to\infty}\frac{A(x)}{x^{c-1}} \le L\lim_{\beta\to 1^+}\frac{\beta^c - 1}{\beta - 1} = cL.$$
Similarly, choosing any $0 < \alpha < 1$, we see that
$$A_1(\alpha x) - A_1(x) = \int_{\alpha x}^{x} A(u)\ du \ge \int_{\alpha x}^{x} A(x)\ du = (1-\alpha)xA(x).$$
This shows
$$xA(x) \le \frac{1}{1-\alpha}(A_1(x) - A_1(\alpha x))$$
or in particular
$$\frac{A(x)}{x^{c-1}} \le \frac{1}{1-\alpha}\left(\frac{A_1(x)}{x^c} - \frac{A_1(\alpha x)}{(\alpha x)^c}\alpha^c\right),$$
and furthermore
$$\liminf_{x\to\infty}\frac{A(x)}{x^{c-1}} \le \frac{1}{1-\alpha}\liminf_{x\to\infty}\left(\frac{A_1(x)}{x^c} - \frac{A_1(\alpha x)}{(\alpha x)^c}\alpha^c\right) = \frac{1}{1-\alpha}(L - L\alpha^c) = \frac{1-\alpha^c}{1-\alpha}L.$$
Given that $0 < \alpha < 1$ was arbitrary, we send $\beta \to 1^-$ to see
$$\liminf_{x\to\infty}\frac{A(x)}{x^{c-1}} \le L\lim_{\alpha\to 1^-}\frac{1-\alpha^c}{1-\alpha} = cL.$$
Hence we conclude that
$$\lim_{x\to\infty}\frac{A(x)}{x^{c-1}} = cL \text{ or } A(x) \sim cLx^{c-1}.$$

□

Theorem 1.18 (Reformulation of the Prime Number Theorem)**.** *Given* $\psi_1(x) = \sum_{n\le x}(x-n)\Lambda(n)$, *we have that the asymptotic relation* $\psi_1(x) \sim \frac{1}{2}x^2$ *implies* $\psi(x) \sim x$ *as* $x \to \infty$.

Proof. By definition, $\psi(x) = \sum_{n \leq z} \Lambda(n)$, and by lemma 1.6,

$$\sum_{n \leq x} (x-n)\Lambda(n) = \int_1^x \psi(t)\ dt = \Psi_1(x).$$

Given that $\lambda(n) \geq 0$ for all n, the assumptions of theorem 1.17 are satisfied. This implies the assertion. □

1.4 The Riemann-Lebesgue Lemma

Theorem 1.19 (The Riemann-Lebesgue Lemma)**.** *If $f(x)$ is a complex function which is Lebesgue integrable on the interval I, $I \subset \mathbb{R}$, then*

$$\lim_{k\to\infty} \int_I f(x)\sin(kx)\ dx = \lim_{k\to\infty} \int_I f(x)\cos(kx)\ dx = 0.$$

Proof. We assume first that the interval I is a finite interval, say $I = [a, b]$. Let $\epsilon > 0$ be given and let g be such that $g \in C^\infty(I)$ and

$$\int_I |f(x) - g(x)|\ dx < \frac{\epsilon}{2}.$$

We partially integrate to see

$$\int_a^b g(x)\sin(kx)\ dx = -\frac{1}{k}\int_a^b g(x)\frac{d}{dx}\cos(kx)\ dx$$

$$= -\frac{1}{k}(g(b)\cos(kb) - g(a)\cos(ka)) + \frac{1}{k}\int_a^b g'(x)\cos(kx)\ dx.$$

Now sending $k \to \infty$, we see this goes to 0. Therefore, for k large enough we continue

$$\left|\int_I f(x)\sin(kx)\ dx\right| = \left|\int_I (f(x) - g(x) + g(x))\sin(kx)\ dx\right|$$

$$\leq \int_I |f(x) - g(x)|\ dx + \left|\int_I g(x)\sin(kx)\ dx\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Similarly, the argument holds for $\cos(kx)$ instead of $\sin(kx)$.
The case of infinite interval I can be reduced to the above as follows. Given $f \in L^1(I)$, we choose a and b such that

$$\int_{I-[a,b]} |f(x)|\ dx < \frac{\epsilon}{2}.$$

In that case, given $\epsilon > 0$, k large enough, we conclude as before

$$\left| \int_{I-[a,b]} f(x) \sin(kx) \; dx \right| \leq \int_{I-[a,b]} |f(x)| \; dx + \left| \int_a^b f(x) \sin(kx) \; dx \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

showing the assertion. □

Remark 1.20. Clearly, theorem 1.19 implies that for $f \in L^1(I)$,

$$\lim_{k \to \infty} \int_I f(x) e^{ikx} \; dx = 0.$$

◇

1.5 Elementary Properties of the Zeta Function of Riemann

Definition 1.21. *For $x \in \mathbb{R}$, $x > 0$, the Gamma function is defined as*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \; dt.$$

Remark 1.22. Given $z \in \mathbb{C}$, we note that

$$|t^{z-1}| = |t^{\mathrm{Re}(z) + i\mathrm{Im}(z) - 1}| = |e^{\mathrm{Re}(z) - 1 + i\mathrm{Im}(z)) \ln t}| = e^{(\mathrm{Re}(z) - 1) \ln t} = t^{\mathrm{Re}(z) - 1}$$

and we can define $\Gamma(z)$ as above for $\mathrm{Re}(z) > 0$. Thus defined, the Gamma function is also called the Euler integral of the second kind. ◇

Remark 1.23. We can also use the Gamma function to define the Beta function, also known as the Euler integral of the first kind,

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \left(= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \right)$$

for $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$. See exercise C.2 for reference, whose result we will use in chapter 2 to complete some of our asymptotic analysis. ◇

Lemma 1.24. *For $\mathrm{Re}(z) > 0$, $\Gamma(z)$ is analytic.*

Proof. Consider any closed curve C in the right half plane. Then

$$\int_C \Gamma(z) \; dz = \int_C \left(\int_0^\infty t^{z-1} e^{-t} \; dt \right) \; dz = \int_0^\infty e^{-t} \left(\int_C t^{z-1} \; dz \right) \; dt = \int_0^\infty e^{-t} (0) \; dt = 0.$$

Hence Morera's theorem (see theorem A.1) implies the assertion. □

Lemma 1.25. *The Gamma function satisfies the functional equation*

$$\Gamma(z+1) = z\Gamma(z).$$

Proof. We calculate

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\, dt = -\int_0^\infty t^z \frac{d}{dt}(e^{-t})\, dt = -t^z e^{-t}\Big|_0^\infty + z\int_0^\infty t^{z-1}e^{-t}\, dt = z\Gamma(z).$$

□

Remark 1.26. In particular, the above result gives that the Gamma function generalizes the factorial:

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!.$$

◇

Lemma 1.27. *For* $\mathrm{Re}(z) > -(n+1)$*, we have*

$$\Gamma(z) = \frac{\Gamma(n+1+z)}{z(1+z)(2+z)\ldots(n+z)}.$$

Proof. We apply the functional equation found in lemma 1.25 repeatedly to see

$$\Gamma(z) = \frac{\Gamma(1+z)}{z} = \frac{\Gamma(2+z)}{z(1+z)} = \cdots = \frac{\Gamma(n+1+z)}{z(1+z)(2+z)\ldots(n+z)}.$$

□

Lemma 1.28. *The Gamma function has singular points at* $z = -n$*,* $n \in \mathbb{N}_0$*, with singular part* $\dfrac{(-1)^n}{n!(n+z)}$.

Proof. The fact that the points $z = -n$ are singular is evident from lemma 1.27. Further, we have that

$$(n+z)\Gamma(z)\Big|_{z=-n} = \frac{\Gamma(n+1+z)}{z(1+z)(2+z)\ldots(n-1+z)}\Bigg|_{z=-n} = \frac{\Gamma(1)}{-n(1-n)\ldots(-1)} = \frac{(-1)^n}{n!}.$$

□

Lemma 1.29. *Let* $\lambda > 0$*, then for* $\mathrm{Re}(s) > 0$ *we have*

$$\frac{1}{\lambda^s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}\, dt.$$

Proof. By definition of the Gamma function, definition 1.21, we have

$$\Gamma(s) = \int_0^\infty \mu^{s-1} e^{-\mu} d\mu = \int_0^\infty (\lambda t)^{s-1} e^{-\lambda t} (\lambda\ dt) = \lambda^s \int_0^\infty t^{s-1} e^{-\lambda t}\ dt$$

where we use the substitution $\mu = \lambda t$ to justify the second equality. Dividing shows the assertion. □

Definition 1.30. *For* $\mathrm{Re}(s) > 1$*, the zeta function of Riemann is defined as*

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Definition 1.31. *For any complex number* x*, we define the Bernoulli polynomials,* $B_n(x)$*, by the equation*

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \text{ where } |z| \leq 2\pi.$$

The numbers $B_n(0)$ *are called the Bernoulli numbers and are denoted by* B_n*. Thus*

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \text{ where } |z| \leq 2\pi.$$

We have that $B_0 = 1$*,* $B_1 = -\frac{1}{2}$*, etc.*
For practice with the Bernoulli polynomials and numbers, see exercises 1.5*,* 1.6*,* 1.7*,* 1.8*, and* 1.9*.*

Theorem 1.32. *The zeta function of Riemann has a meromorphic extension to the whole complex plane and has the following properties:*

i) $Res(\zeta_R(s), 1) = 1$*,*

ii) $\zeta_R(-n) = \begin{cases} -\frac{1}{2} & \text{if } n = 0 \\ -\frac{B_{n+1}}{n+1} & \text{if } n \geq 1. \end{cases}$

Proof. We use lemma 1.29 and evaluate for $\mathrm{Re}(s) > 1$ as follows

$$\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-nt}\ dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n=1}^{\infty} e^{-nt}\ dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}}{1 - e^{-t}}\ dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} \frac{t}{e^t - 1}\ dt$$

$$= \frac{1}{\Gamma(s)} \int_0^1 t^{s-2} \frac{t}{e^t - 1}\, dt + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-2} \frac{t}{e^t - 1}\, dt.$$

We first note that the second integral is analytic everywhere and because $\frac{1}{\Gamma(-n)} = 0$ for nonnegative integers, it does not contribute to any of the properties we are considering.
We analyze the first integral further by using definition 1.31,

$$\frac{1}{\Gamma(s)} \int_0^1 t^{s-2} \frac{t}{e^t - 1}\, dt = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{B_n}{n!} \int_0^1 t^{s-2+n}\, dt = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{1}{s-1+n}.$$

Now we note that while the calculations have been carried out for $\mathrm{Re}(s) > 1$, the above expression can be evaluated for all $s \neq 1$. Hence it provides the analytic continuation for $\zeta_R(s)$ to $\mathbb{C} \backslash \{1\}$.
In particular we find for $s = 1$ we get the pole at $n = 0$ so that

$$\mathrm{Res}(\zeta_R(s), 1) = \frac{1}{\Gamma(1)} \frac{B_0}{0!} = 1$$

and further

$$\zeta_R(-l) = \lim_{s \to -l} \left(\frac{1}{\Gamma(s)} \frac{B_{l+1}}{(l+1)!} \frac{1}{s+l} \right) = (-1)^l \frac{B_{l+1}}{l+1}$$

where we have used lemma 1.28. This shows that $\zeta_R(0) = \frac{B_1}{1} = -\frac{1}{2}$.
To show the remaining property, note that by definition 1.31 and the Bernoulli Number values,

$$\frac{z}{e^z - 1} + \frac{1}{2} z = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n.$$

We will show that the left side of the equation is an even function of z which will imply the assertion.

$$-\frac{z}{e^{-z} - 1} - \frac{1}{2} z = -\frac{z e^z}{1 - e^z} - \frac{1}{2} z = \frac{z(e^z - 1 + 1)}{e^z - 1} - \frac{1}{2} z$$

$$= z + \frac{z}{e^z - 1} - \frac{1}{2} z = \frac{z}{e^z - 1} + \frac{1}{2} z$$

so that $B_{2n+1} = 0$ for $n \in \mathbb{N}$. □

Remark 1.33. Note this implies that $\zeta_R(-2n) = 0$ for $n \in \mathbb{N}$. These are the trivial zeros of the Zeta function of Riemann. The Riemann conjecture states that all other zeros are on the axis with $\mathrm{Re}(s) = \frac{1}{2}$. ⋄

Remark 1.34. We can also acquire more information from the proof of theorem 1.32 that will be useful in the exercises and chapter 2. Notice if we begin by considering $\zeta_R(\alpha s - \beta)$, we can find the residue at $s = \dfrac{1+\beta}{\alpha}$ by noticing when evaluating in the proof we find there is still a pole at $n = 0$ for $s = \dfrac{1+\beta}{\alpha}$, and we calculate

$$\operatorname{Res}\left[\zeta_R(\alpha s - \beta), \frac{1+\beta}{\alpha}\right] = \frac{1}{\Gamma(1)} \frac{B_0}{0!} \lim_{s \to \frac{1+\beta}{\alpha}} \left(s - \frac{1+\beta}{\alpha}\right) \frac{1}{\alpha s - \beta - 1}$$

$$= \frac{1}{\alpha} \lim_{s \to \frac{1+\beta}{\alpha}} \left(s - \frac{1+\beta}{\alpha}\right) \frac{1}{s - \frac{\beta+1}{\alpha}} = \frac{1}{\alpha}.$$

Notice all other values remain the same. ⋄

Remark 1.35. We have the following result:

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}} \text{ for } \operatorname{Re}(s) > 1$$

where p_k is the sequence of prime numbers. ⋄

Proof. Consider the infinite product $\prod_{k=1}^{\infty} \dfrac{1}{1 - p_k^{-s}}$ for $s > 1$. We write $c_k = \dfrac{1}{1 - p_k^{-s}} - 1 = \dfrac{1}{p_k^s - 1}$. Given $p_k > 1$ we have $c_k > 0$. Further, given $p_k \geq 2$ we also have $1 - p_k^{-s}\ \dfrac{1}{2}$ or $c_k < \dfrac{2}{p_k^s}$. Hence,

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} |c_k| < 2 \sum_{k=1}^{\infty} \frac{1}{p_k^s} < 2 \sum_{k=1}^{\infty} \frac{1}{n^s}$$

and the sum $\sum_{k=1}^{\infty} c_k$ is absolutely convergent for $s > 1$, so that $\prod_{k=1}^{\infty} \dfrac{1}{1 - p_k^{-s}}$ is also. Now given $p_k > 1$, each factor of the product can be expanded in a geometric series as below

$$\frac{1}{1 - p_k^{-s}} = \sum_{l=0}^{\infty} p_k^{-l} = (1 + p_k^{-s} + p_k^{-2s} + \dots).$$

If we consider the partial products we calculate

$$\prod_{k=1}^{m} \frac{1}{1 - p_k^{-s}} = \prod_{k=1}^{m} (1 + p_k^{-s} + p_k^{-2s} + \dots) = (1 + p_1^{-s} + p_1^{-2s} + \dots) \dots (1 + p_m^{-s} + p_m^{-2s} + \dots)$$

which implies that

$$\sum_{n=1}^{p_m} \frac{1}{n^s} < \prod_{k=1}^{m} \frac{1}{1-p_k^{-s}} = \sum_{n=1}^{p_m} \frac{1}{n^s} + \sum{}'' \frac{1}{n^s}$$

where $\sum''$ denotes the sum over all natural numbers larger than p_m with prime factors $p_1, \ldots, p_m$. Now as $p_m \to \infty$, we find for $\mathrm{Re}(s) > 1$,

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{k=1}^{\infty} \frac{1}{1-p_k^{-s}}.$$

□

Lemma 1.36. *For* $\mathrm{Re}(s) > 1$*, we have*

$$\zeta_R(s) = e^{G(s)} \text{ with } G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-s} = \sum_p \sum_m \frac{1}{mp^{ms}}.$$

Proof. We have

$$\ln \zeta_R(s) = -\sum_p \ln(1-p^{-s}) = \sum_p \sum_{m=1}^{\infty} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} n^{-s} \Lambda_1(n)$$

where

$$\Lambda_1(n) = \begin{cases} \dfrac{1}{m} & \text{if } n = p^m \text{ for some prime} \\ 0 & \text{otherwise.} \end{cases}$$

But if $n = p^m$ then $\ln n = m \ln p = m\Lambda(n)$ and $\dfrac{1}{m} = \dfrac{\Lambda(n)}{\ln n}$.
This shows $\ln \zeta_R(s) = \sum_{n=2}^{\infty} \dfrac{\Lambda(n)}{\ln n} n^{-s}$. □

Remark 1.37*.* Note that this implies

$$-\frac{\zeta_R'(s)}{\zeta_R(s)} = -\frac{d}{ds}[\ln \zeta_R(s)] = -\frac{d}{ds}[G(s)] = -\frac{d}{ds}\left[\sum_{n=2}^{\infty} \frac{\lambda(n)}{\ln n} n^{-s}\right] = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

◇

1.6 Analytic Proof of the Prime Number Theorem

Lemma 1.38. *If $c > 0$ and $u > 0$, then for every integer $k \geq 1$ we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{u^{-z}}{z(z+1)\dots(z+k)}\ dz = \begin{cases} \frac{1}{k!}(1-u)^k & \text{if } 0 < u \leq 1 \\ 0 & \text{if } u > 1, \end{cases}$$

the integral being absolutely convergent.

Proof. Let $C(R)$ denote the circle centered at $c > 0$ with radius R. Further, denote the right and left circular arcs of this contour by $C_\pm(R)$ respectively. We note

$$\frac{\Gamma(z+k+1)}{\Gamma(z)} = (z+k)(z+k-1)\dots z,$$

such that we want to find

$$\frac{1}{2\pi i}\oint_{C_\pm(R)} \frac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)}\ dz$$

where R is large enough to include all singularities. Along the circular arc of $C_-(R)$ for $0 < u \leq 1$, the integrand behaves like

$$\frac{u^{-z}}{z(z+1)\dots(z+k)} = \mathcal{O}\left(\frac{u^{-c}}{R^{k+1}}\right).$$

So the integral goes to 0 as $R \to \infty$. Similarly, we see along the circular arc of $C_+(R)$ for $u > 1$, we arrive to the same result. This shows the assertion for $u > 1$ as there are no poles enclosed by that contour. Now we calculate the residue of each of the poles for $0 < u \leq 1$:

$$\frac{1}{2\pi i}\oint_{C_-(R)} \frac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)}\ dz = \sum_{n=0}^{k} \text{Res}\left[\frac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)}, -n\right] = \sum_{n=0}^{k} \frac{u^n}{\Gamma(-n+k+1)}\text{Res}\left[\Gamma(z), -n\right]$$

$$= \sum_{n=0}^{k} \frac{u^n}{(k-n)!}\frac{(-1)^n}{n!} = \frac{1}{k!}(1-u)^k,$$

which shows the rest of the assertion. □

Theorem 1.39. *If $c > 1$ and $x \geq 1$ we have*

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta_R'(s)}{\zeta_R(s)}\right) ds.$$

Proof. From theorem 1.18, we have

$$\psi_1(x) = \sum_{n \le x} (x-n)\Lambda(n)$$

such that we see

$$\frac{\psi_1(x)}{x} = \sum_{n \le x} \left(1 - \frac{n}{x}\right) \Lambda(n).$$

From lemma 1.38, with $k = 1$, $u = \dfrac{n}{x} \le 1$, we see

$$1 - \frac{n}{x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)}\, ds.$$

Furthermore,

$$\frac{\psi_1(x)}{x} = \sum_{n \le x} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Lambda(n)(x/n)^s}{s(s+1)}\, ds = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Lambda(n)(x/n)^s}{s(s+1)}\, ds,$$

where we note that we can extend sum because all are zero for $u = \dfrac{n}{x} > 1$. To interchange the sum and integral, we need to see the sum is absolutely convergent. We estimate:

$$\sum_{n=1}^{\infty} \left| \int_{c-i\infty}^{c+i\infty} \frac{\Lambda(n)(x/n)^s}{s(s+1)}\, ds \right| \le \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c} \int_{c-i\infty}^{c+i\infty} \frac{x^c}{|s||s+1|}\, ds \text{ as } \mathrm{Re}(s) = c,$$

where the integral on the right is finite, and for $c > 1$, we have absolute convergence. Hence

$$\frac{\psi_1(x)}{x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds.$$

□

Remark 1.40. Note that we have moved the half plane from $c > 0$ in lemma 1.38 to $c > 1$ in theorem 1.39. This was needed for us to be able to have half of the circle containing all the poles, while the other contains none. ⋄

Theorem 1.41. *If $c > 1$ and $x \ge 1$ we have*

$$\frac{\psi(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{s-1} h(s)\, ds \ \textit{where}\ h(s) = \frac{1}{s(s+1)} \left(-\frac{\zeta_R'(s)}{\zeta_R(s)} - \frac{1}{s-1}\right).$$

Proof. We apply lemma 1.38 with $k = 2$ and have

$$\frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)(s+2)}\ ds.$$

Replacing s with $s-1$ shows the assertion. □

Remark 1.42. Our final step will be to show that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{s-1}h(s)\ ds \to 0 \text{ as } x \to \infty.$$

Substituting $s = c + it$, we find the equivalent

$$\frac{x^{c-1}}{2\pi}\int_{-\infty}^{\infty} e^{it\ln x}h(c+it)\ dt \to 0 \text{ as } x \to \infty.$$

◇

Lemma 1.43. *We have the representations*

$$\zeta_R(s) = \sum_{n=1}^{N}\frac{1}{n^s} - s\int_N^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}\ dx + \frac{N^{1-s}}{s-1}$$

and

$$\zeta_R'(s) = \sum_{n=1}^{N}\frac{\ln n}{n^s} + s\int_N^\infty \frac{(x-\lfloor x\rfloor)\ln x}{x^{s+1}}\ dx - \int_N^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}\ dx - \frac{N^{1-s}\ln n}{s-1} - \frac{N^{1-s}}{(s-1)^2}.$$

Proof. The formula for $\zeta_R'(s)$ follows directly from the one given for $\zeta_R(s)$ by differentiation. The result for $\zeta_R(s)$ follows from theorem 1.4, Euler's Summation formula (let $y = N,\ x = m$), by writing

$$\zeta_R(s) = \sum_{n=1}^{N}\frac{1}{n^s} + \lim_{m\to\infty}\sum_{n=N+1}^{m}\frac{1}{n^s}$$

$$= \sum_{n=1}^{N}\frac{1}{n^s} + \lim_{m\to\infty}\left(\int_N^m \frac{1}{x^s}\ dx + \int_N^m (x-\lfloor z\rfloor)\left(-\frac{s}{x^{s+1}}\right)\ dx + 0 - 0\right)$$

$$= \sum_{n=1}^{N}\frac{1}{n^s} - s\int_N^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}\ dx + \frac{N^{1-s}}{s-1} \text{ for } \mathrm{Re}(s) > 0.$$

□

Theorem 1.44. *Let $s = \sigma + it$. For every $A > 0$ there exists a constant M_A such that*

$$|\zeta_R(s)| \leq M_A \ln t$$

and

$$|\zeta_R'(s)| \leq M_A \ln^2 t$$

for all s with $\sigma \geq \frac{1}{2}$ satisfying $\sigma > 1 - \frac{A}{\ln t}$ and $t \geq e$.

Proof. For $\sigma \geq 2$ we have that

$$\zeta_R(s)| \leq \zeta_R(2) \text{ and } |\zeta_R'(s)| \leq \zeta_R'(2)$$

with both of these being trivially satisfied. We therefore assume that $\sigma < 2$ and $t \geq e$. We then have

$$|s| = \sqrt{\sigma^2 + t^2} = t\sqrt{1 = \frac{\sigma^2}{t^2}} \leq t\left(1 + \frac{1}{2}\frac{\sigma^2}{t^2}\right) = t + \sigma\frac{\sigma}{2t} \leq t + \sigma \leq 2t,$$

and

$$|s-1| = \sqrt{(\sigma-1)^2 + t^2} \geq t, \text{ so } \frac{1}{t} \geq \frac{1}{|s-1|}.$$

From lemma 1.43, we therefore estimate

$$|\zeta_R(s)| \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} + 2t\int_N^\infty \frac{1}{x^{\sigma+1}}\,dx + \frac{N^{1-\sigma}}{t} = \sum_{n=1}^{N} \frac{1}{n^\sigma} + \frac{2t}{\sigma N^\sigma} + \frac{N^{1-\sigma}}{t}.$$

Let $N = \lfloor t \rfloor$ such that $N \leq t < N+1$ and $\ln n \leq \ln t$ if $n \leq N$. We consider the region of the complex plane where

$$1 - \sigma < \frac{A}{\ln t}$$

so

$$\frac{1}{n^\sigma} = \frac{n^{1-\sigma}}{n} = \frac{1}{n}e^{(1-\sigma)\ln n} < \frac{1}{n}e^{A\frac{\ln n}{\ln t}} \leq \frac{1}{n}e^A = \mathcal{O}(n^{-1}).$$

Given $t = \mathcal{O}(N)$, we see

$$\frac{2t}{\sigma N^\sigma} = \mathcal{O}(1) \text{ and } \frac{N^{1-\sigma}}{t} = \mathcal{O}(N^{-\sigma}) = \mathcal{O}(N^{-1}) = \mathcal{O}(1).$$

This shows

$$|\zeta_R(s)| = \mathcal{O}\left(\sum_{n=1}^{N} \frac{1}{n^\sigma}\right) = \mathcal{O}\left(\sum_{n=1}^{N} \frac{1}{n}\right) = \mathcal{O}(\ln N) = \mathcal{O}(\ln t)$$

which implies the assertion for $|\zeta_R(s)|$.

For $|\zeta_R'(s)|$ we proceed along the same lines.

$$|\zeta_R'(s)| \leq \sum_{n=1}^{N} \frac{\ln n}{n^\sigma} + 2t \int_N^\infty \frac{\ln x}{x^{\sigma+1}}\ dx + \int_N^\infty \frac{dx}{x^{\sigma+1}} + \frac{N^{1-\sigma}\ln N}{t} + \frac{N^{1-\sigma}}{t^2}$$

$$= \sum_{n=1}^{N} \frac{\ln n}{n^\sigma} + 2t\left(\frac{1+\sigma\ln N}{\sigma^2 N^\sigma}\right) + \frac{1}{\sigma N^\sigma} + \frac{N\ln N}{tN^\sigma} + \frac{N}{N^\sigma t^2}.$$

We consider each of the single terms to see

$$\sum_{n=1}^{N} \frac{\ln n}{n^\sigma} = \mathcal{O}\left(\sum_{n=1}^{N} \frac{\ln n}{n}\right) = \mathcal{O}(\ln^2 N), \quad 2t\left(\frac{1+\sigma\ln N}{\sigma^2 N^\sigma}\right) = \mathcal{O}(\ln N),$$

$$\frac{1}{\sigma N^\sigma} = \mathcal{O}(N^{-1}), \quad \frac{N\ln N}{tN^\sigma} = \mathcal{O}\left(\frac{\ln N}{N}\right), \quad \frac{N}{N^\sigma t^2} = \mathcal{O}(N^{-2}),$$

thus implying the assertion. □

Theorem 1.45. *If $\sigma > 1$ we have*

$$\zeta_R^3(\sigma)|\zeta_R(\sigma+it)|^4|\zeta_R(\sigma+2it)| \geq 1.$$

Proof. From lemma 1.36 we have with $s = \sigma + it$,

$$\zeta_R(s) = \exp\left\{\sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}\right\} = \exp\left\{\sum_p \sum_{m=1}^{\infty} \frac{e^{-imt\ln p}}{mp^{m\sigma}}\right\}.$$

This shows

$$|\zeta_R(s)| = \exp\left\{\sum_p \sum_{m=1}^{\infty} \frac{\cos(mt\ln p)}{mp^{m\sigma}}\right\}.$$

We will use the following trig identity which will produce the powers in the theorem, which we also note is nonnegative,

$$2(1+\cos\theta)^2 = 3 + 4\cos\theta + \cos(s\theta)it.$$

This shows

$$\zeta_R^3(\sigma)|\zeta_R(\sigma+it)|^4|\zeta_R(\sigma+2it)| = \exp\left\{\sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{m\sigma}}(3 + 4\cos(mt\ln p) + \cos(2mt\ln p))\right\} \geq 1.$$

□

Theorem 1.46. *We have $\zeta_R(1+it) \neq 0$ for $t \in \mathbb{R}$.*

Proof. From theorem 1.45, for $\sigma > 1$ we have

$$[(\sigma-1)\zeta_R(\sigma)]^3 \left|\frac{\zeta_R(\sigma+it)}{(\sigma-1)}\right| |\zeta_R(\sigma+2it)| \geq \frac{1}{\sigma-1}.$$

We need only consider $t \neq 0$. Assume there exists a $t \neq 0$ with $\zeta_R(1+it) = 0$. For this specific value of t, as $\sigma \to 1^+$, we find

$$\lim_{\sigma\to 1^+}[(\sigma-1)\zeta_R(\sigma)]^3 \left|\frac{\zeta_R(\sigma+it)-\zeta_R(1+it)}{(\sigma-1)}\right|^4 |\zeta_R(\sigma+2it)|$$

$$= 1 \cdot |\zeta_R'(1+it)|^4 |\zeta_R(1+2it)| < \infty.$$

But

$$\lim_{\sigma\to 1^+}\frac{1}{\sigma-1} = \infty,$$

a contradiction. □

Remark 1.47. From the proof of lemma 1.36, for $\text{Re}(s) > 1$, we have

$$\ln\zeta_R(s) = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{\ln n}n^{-s}.$$

Given that the right hand side is absolutely convergent for $\text{Re}(s) > 1$ (in particular it is finite), this shows there are no zeros of $\zeta_R(s)$ for $\text{Re}(s) > 1$. ⋄

Theorem 1.48. *There is a constant $M > 0$ such that*

$$\left|\frac{1}{\zeta_R(s)}\right| < M\ln^7 t \text{ and } \left|\frac{\zeta_R'(s)}{\zeta_R(s)}\right| < M\ln^9 t \text{ where } \sigma \geq 1 \text{ and } t \geq e.$$

Proof. We have

$$\zeta_R(s) = 1 + \sum_{n=2}^{\infty}\frac{1}{n^s}$$

and for $\sigma \geq 2$,

$$|\zeta_R(s)| \geq 1 - \left|\sum_{n=2}^{\infty}\frac{1}{n^s}\right| \geq 1 - \sum_{n=2}^{\infty}\frac{1}{n^\sigma} = 1 - [\zeta_R(\sigma) - 1] \geq 2 - \zeta_R(2).$$

Thus

$$\left|\frac{1}{\zeta_R(s)}\right| \leq \frac{1}{2-\zeta_R(2)}.$$

Furthermore, from remark 1.37,

$$\left|\frac{\zeta_R'(s)}{\zeta_R(s)}\right| \le \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \le \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2}.$$

This shows the inequality for $\sigma \ge 2$.

Suppose now that $1 < \sigma \le 2$ and $t \ge e$. From theorem 1.45 we have

$$\frac{1}{|\zeta_R(\sigma+it)|} \le \zeta_R(\sigma)^{3/4}|\zeta_R(\sigma+2it)|^{1/4},$$

and theorem 1.44 shows that $\zeta_R(\sigma+2it) = \mathcal{O}(\ln t)$ for $1 \le \sigma \le 2$. Further, we know there is a suitable constant, M, such that

$$(\sigma-1)\zeta_R(\sigma) \le M,$$

so

$$\zeta_R(\sigma) \le \frac{M}{(\sigma-1)} \text{ for } 1 < \sigma \le 2.$$

Thus we can continue for $1 < \sigma \le 2$,

$$\frac{1}{|\zeta_R(\sigma+it)|} \le \frac{M^{3/4}}{(\sigma-1)^{3/4}}(\ln t)^{1/4} = \frac{A(\ln t)^{1/4}}{(\sigma-1)^{3/4}}, \text{ where } A = M^{3/4}.$$

So we have for some constant, B,

$$|\zeta_R(\sigma+it)| > \frac{B(\sigma-1)^{3/4}}{(\ln t)^{1/4}}$$

which is valid for all of $1 \le \sigma \le 2$, as it trivially holds for $\sigma = 1$.

Now, let $1 \le \sigma \le \alpha \le 2$ and $t \ge e$. Then theorem 1.44 implies

$$|\zeta_R(\sigma+it) - \zeta_R(\alpha+it)| = \left|\int_\sigma^\alpha \frac{d}{du}\zeta_R(u+it)\ du\right| \le \int_\sigma^\alpha |\zeta_R'(u+it)|\ du$$

$$\le (\alpha-\sigma)M\ln^2 t \le (\alpha-1)M\ln^2 t.$$

By the triangle inequality we now find using the above

$$|\zeta_R(\sigma+it)| = |\zeta_R(\alpha+it) + \zeta_R(\sigma+it) - \zeta_R(\alpha+it)| \ge |\zeta_R(\alpha+it)| - |\zeta_R(\sigma+it) - \zeta_R(\alpha+it)|$$

$$\ge |\zeta_R(\alpha+it)| - (\alpha-1)M\ln^2 t \ge \frac{B(\alpha-1)^{3/4}}{(\ln t)^{1/4}} - (\alpha-1)M\ln^2 t.$$

Similarly, we have the same inequality for $1 \leq \alpha \leq \sigma \leq 2$,

$$|\zeta_R(\sigma+it)| \geq \frac{B(\sigma-1)^{3/4}}{(\ln t)^{1/4}} \geq \frac{B(\alpha-1)^{3/4}}{(\ln t)^{1/4}} - (\alpha-1)M\ln^2 t.$$

So for $1 \leq \sigma \leq 2$, $t \geq e$, and $1 < \alpha < 2$, we have

$$|\zeta_R(\sigma+it)| \geq \frac{B(\alpha-1)^{3/4}}{(\ln t)^{1/4}} - (\alpha-1)M\ln^2 t.$$

To simplify the inequality, we try to find $\alpha = \alpha(t)$ such that

$$\frac{B(\alpha-1)^{3/4}}{(\ln t)^{1/4}} = 2(\alpha-1)M\ln^2 t$$

or

$$\alpha = \left(\frac{B}{2M}\right)^4 \frac{1}{\ln^9 t} + 1, \text{ for } t \text{ large enough and } 1 < \alpha < 2.$$

With this choice of α, we find

$$|\zeta_R(\sigma+it)| \geq (\alpha-1)M\ln^2 t = \frac{C}{\ln^7 t}, \text{ with } 1 \leq \sigma \leq 2,\ t \geq e$$

so that

$$\left|\frac{1}{\zeta_R(\sigma+it)}\right| \leq \tilde{M}\ln^7 t$$

for suitable $\tilde{M}$. The remaining information of the theorem follows from theorem 1.44,

$$\left|\frac{\zeta_R'(s)}{\zeta_R(s)}\right| \leq |\zeta_R'(s)|\tilde{M}\ln^7 t \leq \tilde{M}\ln^9 t$$

for suitable $\tilde{M}$. □

Theorem 1.49. *The function*

$$F(s) = -\frac{\zeta_R'(s)}{\zeta_R(s)} - \frac{1}{s-1} \text{ is analytic at } s = 1.$$

Proof. This is an immediate consequence of theorem 1.32, namely $\text{Res}(\zeta_R(s), 1) = 1$ and $s = 1$ is a first order pole. □

Theorem 1.50. *For $x \geq 1$ we have*

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{it\ln x}\, dt$$

where the integral $\displaystyle\int_{-\infty}^{\infty} |h(1+it)|\, dt$ *converges.*
Therefore by the Riemann-Lebesgue Lemma (theorem 1.19),

$$\psi_1(x) \sim \frac{1}{2}x^2 \text{ and hence } \psi(x) \sim x \text{ as } x \to \infty.$$

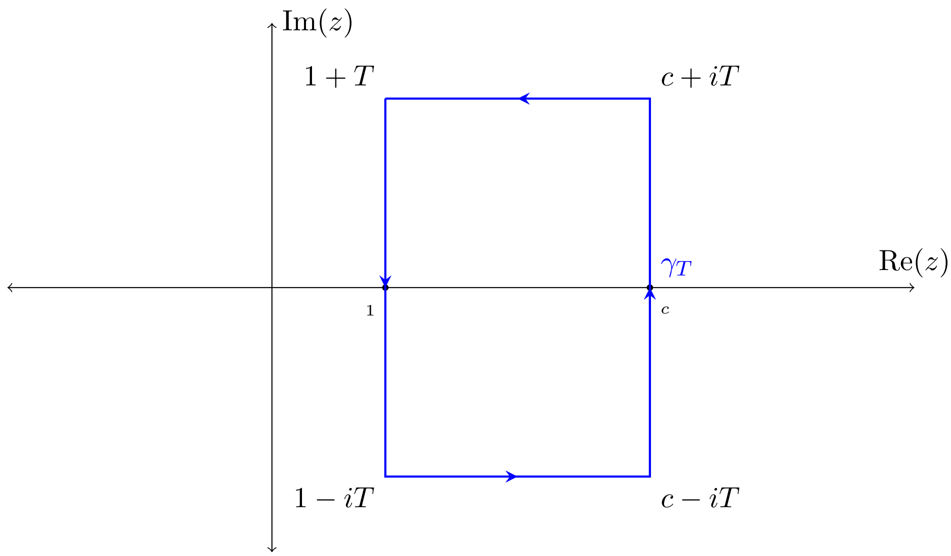
Proof. In theorem 1.41, we showed that if $c > 1,\ x \geq 1$, we have

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{s-1}h(s)\, ds$$

where

$$h(s) = \frac{1}{s(s+1)}\left(-\frac{\zeta_R'(s)}{\zeta_R(s)} - \frac{1}{s-1}\right).$$

We first show that we can shift the contour to the line $c = 1$. To see this, consider the follow contour as $T \to \infty$,



Note, as the integrand has poles at $s = 0, -1, \ldots$ so the integral along γ_T equals zero. On the upper segment, we have the following estimates for T large enough:

$$\left|\frac{1}{s(s+1)}\right| \leq \frac{1}{T^2}, \quad \left|\frac{1}{s(s+1)(s-1)}\right| \leq \frac{1}{T^3} \leq \frac{1}{T^2},$$

from which we see by applying theorem 1.48,

$$|h(s)| \le \frac{M \ln^9 T}{T^2}.$$

In particular, this shows for $s = \sigma + iT$,

$$\left| \int_{1+iT}^{c+iT} x^{s-1} h(s)\ ds \right| \le \int_1^c x^{c-1} \frac{M \ln^9 T}{T^2}\ d\sigma = M x^{c-1} \frac{\ln^9 T}{T^2}(c-1).$$

The same estimate holds for the lower segment as $|h(s)| = h(\overline{s})|$, since

$$|\zeta_R(\overline{s})| = |\overline{\zeta_R(s)}| = |\zeta_R(s)|.$$

This shows as $T \to \infty$, contributions from the horizontal segments vanish. Hence we can continue,

$$\int_{c-i\infty}^{c+i\infty} x^{s-1} h(s)\ ds = \int_{1-i\infty}^{1+i\infty} x^{s-1} h(s)\ ds = i \int_{-\infty}^{\infty} x^{it} h(1+it)\ dt,$$

where we have used the substitution $s = 1 + it$. To conclude the proof we need only show that

$$\int_{-\infty}^{\infty} |h(1+it)|\ dt$$

exists. But for $t \ge e$, we have

$$|h(1+it)| \le \frac{M \ln^9 t}{t^2},$$

and the assertion follows from the fact

$$\int_{-\infty}^{\infty} |h(1+it)|\ dt = \int_{-\infty}^{-e} |h(1+it)|\ dt + \int_{-e}^{e} |h(1+it)|\ dt + \int_{e}^{\infty} |h(1+it)|\ dt.$$

□

Remark 1.51. This concludes our analytic proof of the Prime Number Theorem! ⋄

1.7 Exercises

Exercise 1.1. Let

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } s > 1.$$

Show for $x \ge 1$ that the following holds.

i) Let γ be Euler's constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = 0.577216\ldots.$$

Then we have

$$\sum_{n\le x}\frac{1}{n} = \ln x + \gamma + \mathcal{O}\left(\frac{1}{x}\right).$$

ii)

$$\sum_{n\le x}\frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta_R(s) + \mathcal{O}(x^{-s}) \text{ for } s>1.$$

iii)

$$\sum_{n>x}\frac{1}{n^s} = \mathcal{O}(x^{1-s}) \text{ for } s>1.$$

iv)

$$\sum_{n\le x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + \mathcal{O}(x^{\alpha}) \text{ for } \alpha\ge 0.$$

Exercise 1.2. Show for $x\ge 2$ the following holds for constants A, B.

i)

$$\sum_{n\le x}\frac{\ln n}{n} = \frac{1}{2}\ln^2 x + A + \mathcal{O}\left(\frac{\ln x}{x}\right).$$

ii)

$$\sum_{2\le n\le x}\frac{1}{n\ln n} = \ln(\ln x) + B + \mathcal{O}\left(\frac{1}{x\ln x}\right).$$

Exercise 1.3. If $x\ge 2$, define the logaritheoremic integral of x as

$$Li(x) = \int_2^x \frac{dt}{\ln t}.$$

i) Prove that

$$Li(x) = \frac{x}{\ln x} + \int_2^x \frac{dt}{\ln^2 t} - \frac{2}{\ln 2}$$

and that more generally

$$Li(x) = \frac{x}{\ln x}\left(1 + \sum_{k=1}^{n-1}\frac{k!}{\ln^k x}\right) + n!\int_2^x \frac{dt}{\ln^{n+1} t} + C_n.$$

ii) If $x \geq 2$, prove that

$$\int_2^x \frac{dt}{\ln^n t} = \mathcal{O}\left(\frac{x}{\ln^n x}\right).$$

Exercise 1.4. Prove that the following are equivalent:

i)

$$\pi(x) = \frac{x}{\ln x} + \mathcal{O}\left(\frac{x}{\ln^2 x}\right),$$

ii)

$$\theta(x) = x + \mathcal{O}\left(\frac{x}{\ln x}\right).$$

Exercise 1.5. Show that the Bernoulli Polynomials satisfy

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Exercise 1.6. Show that the Bernoulli Polynomials $B_n(x)$ satisfy the difference equation

$$B_n(x+1) - B_n(x) = nx^{n-1} \text{ if } n \geq 1, \text{ in particular } B_n(0) = B_n(1) \text{ for } n \geq 2.$$

Exercise 1.7. Show that if $n \geq 2$ we have

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Exercise 1.8. Show that for $n \in \mathbb{N}_0$,

$$B_n(1-x) = (-1)^n B_n(x),$$

$$(-1)^n B_n(-x) = B_n(x) + nx^{n-1}.$$

Exercise 1.9. For $a \in \mathbb{C}$, $\mathrm{Re}(a) > 0$, we define the Hurtwitz zeta function by

$$\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

Show the following properties:

$$\mathrm{Res}[\zeta_H(s,a), 1] = 1, \qquad \zeta_H(-n,a) = -\frac{B_{n+1}(a)}{n+1}.$$

Exercise 1.10. Given that

$$\lim_{|y|\to\infty} |\Gamma(x+iy)| e^{\pi/2|y|} y^{1/2-x} = \sqrt{2\pi}, \quad x, y \in \mathbb{R}$$

and

$$\Gamma(z) = \sqrt{2\pi} e^{(x-1/2)\ln z - z}(1+o(1))$$

as $|z| \to \infty$, show that for $\sigma > 0,\ |arg(a)| < \frac{\pi}{2} - \delta,\ 0 < \delta \leq \frac{\pi}{2}$, we have

$$e^{-a} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} a^{-t}\Gamma(t)\ dt.$$

Exercise 1.11. If $F(x)$ is continuous and $f \in L^1(\mathbb{R})$ then we define the Fourier transform by

$$\hat{F}(u) = \int_{\mathbb{R}} F(x) e^{-2\pi i x u}\ dx.$$

If $\hat{F} \in L^1(\mathbb{R})$ then we have the Fourier inversion formula

$$F(x) = \int_{\mathbb{R}} \hat{F}(x) e^{2\pi i x u}\ du.$$

Show the following theorem:

Let $F \in L^1(\mathbb{R})$. Suppose that the series

$$\sum_{n\in\mathbb{Z}} F(n+v)$$

converges absolutely and uniformly in v, and that

$$\sum_{m\in\mathbb{Z}} |\hat{F}(m)| < \infty.$$

Then

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \hat{F}(n) e^{2\pi i n v}.$$

Exercise 1.12. Apply exercise 1.11 to show the Poisson Resummation Formula

$$\sum_{n\in\mathbb{Z}} e^{-(n+v)^2\pi/x} = \sqrt{x} \sum_{n\in\mathbb{Z}} e^{-n^2\pi x + 2\pi i n v}.$$

Exercise 1.13. Use exercise 1.10 to consider

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi x}.$$

What is the condition on σ in order to interchange summation and integration? Shifting the resulting contour to the left,find the correct polynomial terms in x describing the leading term in the small-x behavior. Note you might check your answer with exercise 1.12.

Exercise 1.14. Argue how the following behaves as $\beta \to 0$,

$$\sum_{n=1}^{\infty} e^{-\beta n^{\alpha}}, \ \alpha, \beta > 0.$$

Determine the leading three terms in the expansion.

Exercise 1.15. Find the leading three terms of the small-β behavior of

$$\sum_{n=1}^{\infty} \ln\left(1 - e^{-\beta n}\right).$$

Use that for $z \to 0$ we have

$$\zeta_R(1+z) = \frac{1}{z} + \gamma + \mathcal{O}(z), \quad \Gamma(z) = \frac{1}{z} - \gamma + \mathcal{O}(z),$$

and furthermore

$$\zeta_R'(0) = -\frac{1}{2}\ln(2\pi), \quad \zeta_R(-1) = -\frac{1}{12}.$$

Here, γ is Euler's constant given in exercise 1.1 i).

Remark 1.52. The above technique can be used very efficiently to derive the famous result by Hardy-Ramanujan,

$$p(n) = \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}} \left[1 + \mathcal{O}(n^{-1/2})\right],$$

giving the number $p(n)$ of ways to partition a large integer n into any number of smaller integers.
◇

Chapter 2

Spectral Functions of Differential Operators

In this chapter, we will explore spectral functions and a few of their uses. This will be motivated by considering certain differential operators and their eigenvalue problems.

2.1 Zeta Function of Riemann, revisited

Let $M = [0, L]$ and consider the one-dimensional Laplacian, that is $P = -\Delta = -\frac{\partial^2}{\partial x^2}$. We consider the associated Dirichlet boundary-value problem (BVP)

$$P\phi_l = -\frac{\partial^2}{\partial x^2}\phi_l(x) = \lambda_l \phi_l(x), \text{ with } \phi_l(0) = 0 = \phi_l(L).$$

Next we find solutions of this BVP:
$\lambda = 0:\ \phi(x) = Ax + B \Rightarrow A = 0 = B$ by boundary conditions (BC), so $\lambda = 0$ is not a solution.
$\lambda < 0,\ \lambda = -p^2:\ \phi(x) = A\cosh(px) + B\sinh(px) \Rightarrow A = 0 = B$ by BC, so $\lambda < 0$ is not a solution.
$\lambda > 0,\ \lambda = p^2:\ \phi(x) = A\cos(px) + B\sin(px)$, which by imposing BC we find

$$\phi(0) = 0 = A\cos(p \cdot 0) \Rightarrow A = 0$$

$$\phi(L) = 0 = B\sin(pL) \Rightarrow pL = n\pi,\ n \in \mathbb{Z}.$$

Now writing $\lambda_n = p_n^2$ we have

$$p_n = \frac{n\pi}{L},\ \lambda_n = \left(\frac{n\pi}{L}\right)^2,\ \phi(x) = \sin\left(\frac{n\pi x}{L}\right),\ n \in \mathbb{N}$$

as $n = 0$ gives a trivial solution, n and $-n$ give same eigenfunctions ϕ_n, and we want to consider normalizable linearly independent eignefunctions.

We denote the Zzeta function associated with this BVP for the given operator P as

$$\zeta_P(s) = \sum_{n=1}^{\infty} \left[\left(\frac{n\pi}{L} \right)^2 \right]^{-s} = \left(\frac{L}{\pi} \right)^{2s} \zeta_R(2s).$$

Further, we can find the following extremely useful representation of $\zeta_P(s)$,

$$\zeta_P(s) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} \right)^{-2} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \sum_{n=1}^{\infty} e^{-(n\pi/L)^2 t} \, dt$$

where

$$\sum_{n=1}^{\infty} e^{-(n\pi/L)^2 t} = \sum_{n=1}^{\infty} e^{-\lambda_n t} = K_P(t)$$

is the heat kernel, i.e. the fundamental solution to the heat equation.

Example 2.1. *The BVP*

$$-\Delta\phi_l(x) = \lambda_l \phi_l(x)$$

with Dirichlet BC results when describing a vibrating string. The wave equation in (1+1) *dimensions is given by*

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t), \;\; u(0,t) = 0 = u(L,t), \;\; u(x,0) = \psi_1(x), \;\; u_t(x,0) = \psi_2(x)$$

where u_t denotes the partial derivative of u with respect to t.
We begin searching for solutions by separation of variables. Let us assume we have

$$u(x,t) = \phi(x)T(t)$$

so that substituting into the differential equation we have

$$\phi(x)T''(t) = c^2\phi''(x)T(t) \Rightarrow \frac{\phi''(x)}{\phi(x)} = \frac{T''(t)}{c^2T(t)}.$$

From this we see that both sides must be equal to a constant, let us say $-\lambda$. Now letting $\lambda = p^2$ results in following two ordinary differential equations

$$\phi''(x) + p^2\phi(x) = 0,$$

$$T''(t) + c^2p^2T(t) = 0.$$

Now, reasoning as before, we find solutions are of the form

$$\phi(x) = C\cos(px) + D\sin(px),$$

$$T(t) = A\cos(cpt) + B\sin(cpt)$$

where A, B, C, D *are constants. Imposing Dirichlet BC produces* $\phi(0) = 0 = \phi(L)$ *so that we find*

$$\lambda_l = \left(\frac{l\pi}{L}\right)^2,\ l \in \mathbb{N}.$$

Further, the set of separated solutions is

$$u_l(x,t) = \left[A_l \cos\left(\frac{lc\pi t}{L}\right) + B_l \sin\left(\frac{lc\pi t}{L}\right)\right] \sin\left(\frac{l\pi x}{L}\right),\ l \in \mathbb{N}.$$

Remark 2.2. $c\sqrt{\lambda_l} = \dfrac{lc\pi}{L}$ plays the role of the frequency of the vibration of the string. ⋄

Remark 2.3. We can similarly describe the vibration of a drum, M, by considering

$$-\Delta\phi = \left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = \lambda\phi(x,y),\ \phi(x,y)|_{(x,y)\in\partial M} = 0.$$

⋄

2.2 What are Spectral Functions?

Let λ_l be eigenvalues of a suitable differential operator P with appropriate boundary conditions,

$$P\phi_l(x) = \lambda_l \phi_l(x). \tag{2.1}$$

The following quantities associated to the operator P, which we will naturally call spectral functions, seem to be of interest:

Functional Determinant:

$$e^{-\zeta_P'(0)} = \prod_{l=1}^{\infty} \lambda_l \tag{2.2}$$

Casimir Energy:

$$\zeta_P\left(-\frac{1}{2}\right) = \sum_{l=1}^{\infty} \lambda_l^{1/2} \tag{2.3}$$

Heat Kernel:

$$K_P(t) = \sum_{l=1}^{\infty} e^{-\lambda_l t} \tag{2.4}$$

Zeta Function:

$$\zeta_P(s) = \sum_{l=1}^{\infty} \lambda_l^{-s} \tag{2.5}$$

Remark 2.4. The first two quantities here should make one "uneasy." We will now motivate both of these to see why they, while at first glance seem problematic, do make sense. ⋄

2.2.1 Motivating the Functional Determinant

First, we shall motivate the definition of the Functional Determinant:

Let P be a finite dimensional $(N \times N)$ matrix with real eigenvalues, $\lambda_1, \ldots, \lambda_N$. Then

$$\ln(\det P) = \ln \prod_{l=1}^{N} \lambda_l = \sum_{l=1}^{N} \ln\lambda_l = -\frac{d}{ds} \sum_{l=1}^{N} e^{-s\ln\lambda_l}\Big|_{s=0} = -\frac{d}{ds} \sum_{l=1}^{N} \lambda_l^{-s}\Big|_{s=0} = -\zeta_P'(0).$$

So we see that $\det P = e^{-\zeta_P'(0)}$, so the definition does indeed make sense.

Remark 2.5. This concept generalizes to a large class of (partial) differential operators. ⋄

Example 2.6. *Find* $\det P$ *for* $P = -\frac{\partial^2}{\partial x^2}$, $M = [0, \pi]$, *with Dirichlet BC.*

Main Idea: We will use the transcendental equation for eigenvalues, $\lambda = \beta^2$, *instead of the eigenvalues themselves, i.e.*

$$\frac{\sin(\pi\beta)}{\beta} = 0,$$

where we divide by β *to ensure we remove the zero, and hence later pole, at zero. Then we write*

$$\zeta_P(s) = \frac{1}{2\pi i} \oint_\gamma \beta^{-2s} \frac{d}{d\beta} \ln \left[\frac{\sin(\pi\beta)}{\beta} \right] \, d\beta,$$

where γ *encloses all eigenvalues counterclockwise. Note that*

$$\frac{d}{d\beta} \ln \left[\frac{\sin(\pi\beta)}{\beta} \right]$$

will always give a singularity at β.

Remark 2.7. We analysis to see

$$\frac{d}{d\beta} \ln \left[\frac{\sin(\pi\beta)}{\beta} \right] = \frac{\pi \cos(\pi\beta)}{\sin \pi\beta} - \frac{1}{\beta} = \pi \cot(\pi\beta) - \frac{1}{\beta}$$

has poles at $\beta = l,\ l \in \mathbb{Z}\backslash\{0\}$ with residues

$$\operatorname{Res}\left[\pi \cot(\pi\beta), l\right] = 1.$$

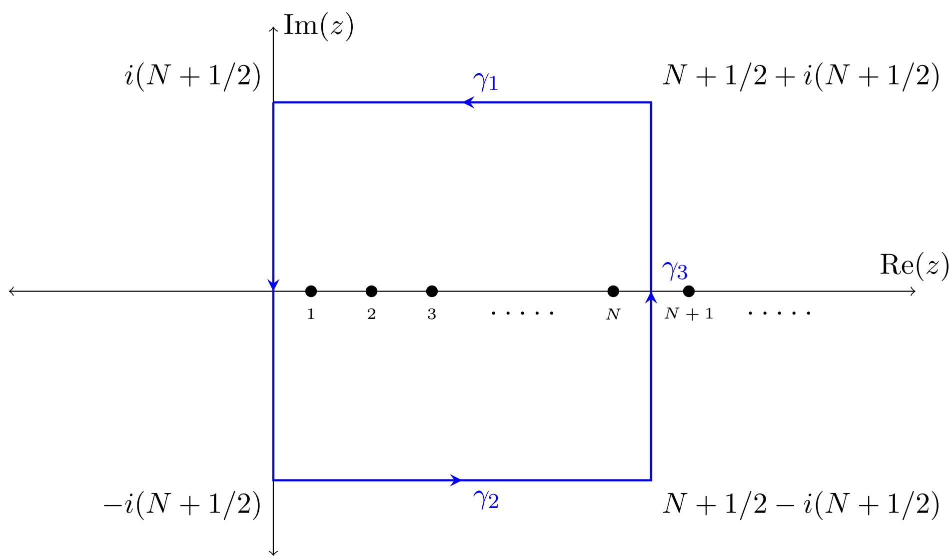
So by the Cauchy Residue theorem, A.2, we see the integrand has poles at each β with residue 1, and hence will be the sum of all these β raise to $-2s$, i.e. the zeta function associated to this operator. Therefore this representation is correct provided the infinite contour is justified. For reference, see [KL08]. ⋄

Remark 2.8. To gain intuition on this problem, notice for large β, $\dfrac{\sin(\pi\beta)}{\beta}$ gives exponential growth, however we take the natural logaritheorem to reduce to polynomial growth in β. Then by taking the derivative and multiplying by β^{-2s}, for s large enough we will have the limit go to zero. ⋄

We continue our example by considering

$$\zeta_P(s) = \frac{1}{2\pi i}\oint_{\gamma_N} \beta^{-2s}\left[\pi\cot(\pi\beta) - \frac{1}{\beta}\right]\, d\beta = \frac{1}{2i}\oint_{\gamma_N} \beta^{-2s}\cot(\pi\beta)\, d\beta + \frac{1}{2\pi i}\oint_{\gamma_N} \beta^{-2s-1}\, d\beta$$

with the contour γ_N given below



First we consider each of the contours γ_1, γ_2, and γ_3.

γ_1: We let $\beta = x + i(N+1/2)$ for $N + 1/2 > x > 0$ to see

$$|\cot(\pi\beta)| = \left|\frac{e^{i\pi\beta}+e^{-i\pi\beta}}{e^{i\pi\beta}-e^{-i\pi\beta}}\right| = \left|\frac{e^{2i\pi\beta}+1}{e^{2i\pi\beta}-1}\right| = \left|\frac{e^{-2\pi(N+1/2)}e^{2i\pi x}+1}{e^{-2\pi(N+1/2)}e^{2i\pi x}-1}\right| \le \frac{1+e^{-2\pi(N+1/2)}}{1-e^{-2\pi(N+1/2)}}.$$

γ_2: We let $\beta = x - i(N+1/2)$ for $0 < x < N+1/2$ to see

$$|\cot(\pi\beta)| = \left|\frac{e^{i\pi\beta}+e^{-i\pi\beta}}{e^{i\pi\beta}-e^{-i\pi\beta}}\right| = \left|\frac{1+e^{-2i\pi\beta}}{1-e^{-2i\pi\beta}}\right| = \left|\frac{1+e^{-2\pi(N+1/2)}e^{-2i\pi x}}{1-e^{-2\pi(N+1/2)}e^{-2i\pi x}}\right| \le \frac{1+e^{-2\pi(N+1/2)}}{1-e^{-2\pi(N+1/2)}}.$$

γ_3: We let $\beta = (N+1/2) + iy$ for $-N-1/2 < y < N+1/2$ to see

$$|\cot(\pi\beta)| = |\cot[\pi(N+1/2)+\pi i y]| = |\cot(\pi/2+\pi i y)| = |\tanh(\pi y)| \le 1.$$

So we can conclude that

$$\left|\oint_{\gamma_1+\gamma_2+\gamma_3} \beta^{-2s}\cot(\pi\beta)\ d\beta\right| \leq 2|N+1/2|^{-2s+1}\frac{1+e^{-2\pi(N+1/2)}}{1-e^{-2\pi(N+1/2)}} + |N+1/2|^{-2s+1},$$

which goes to 0 as $N\to\infty$ for $\mathrm{Re}(s) > \frac{1}{2}$, and

$$\left|\oint_{\gamma_1+\gamma_2+\gamma_3} \beta^{-2s-1}\ d\beta\right| = \mathcal{O}(N^{-2s}),$$

which goes to 0 as $N\to\infty$ for $\mathrm{Re}(s)>0$. Hence the only contribution will come from along the imaginary axis, which we will treat next.
Along the imaginary axis: For $\mathrm{Re}(s) > \frac{1}{2}$ we calculate

$$\zeta_P(s) = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} \beta^{-2s}\frac{d}{d\beta}\ln\left[\frac{\sin(\pi\beta)}{\beta}\right]\ d\beta$$

$$\underset{\beta=\pm iy}{=} \frac{1}{2\pi i}\left[-\int_0^\infty (iy)^{-2s}\frac{d}{dy}\ln\left[\frac{\sin(i\pi y)}{iy}\right]\ dy + \int_0^\infty (-iy)^{-2s}\frac{d}{dy}\ln\left[\frac{\sin(-i\pi y)}{-iy}\right]\ dy\right]$$

$$= \frac{1}{2\pi i}\left(-e^{-i\pi s}+e^{i\pi s}\right)\int_0^\infty y^{-2s}\frac{d}{dy}\ln\left[\frac{\sin(i\pi y)}{iy}\right]\ dy$$

$$= \frac{\sin(\pi s)}{\pi}\int_0^\infty y^{-2s}\frac{d}{dy}\ln\left[\frac{\sin(i\pi y)}{iy}\right]\ dy$$

$$= \frac{\sin(\pi s)}{\pi}\int_0^1 y^{-2s}\frac{d}{dy}\ln\left[\frac{\sin(i\pi y)}{iy}\right]\ dy + \frac{\sin(\pi s)}{\pi}\int_1^\infty y^{-2s}\frac{d}{dy}\ln\left[\frac{\sin(i\pi y)}{iy}\right]\ dy$$

where the first integral is valid for $\mathrm{Re}(s)<1$ and the second integral is valid for $\mathrm{Re}(s) > \frac{1}{2}$.

Remark 2.9. To conclude where each integral is valid in the last equality, note we must treat the large- and small-y behavior of the previous integral separately. See exercise 2.2 for reference. ⋄

We continue by computing for the strip $\frac{1}{2} < \mathrm{Re}(s) < 1$ in order to find the analytic continuation desired. Let us consider the large-y behavior of the integrand:

$$\frac{\sin(i\pi y)}{iy} = \frac{e^{-\pi y}-e^{\pi y}}{(2i)iy} = \frac{e^{\pi y}-e^{-\pi y}}{2y} = \frac{e^{\pi y}}{2y}\left(1-e^{-2\pi y}\right).$$

Remark 2.10. Take note that the goal here is to always factor into something multiplied by a factor of the form $1 \pm (\textit{small})$ above. ⋄

We therefore write

$$\int_1^\infty y^{-2s}\frac{d}{dy}\ln\left[\frac{\sin(i\pi y)}{iy}\right]\ dy = \int_1^\infty y^{-2s}\frac{d}{dy}\ln\left[\frac{e^{\pi y}}{2y}\left(1-e^{-2\pi y}\right)\right]\ dy$$

$$= \pi\int_1^\infty y^{-2s}\ dy - \int_1^\infty y^{-2s-1}\ dy + \int_1^\infty y^{-2s}\frac{d}{dy}\ln\left(1-e^{-2\pi y}\right)\ dy$$

$$= \frac{\pi}{2s-1} - \frac{1}{2s} + \int_1^\infty y^{-2s}\frac{d}{dy}\ln\left(1-e^{-2\pi y}\right)\ dy.$$

Summarizing, we have found the following using the analytic continuation above

$$\zeta_P(s) = \zeta_R(2s) = \frac{\sin(\pi s)}{\pi}\int_0^1 y^{-2s}\frac{d}{dy}\ln\left[\frac{e^{\pi y}}{2y}\left(1-e^{-2\pi y}\right)\right]\ dy +$$

$$+\frac{\sin(\pi s)}{2s-1} - \frac{\sin(\pi s)}{2\pi s} + \frac{\sin(\pi s)}{\pi}\int_1^\infty y^{-2s}\frac{d}{dy}\ln\left(1-e^{-2\pi y}\right)\ dy.$$

Now we can finally use this form to calculate what is needed to find the functional determinant of our operator P! We get the following result, noting the all pieces above can now be evaluated at $s=0$, as the last integral will now be exponentially damped for large y values, so the values of s will not effect the integral, and it will hence be finite. We compute

$$\zeta_P'(0) = 2\zeta_R'(0) = \ln\left[\frac{e^\pi}{2}(1-e^{-2\pi})\right] - \ln\pi - \pi - \ln(1-e^{-2\pi}) = -\ln(2\pi).$$

In the expression above, we note the first two terms come from the first integral in the formula found for $\zeta_P(s)$, the third term comes from the second term in the formula found for $\zeta_P(s)$, and the last comes from the last integral in the formula found for $\zeta_P(s)$ evaluated at 1. Hence we can conclude that

$$\det P = e^{-\zeta_P'(0)} = 2\pi.$$

Remark 2.11. Note we also find a very helpful value for later from here, namely

$$\zeta_R'(0) = -\frac{1}{2}\ln(2\pi). \tag{2.6}$$

◇

2.2.2 Motivating the Casimir Energy

Notice first that by the formula given in (2.3) we have

$$\frac{1}{2}\zeta_P\left(-\frac{1}{2}\right) = \frac{1}{2}\sum_{l=1}^\infty \lambda_l^{1/2} := E^P_{Cas}.$$

We will motivate this definition through an example by the Dutch physicist Hendrik Casimir.

Example 2.12 (Casimir 1948)**.** *Suppose we have two plates in a vacuum parallel to one another in the x-coordinate. Further, let one be positioned at $x = 0$ and the other at $x = a$. Then we have that virtual particles satisfy*

$$P_{u_k} = -\Delta u_k = \lambda_k u_k, \quad u_k(0, y, z) = 0 = u_k(a, y, z),$$

$$u_k(x, y, z) = u_k(x, y + nL, z) = u_k(x, y, z + nL), \; n \in \mathbb{N},$$

where for the last conditions we have rolled up into a taurus with periodic BC. Now we find that

$$u_k(x, y, z) = Ae^{(2\pi i l_1/L)y} e^{(2\pi i l_2/L)z} \sin\left(\frac{\pi l}{a} x\right), \quad (l_1, l_2) \in \mathbb{Z}^2, \quad l \in \mathbb{N}$$

and the eigenvalues are given by

$$\lambda_k = \left(\frac{2\pi l_1}{L}\right)^2 + \left(\frac{2\pi l_2}{L}\right)^2 + \left(\frac{\pi l}{L}\right)^2.$$

Now we consider the zeta function for these eigenvalues. Then for $\mathrm{Re}(s) > \frac{3}{2}$ *we have*

$$\frac{\zeta_P(s)}{L^2} = \frac{1}{L^2} \sum_{(l_1,l_2)\in\mathbb{Z}^2} \sum_{l=1}^{\infty} \left[\left(\frac{2\pi l_1}{L}\right)^2 + \left(\frac{2\pi l_2}{L}\right)^2 + \left(\frac{\pi l}{L}\right)^2\right]^{-s}$$

and sending $L \to \infty$ and changing to polar coordinates we continue

$$= \frac{1}{L^2} \left(\frac{L}{2\pi}\right)^2 \sum_{l=1}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[k_1^2 + k_2^2 + \left(\frac{\pi l}{a}\right)^2\right]^{-s} \; dk_1 \; dk_2$$

$$= \frac{1}{(2\pi)^2} \sum_{l=1}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left[r^2 + \left(\frac{\pi l}{a}\right)^2\right]^{-s} r \; dr \; d\theta = \frac{1}{(2\pi)^2} 2\pi \frac{1}{2(1-s)} \sum_{l=1}^{\infty} \left[r^2 + \left(\frac{\pi l}{a}\right)^2\right]^{-s+1} \Bigg|_0^{\infty}$$

$$= -\frac{1}{4\pi(1-s)} \sum_{l=1}^{\infty} \left(\frac{\pi l}{a}\right)^{2-2s} = -\frac{1}{4\pi(1-s)} \left(\frac{\pi}{a}\right)^{2-2s} \zeta_R(2s-2).$$

Now we consider $s = -\frac{1}{2}$ *and calculate to find*

$$\zeta_P\left(-\frac{1}{2}\right) = -\frac{2}{12\pi}\left(\frac{\pi}{a}\right)^3 \zeta_R(-3) = -\frac{\pi^2}{720a^3}. \tag{2.7}$$

Further we can also find the Casimir force defined as

$$F_{Cas}^P = \frac{\partial}{\partial a}\left(\frac{E_{Cas}^P}{L^2}\right) = \frac{\pi^2}{240a^4}.$$

2.3 Functional Determinant of Ordinary Second Order Differential Operators

We now introduce a few elements of the problem that we will be studying.
We begin by considering a simple Sturm-Liouville Operator BVP, namely

$$Ly(x) = -\frac{d^2}{dx^2}y(x) + V(x)y(x) = \lambda y(x), \quad I = [a,b], \quad V \text{ real}, \quad V \in C^0([a,b]), \tag{2.8}$$

with one of the following two BC

$$\text{(Separated BC)} \quad y(a)\cos\alpha + y'(a)\sin\alpha = 0, \quad y(b)\cos\beta + y'(b)\sin\beta = 0, \tag{2.9}$$

$$\text{(Nonseparated BC)} \quad y(a) = y(b), \quad y'(a) = y'(b), \tag{2.10}$$

where WLOG we may suppose that $a = 0$, $b = \pi$ by way of the substitution $t = \dfrac{\pi(x-a)}{b-a}$.

Remark 2.13. This brings up an interesting question in the real of Inverse Spectral Problems. Namely,

If we know eigenvalues (EV), what can we say about the potential $V(x)$?

This is a question we will explore presently. ◇

2.3.1 Eigenvalues of Sturm-Liouville Type Operators

Lemma 2.14. *Eigenfunctions $y_{\lambda_1}(x)$ and $y_{\lambda_2}(x)$, $\lambda_1 \neq \lambda_2$ are orthogonal over the Hilbert space, $L^2([0,\pi])$, of square-integrable functions where the inner product is given by*

$$\langle y_{\lambda_1}, y_{\lambda_2}\rangle_{L^2} = \int_0^\pi y_{\lambda_1}(x) y_{\lambda_2}(x)\ dx = 0 \text{ if } \lambda_1 \neq \lambda_2.$$

Proof. For $f, g \in C^2([0,\pi])$, we evaluate

$$\int_0^\pi (Lf(x))g(x)\ dx = \int_0^\pi \left(-\frac{d^2}{dx^2}f(x) + V(x)f(x)\right) g(x)\ dx$$

$$= -f'(x)g(x)\Big|_0^\pi + \int_0^\pi \left(\frac{d}{dx}f(x)\frac{d}{dx} + v(x)f(x)g(x)\right)\ dx$$

$$= -f'(x)g(x)\Big|_0^\pi + -f(x)g'(x)\Big|_0^\pi + \int_0^\pi \left(-f(x)\frac{d^2}{dx^2} + v(x)f(x)g(x)\right)\ dx$$

$$= -f'(\pi)g(\pi) + f'(0)g(0) + f(\pi)g'(\pi) - f(0)g'(0) + \int_0^\pi f(x)(Lg(x))\ dx$$

$$= W_\pi\{f,g\} - W_0\{f,g\} + \int_0^\pi f(x)(Lg(x))\ dx,$$

where $W_a\{f,g\}$ represents the Wronski determinant (or Wronskian) given by

$$W_a\{f,g\} = \det\begin{pmatrix} f(a) & g(a) \\ f'(a) & g'(a) \end{pmatrix}.$$

Now if f,g satisfy the separated BC (2.10), then

$$W_\pi\{f,g\} = W_0\{f,g\}.$$

So for $f(x) = y_{\lambda_1}(x)$ and $g(x) = y_{\lambda_2}(x)$ we see

$$\int_0^\pi (Lf(x))g(x)\ dx = \lambda_1 \int_0^\pi f(x)g(x)\ dx = \lambda_2 \int_0^\pi f(x)g(x)\ dx,$$

so that

$$(\lambda_1 - \lambda_2)\int_0^\pi f(x)g(x)\ dx = 0.$$

As $lambda_1 \neq \lambda_2$ by assumption, it must be

$$\langle f,g\rangle_{L^2} = \int_0^\pi f(x)g(x)\ dx = 0.$$

□

Lemma 2.15. *Eigenvalues of the BVP* (2.8) *with BC* (2.9) *are real.*

Proof. Let $\lambda_1 = u + iv$, $v \neq 0$ be a complex eigenvalue. Taking the complex conjugate of (2.8) and (2.9) shows

$$-\frac{d^2}{dx^2}\overline{y}(x) + V(x)\overline{y}(x) = \overline{\lambda}\overline{y}(x), \quad \overline{y}(a)\cos\alpha + \overline{y}'(a)\sin\alpha = 0, \quad \overline{y}(b)\cos\beta + \overline{y}'(b)\sin\beta = 0,$$

therefore $\overline{y}_{\lambda_1}(x)$ is an eigenfunction with eigenvalue $\overline{\lambda}_1$. From lemma 2.14 we have

$$\int_0^\pi y_{\lambda_1}(x)\overline{y}_{\lambda_1}(x)\ dx = \int_0^\pi |y_{\lambda_1}(x)|^2\ dx = 0$$

which implies $y_{\lambda_1} \equiv 0$ and thus the assertion holds. □

Remark 2.16. This theorem can easily be seen if one knows a bit of operator theory, specifically the basics of spectral theory of self-adjoint operators. For reference, see [Wei80]. For our discussion here, it is not necessary to use such strong results. $\diamond$

Theorem 2.17. *If $V(x)$ (the potential energy) is a continuous function on the interval $I = [a, b]$, then for every α there exists an unique solution $y(x, \lambda)$, $a \leq x \leq b$, of equation (2.8) such that*

$$y(\alpha, \lambda) = \sin\alpha, \quad y'(\alpha, \lambda) = -\cos\alpha.$$

Furthermore, for every fixed $x \in [a, b]$, the function $y(x, \lambda)$ is an entire function of λ.

Remark 2.18. We will first motivate the proof of theorem 2.17.

The differential equation reads $y''(x, \lambda) = (V(x) - \lambda)y(x, \lambda)$. Integrating once we have

$$y'(x, \lambda) = y'(a, \lambda) + \int_a^x (V(u) - \lambda)y(u, \lambda)\ du,$$

and integrating a second time gives

$$\begin{aligned} y(x, \lambda) &= y(a, \lambda) + \int_a^x y'(a, \lambda)\ dt + \int_a^x \int_a^t (V(u) - \lambda)y(u, \lambda)\ du\ dt \\ &= y(a, \lambda) + (x - a)y'(a, \lambda) + \int_a^x \int_a^t (V(u) - \lambda)y(u, \lambda)\ du\ dt. \end{aligned}$$

We now change the integration range such that for $u \in [a, x]$, t is integrated from u to x. Then we have that

$$\begin{aligned} \int_a^x \int_a^t (V(u) - \lambda)y(u, \lambda)\ du\ dt &= \int_a^x \int_u^x (V(u) - \lambda)y(u, \lambda)\ dt\ du \\ &= \int_a^x (x - u)(V(u) - \lambda)y(u, \lambda)\ du. \end{aligned}$$

Summarizing, we have the integral equation

$$y(x, \lambda) = y(a, \lambda) + (x - a)y'(a, \lambda) + \int_a^x (x - u)(V(u) - \lambda)y(u, \lambda)\ du.$$

We will let this be the starting point of our proof. $\diamond$

Proof. (Picard's Method of Successive Approximation) Let $y_0(x, \lambda) = \sin\alpha - (x - a)\cos\alpha$ and define for $n > 0$,

$$y_n(x, \lambda) = y_0(x, \lambda) + \int_a^x (x - t)(V(t) - \lambda)y_{n-1}(t, \lambda)\ dt.$$

Since V is continuous, we have that $|V(x)| < M$ on $[a,b]$ for some M. Furthermore, let N and K be such that $|\lambda| \leq N$ and $|y_0(x,\lambda)| \leq K$. We estimate

$$|y_1(x,\lambda) - y_0(x,\lambda)| \leq \int_a^x (M+N)K(x-t)\ dt = \frac{1}{2}(M+N)K(x-a)^2.$$

For $n \geq 2$ we have

$$y_n(x,\lambda) - y_{n-1}(x,\lambda) = \int_a^x (x-t)(V(t)-\lambda)(y_{n-1}(t,\lambda) - y_{n-2}(t,\lambda))\ dt$$

so

$$|y_n(x,\lambda) - y_{n-1}(x,\lambda)| \leq (M+N)(b-a)\int_a^x |y_{n-1}(t,\lambda) - y_{n-2}(t,\lambda)|\ dt.$$

This shows

$$|y_2(x,\lambda) - y_1(x,\lambda)| \leq \frac{1}{2}(M+N)^2K(b-a)\int_a^x (t-a)^2\ dt = \frac{K}{3!}(M+N)^2(b-a)(x-a)^3,$$

and inductively

$$|y_n(x,\lambda) - y_{n-1}(x,\lambda)| \leq \frac{K}{(n+1)!}(M+N)^n(b-a)^{n-1}(x-a)^{n+1}.$$

Therefore the series

$$y(x,\lambda) = y_0(x,\lambda) + \sum_{n=1}^{\infty}(y_n(x,\lambda) - y_{n-1}(x,\lambda))$$

converges uniformly in λ for $|\lambda| \leq N$ and x for $a \leq x \leq b$. Further, noting for $n \geq 2$ we have

$$y_n'(x,\lambda) - y_{n-1}'(x,\lambda) = \int_a^x (V(t)-\lambda)(y_{n-1}(x,\lambda) - y_{n-2}(x,\lambda))\ dt$$

and

$$y_n''(x,\lambda) - y_{n-1}''(x,\lambda) = (V(x)-\lambda)(y_{n-1}(x,\lambda) - y_{n-2}(x,\lambda)),$$

the same remarks on convergence hold for the series encountered for $y'(x,\lambda)$ and $y''(x,\lambda)$.

The function $y(x,\lambda)$ so constructed satisfies the equation

$$y''(x,\lambda) = \sum_{n=1}^{\infty}(y_n''(x,\lambda) - y_{n-1}''(x,\lambda)) = (V(x)-\lambda)y(x,\lambda).$$

It also clearly satisfies the initial condition by construction and the statement about being analytic follows from the uniform convergence shown. It remains to show uniqueness.

Let $\tilde{y}(x,\lambda)$ be another solution. It follows that

$$|\tilde{y}(x,\lambda) - y(x,\lambda)| = \left| \int_a^x (x-u)(V(u)-\lambda)(\tilde{y}(u,\lambda) - y(u,\lambda)) \, du \right|$$
$$\leq \frac{1}{2}(M+N)(x-a)^2 \max_{u\in[a,x]} |\tilde{y}(u,\lambda) - y(u,\lambda)|$$

which implies

$$\max_{u\in[a,x]} |\tilde{y}(u,\lambda) - y(u,\lambda)| \leq \frac{1}{2}(M+N)(x-a)^2 \max_{u\in[a,x]} |\tilde{y}(u,\lambda) - y(u,\lambda)|.$$

As $\tilde{y} \neq y$ by assumption, we divide to conclude

$$1 \leq \frac{1}{2}(M+N)(x-a)^2,$$

which for x close enough to a is a contradiction. Thus the solution is indeed unique. □

Lemma 2.19. *Let $\psi(x,\lambda)$ denote the solution of the IVP*

$$-\frac{\partial^2}{\partial x^2}\psi + V(x)\psi = \lambda\psi, \quad \psi(0,\lambda) = 0, \quad \psi'(0,\lambda) = 1.$$

If $\lambda = s^2$, then we find the integral equation

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda) \, d\tau.$$

Proof. We multiply the differential equation by $\sin[s(x-\tau)]$ and integrate,

$$\int_0^x \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda) \, d\tau = \int_0^x \sin[s(x-\tau)]\psi''(\tau,\lambda) \, d\tau + s^2 \int_0^x \sin[s(x-\tau)]\psi(\tau,\lambda) \, d\tau.$$

Two partial integrations of the first term on the right produces

$$\int_0^\pi \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda) \, d\tau = \sin[s(x-\tau)]\psi'(\tau,\lambda)\Big|_0^x - \left[\frac{d}{d\tau}\sin[s(x-\tau)]\right]\psi(\tau,\lambda)\Big|_0^x$$
$$= -\sin(sx) + s\psi(x,\tau).$$

Solving for $\psi(x,\lambda)$ gives the desired result. □

Remark 2.20. We call

$$\frac{\sin(sx)}{s}$$

from lemma 2.19 the free solution (as the potential energy, V, is 0). The integral portion is the Green's function. ⋄

Lemma 2.21. *Let $s = \sigma + it$. Then there exists $s_0 > 0$ such that for $|s| > s_0$ one has the estimates*

$$\psi(x,\lambda) = \mathcal{O}(|s|^{-1}e^{|t|x})$$

and more precisely

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} + \mathcal{O}(|s|^{-2}e^{|t|x}).$$

All these estimates hold uniformly in x for $0 \le x \le \pi$.

Proof. We write

$$\psi(x,\lambda) = |s|^{-1}e^{|t|x}F(x).$$

Then from lemma 2.19 we have

$$F(x) = \frac{\sin(sx)}{s}|s|e^{-|t|x} + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)e^{|t|(\tau-x)}F(\tau)\ d\tau.$$

It follows for $\mu = \max\limits_{0\le x\le\pi} |F(x)|$ that

$$\mu \le 1 + \frac{\mu}{|s|}\int_0^\pi |V(\tau)|\ d\tau \Rightarrow \mu \le \left(1 - |s|^{-1}\int_0^\pi |V(\tau)|\ d\tau\right)^{-1}$$

as long as $|s| > \displaystyle\int_0^\pi |V(\tau)|\ d\tau$. Thus, for $V \in C^0([0,\pi])$ we have shown

$$\psi(x,\lambda) = \mathcal{O}(|s|^{-1}e^{|t|x})$$

for $|s| > s_0 = \displaystyle\int_0^\pi |V(\tau)|\ d\tau$ large enough.
Substituting back into lemma 2.19 we continue

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)\mathcal{O}(|s|^{-1}e^{|t|x})\ d\tau$$

$$= \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^x V(\tau)\mathcal{O}(|s|^{-1}e^{|t|x})\ d\tau = \frac{\sin(sx)}{s} + \mathcal{O}(|s|^{-2}e^{|t|x}).$$

□

Remark 2.22. Lemma 2.21 shows that for large λ, $\lambda\psi$ will be much larger than $V(x)\psi$ (if no singular potential). The leading term of the λ-expansion comes from the ODE $-\dfrac{\partial^2\psi}{\partial x^2} = \lambda\psi$ or the $\dfrac{\sin(xs)}{s}$ part. See exercise 2.6 for improved asymptotic estimates. ◇

Theorem 2.23. *Consider the eigenvalue problem (for simplicity we assume $s_n^2 > 0$ and $V \in C^1([0,\pi])$)*

$$-\frac{d^2\psi_n(x)}{dx^2} + V(x)\psi_n(x) = s_n^2\psi_n(x), \quad \psi_n(0) = 0 = \psi_n(\pi).$$

For large n, the eigenvalues have the form

$$s_n = n + \frac{\alpha_1}{n} + \mathcal{O}(n^{-2}), \text{ where } \alpha_1 = \frac{1}{2\pi}\int_0^\pi V(\tau)\ d\tau.$$

Further, the normalized eigenfunctions satisfy

$$V_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx) + \mathcal{O}(n^{-1}).$$

Proof. In lemma 2.19 we have shown that

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^\infty \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda)\ d\tau$$

solves the IVP

$$-\frac{\partial^2}{\partial x^2}\psi + V(x)\psi = \lambda\psi, \quad \psi(0,\lambda) = 0, \quad \psi'(0,\lambda) = 1.$$

We are going to relate this IVP to the previous BVP by noting we already have $\psi(0) = 0$ so now we look for values of λ such that $\psi(\pi,\lambda) = 0$ in order to satisfy the needed boundary condition. Therefore, by substituting $\psi(x,\lambda)$ into the second boundary condition we obtain an equation for eigenvalues λ_n. The equation now reads

$$0 = s\psi(\pi,\lambda) = \sin(s\pi) + \int_0^\pi \sin[s(\pi-\tau)]V(\tau)\psi(\tau,s^2)\ d\tau$$

$$= \sin(s\pi)\left[1 + \int_0^\pi \cos(s\tau)V(\tau)\psi(\tau,s^2)\ d\tau\right] - \cos(s\pi)\int_0^\pi \sin(s\tau)V(\tau)\psi(\tau,s^2)\ d\tau.$$

In lemma 2.21 (for $s \in \mathbb{R}$), we have shown

$$\psi(\tau,s^2) = \frac{\sin(s\tau)}{s} + \mathcal{O}(s^{-2}).$$

We continue our calculation,

$$0 = \sin(s\pi)\left[1 + \frac{1}{s}\int_0^\pi \cos(s\tau)\sin(s\tau)V(\tau)\ d\tau\right] - \frac{\cos(s\pi)}{s}\int_0^\pi \sin^2(s\tau)V(\tau)\ d\tau + \mathcal{O}(s^{-2})$$

$$= \sin(s\pi) - \frac{\cos(s\pi)}{2s}\int_0^\pi V(\tau)d\tau + \frac{\sin(s\pi)}{2s}\int_0^\pi \sin(2s\tau)V(\tau)d\tau + \frac{\cos(s\pi)}{2s}\int_0^\pi \cos(2s\tau)V(\tau)d\tau + \mathcal{O}(s^{-2}).$$

Now let us analyze how the right integrals go to zero:

$$\int_0^\pi \sin(2s\tau)V(\tau)\ d\tau = -\frac{1}{2s}\int_0^\pi \left[\frac{d}{d\tau}\cos(2s\tau)\right] V(\tau)\ d\tau$$

$$= -\frac{1}{2s}\left[\cos(2s\tau)V(\tau)\Big|_0^\pi - \int_0^\pi \cos(2s\tau)\frac{d}{d\tau}(V(\tau))\ d\tau\right] = \mathcal{O}(s^{-1})$$

with the last integral following similarly. Hence we have found

$$\sin(s\pi) - \frac{\cos(s\pi)}{2s}\int_0^\pi V(\tau)\ d\tau + \mathcal{O}(s^{-2}) = 0$$

or

$$\sin(s\pi) - \pi\alpha_1\frac{\cos(s\pi)}{s} + \mathcal{O}(s^{-2}) = 0.$$

For large s, this vanishes close to the integers, which we note implies the existence of infinitely many eigenvalues. Further, for sufficiently large n, the corresponding eigenvalues are simple, which we see by taking the derivative with respect to s of the previous equation to see

$$\pi\cos(s\pi) + \pi^2\alpha_1\frac{\sin(s\pi)}{s} + \pi\alpha_1\frac{\cos(s\pi)}{s^2} + \mathcal{O}(s^{-3}) = 0,$$

which does not vanish for $s \approx n,\ n$ large.
Now we determine the large-n behavior of $s_n = n + \delta_n$ by solving the equation found for $\sin(s\pi)$ and then dividing by $\cos(s\pi)$ to see

$$\tan(s\pi) = \frac{\pi\alpha_1}{s} + \mathcal{O}(s^{-2}),$$

and substituting we have

$$\tan((n+\delta_n)\pi) = \tan(\delta_n\pi) = \frac{\pi\alpha_1}{n+\delta_n} + \mathcal{O}(n^{-2}).$$

Finally, by considering the series expansion of inverse tangent and comparing powers, one obtains

$$\delta_n\pi = \frac{\pi\alpha_1}{n} + \mathcal{O}(n^{-2})$$

so that we conclude

$$\delta_n = \frac{\alpha_1}{n} \text{ and } s_n = n + \frac{\alpha_1}{n} + \mathcal{O}(n^{-2}).$$

For eigenfunctions $\psi_n(x) = \psi_n(x, \lambda_n)$, this implies

$$\psi_n(x) = \sin(nx + \alpha_1 x n^{-1} + \mathcal{O}(n^{-2})) + \mathcal{O}(n^{-2}) = \frac{\sin(nx)}{n} + \mathcal{O}(n^{-2}).$$

The normalization follows from

$$\alpha_n := \int_0^\pi \psi_n(x)\ dx = \frac{1}{n^2}\int_0^\pi \sin^2(nx)\ dx + \mathcal{O}(n^{-3}) = \frac{\pi}{2n^2} + \mathcal{O}(n^{-3})$$

so that

$$\frac{1}{\alpha_n} = \sqrt{\frac{2}{\pi}} n + \mathcal{O}(n^0)$$

and

$$V_n(x) = \frac{\psi_n(x)}{\alpha_n} = \sqrt{\frac{2}{\pi}} \sin(nx) + \mathcal{O}(n^0).$$

□

Remark 2.24. We can improve the asymptotic estimates in theorem 2.23 further. See exercise 2.7 for reference. ⋄

Now we recall the following result from basic complex analysis. For the proof, see theorem A.7. Notation: Given any closed, simple (i.e. not self-intersecting) curve Γ, we will denote the interior of the region enclosed by Γ as (Γ). The union of the curve (the boundary of the interior) and the interior will be denoted $[\Gamma]$.

Theorem 2.25 (Rouché's theorem)**.** *Suppose f, g are holomorphic on $[\Gamma]$, where Γ is a closed, simple curve, and $|f| > |g|$ for all z on Γ. Then the functions f and $f+g$ have the same number of roots inside Γ.*

Theorem 2.26. *Inside a disc, D_R, of radius R, the Dirichlet eigenvalue problem described in theorem 2.23 has only finitely many eigenvalues.*

Proof. Recall that by lemma 2.21 we have the estimate

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} + \mathcal{O}(|s|^{-2}e^{|t|x}).$$

Choose $R = N + \dfrac{1}{2}$. For N large enough the first term dominates the second on D_R and eigenvalues are determined by $\psi(\pi,\lambda) = 0$. Denote

$$f(s) = \frac{\sin(\pi s)}{s} \text{ and } g(s) = \mathcal{O}(|s|^{-2}e^{|t|\pi}).$$

Then we have be Rouché's theorem that $\psi(\pi,\lambda)$ has the same number of zeros in D_R as $f(s)$. Hence there are exactly $2N$ zeros inside of D_R, showing the assertion. □

2.3.2 Zeta Functions of Sturm-Liouville Type Operators

Remark 2.27. The method illustrated below is justified for any general Sturm-Liouville operator by exercise 2.5. ◇

As before, we consider the BVP given by

$$P\psi_n(x) = -\frac{d^2}{dx^2}\psi_n(x) + V(x)\psi_n(x) = \lambda_n\psi_n(x), \quad \psi_n(0) = 0 = \psi_n(\pi).$$

Definition 2.28. *The zeta function associated with the operator P above is given by*

$$\zeta_P(s) = \sum_{n=1}^{\infty} \lambda_n^{-s} \textit{ for } \mathrm{Re}(s) > \frac{1}{2}.$$

The procedure for the evaluation of $\zeta_P'(0)$ will be as follows for

$$\psi(x,\lambda) = \lambda\psi(x,\lambda), \quad \lambda \in \mathbb{C}.$$

i) Initial Conditions: $\psi(0,\lambda) = 0, \quad \frac{d}{dx}\psi(x,\lambda)\Big|_{x=0} = 1$

ii) Unique Solution: By theorem 2.17, given the initial conditions above we have a unique solution $\psi(x,\lambda)$ whose analytic dependence is on λ.

iii) Eigenvalues: Eigenvalues for this BVP are determined by $\psi(\pi,\lambda) = 0$. Note eigenvalues are poles of the equation

$$\frac{d}{d\lambda}\ln\psi(\pi,\lambda) = \frac{\psi'(\pi,\lambda)}{\psi(\pi,\lambda)}.$$

iv) Expansion about $\lambda = \lambda_l$ for simple eigenvalue:

$$\frac{d}{d\lambda}\ln\psi(\pi,\lambda) = \frac{\psi'(\pi,\lambda_l) + \mathcal{O}(\lambda-\lambda_l)}{(\lambda-\lambda_l)\psi'(\pi,\lambda_l) + \mathcal{O}((\lambda-\lambda_l)^2)} = \frac{1}{\lambda-\lambda_l} + \mathcal{O}((\lambda-\lambda_l)^0).$$

v) Expansion about $\lambda = \lambda_l$ for n-fold degenerate eigenvalue:

$$\frac{d}{d\lambda}\ln\psi(\pi,\lambda) = \frac{\frac{1}{(n-1)!}(\lambda-\lambda_l)^n\psi^{(n)}(\pi,\lambda_l) + \mathcal{O}((\lambda-\lambda_l)^n)}{\frac{1}{n!}(\lambda-\lambda_l)^n\psi^{(n)}(\pi,\lambda_l) + \mathcal{O}((\lambda-\lambda_l)^{n+1})} = \frac{n}{\lambda-\lambda_l} + \mathcal{O}((\lambda-\lambda_l)^0).$$

Theorem 2.29.

$$\zeta_P(s) = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} \beta^{-2s}\frac{d}{d\beta}\ln\psi(\pi,\beta^2)\, d\beta \textit{ for } \frac{1}{2} < \mathrm{Re}(s) < 1.$$

Proof. Proceed as in the evaluation of $\zeta_R'(0)$; note the leading terms do not depend on the potential. □

Lemma 2.30.

$$\zeta_P(s) = \frac{\sin(\pi s)}{\pi}\int_0^\infty k^{-2s}\frac{d}{dk}\ln\psi(\pi,-k^2)\ dk \textit{ for } \frac{1}{2} < \mathrm{Re}(s) < 1.$$

Proof. We substitute $\beta = \pm ik$ in theorem 2.29 with the upper/lower part of the contour respectively to find

$$\zeta_P(s) = \frac{1}{2\pi i}\left[\int_\infty^0 (e^{i\pi/2}k)^{-2s}\frac{d}{dk}\ln\psi(\pi,-k^2)\ dk + \int_0^\infty (e^{-i\pi/2}k)^{-2s}\frac{d}{dk}\ln\psi(\pi,-k^2)\ dk\right]$$
$$= \frac{e^{i\pi s}-e^{-i\pi s}}{2\pi i}\int_0^\infty k^{-2s}\frac{d}{dk}\ln\psi(\pi,-k^2)\ dk = \frac{\sin(\pi s)}{\pi}\int_0^\infty k^{-2s}\frac{d}{dk}\ln\psi(\pi,-k^2)\ dk.$$

□

Lemma 2.31. *Let* $P_i = -\frac{d^2}{dx^2} + V_i(x),\ V_i \in C^0([0,\pi]),\ i = 1,2.$ *Then*

$$\zeta_{P_1}(s) - \zeta_{P_2}(s) = \frac{\sin(\pi s)}{\pi}\int_0^\infty k^{-2s}\frac{d}{dk}\ln\left[\frac{\psi_1(\pi,-k^2)}{\psi_2(\pi,-k^2)}\right]\ dk \textit{ for } -\frac{1}{2} < \mathrm{Re}(s) < 1.$$

Proof. From previous statements we obtain

$$\frac{\psi_1(\pi,-k^2)}{\psi_2(\pi,-k^2)} = 1 + \mathcal{O}(k^{-1})$$

and the assertion follows directly from lemma 2.30. □

Lemma 2.32 (Relative Determinant)**.**

$$\zeta_{p_1}'(0) - \zeta_{P_2}'(0) = -\ln\left[\frac{\psi_1(\pi,0)}{\psi_2(\pi,0)}\right].$$

Proof. By construction, the integral in lemma 2.31 is analytic about $s = 0$ and therefore,

$$\zeta_{P_1}'(s) - \zeta_{P_2}'(s)$$
$$= \cos(\pi s)\int_0^\infty k^{-2s}\frac{d}{dk}\ln\left[\frac{\psi_1(\pi,-k^2)}{\psi_2(\pi,-k^2)}\right]\ dk + \frac{\sin(\pi s)}{\pi}\frac{d}{ds}\int_0^\infty k^{-2s}\frac{d}{dk}\ln\left[\frac{\psi_1(\pi,-k^2)}{\psi_2(\pi,-k^2)}\right]\ dk$$

where we notice the last integral is finite so that the last term goes to 0 when evaluating at $s = 0$. Hence we find that

$$\zeta_{P_1}'(0) - \zeta_{P_2}'(0) = -\ln\left[\frac{\psi_1(\pi,0)}{\psi_2(\pi,0)}\right].$$

□

Remark 2.33. Lemma 2.32 implies something particularly deep and helpful. It shows that to find the determinant, we do not need the eigenvalues, only the homogeneous solution of the IVP! ⋄

Lemma 2.34. *Let* $P = -\frac{d^2}{dx^2} + V(x)$. *Then*

$$\zeta_P'(0) = -\ln(2\psi(\pi,0)),$$

where $\psi(\pi,0)$ *is the solution to the Sturm-Liouville homogeneous problem,* $P\psi = 0$.

Proof. In lemma 2.32, let $P_1 = P$ and $P_2 = -\frac{d^2}{dx^2}$. By our work in subsection 2.2.1, we know that $\zeta_{P_2}'(0) = -\ln(2\pi)$. Further, the solution to

$$-\frac{d^2}{dx^2}\psi_2(x,0) = 0, \quad \psi_2(0,0) = 0, \quad \frac{d}{dx}\psi_2(x,0)\Big|_{x=0} = 1$$

is $\psi(x,0) = x$, so

$$\zeta_P'(0) = -\ln\left[\frac{\psi(\pi,0)}{\pi}\right] - \ln(2\pi) = -\ln(2\psi(\pi,0)).$$

□

Remark 2.35. To verify this result through a different approach, see exercises 2.8 and 2.9. ⋄

2.4 Heat Kernel of Second Order (Partial) Differential Operators

Lemma 2.36.

$$\zeta_P(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_P(t)\ dt \textit{ for } \mathrm{Re}(s) > \frac{1}{2},\ \lambda_l > 0,\ l \in \mathbb{N},$$

where

$$K_P(t) = \sum_{l=1}^\infty e^{-\lambda_l t}$$

is the (global) heat kernel.

Proof. By lemma 1.29, we can write

$$\zeta_P(s) = \sum_{l=1}^\infty \lambda_l^{-s} = \frac{1}{\Gamma(s)}\sum_{l=1}^\infty \int_0^\infty t^{s-1}e^{-\lambda_l t}\ dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_P(t)\ dt.$$

The region of convergence follows from the fact that

$$K_P(t) = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \text{ as } t \to 0,$$

which was found in exercise 1.13. □

Now we look at the asymptotics of the heat kernel. Let $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ be a sequence of real numbers such that as $t \to 0$,

$$K_P(t) = \sum_{l=1}^{\infty} e^{-\lambda_l t} \underset{t\to 0}{\sim} \sum_{n\in\mathbb{N}_0} a_{i_n} t^{i_n}, \; i_n \in \mathbb{R} \text{ with } i_{n+1} > i_n, \; i_0 < 0$$

and

$$i_n \to \infty \text{ as } n \to \infty.$$

The asymptotic above means there exists an N large enough such that as $t \to 0$,

$$K_P(t) - \sum_{n=0}^{N} a_{i_n} t^{i_n} = \mathcal{O}(t^{i_{N+1}}).$$

Theorem 2.37. *The zeta function, $\zeta_P(s) = \sum_{l=1}^{\infty} \lambda_l^{-s}$ has the following properties:*

i) $Res[\zeta_p(s), -i_n] = \dfrac{a_{i_n}}{\Gamma(-i_n)}$,

ii) $\zeta_P(-n) = (-1)^n n! a_n$.

Proof. From lemma 2.36 we can write for $\text{Re}(s) > -i_0$,

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} K_P(t)\, dt = \frac{1}{\Gamma(s)} \infty_0^{\infty} t^{s-1+i_0} t^{-i_0} K_P(t)\, dt$$

$$= \frac{1}{\Gamma(s)} \frac{1}{s+i_0} \int_0^{\infty} \frac{d}{dt}\left(t^{s+i_0}\right) t^{-i_0} K_P(t)\, dt$$

$$= \frac{1}{\Gamma(s)} \frac{1}{s+i_0} \left[t^{s+i_0} t^{-i_0} K_P(t) \Big|_0^{\infty} - \int_0^{\infty} t^{s+i_0} \frac{d}{dt}\left(t^{-i_0} K_P(t)\right)\, dt \right]$$

where the first term goes to 0 as $t \to 0$, so that

$$= \frac{1}{\Gamma(s)} \left[-\frac{1}{s+i_0} \int_0^{\infty} t^{s+i_0} \frac{d}{dt}\left(t^{-i_0} K_P(t)\right)\, dt \right], \text{ for } \text{Re}(s) > -i_1$$

and we continue computing the same way to find

$$= \frac{1}{\Gamma(s)} \left[-\frac{1}{s+i_0} \int_0^{\infty} t^{s+i_1-1} t^{i_0-i_1+1} \frac{d}{dt}\left(t^{-i_0} K_P(t)\right)\, dt \right]$$

$$= \frac{1}{\Gamma(s)} \left[-\frac{1}{(s+i_0)(s+i_1)} \int_0^{\infty} \frac{d}{dt}(t^{s+i_1}) t^{i_0-i_1+1} \frac{d}{dt}\left(t^{-i_0} K_P(t)\right)\, dt \right]$$

$$= \frac{1}{\Gamma(s)}\left[-\frac{1}{(s+i_0)(s+i_1)}\int_0^\infty t^{s+i_1}\frac{d}{dt}\left(t^{i_0-i_1+1}\frac{d}{dt}\left(t^{-i_0}K_P(t)\right)\right)\,dt\right],\ \text{for } \mathrm{Re}(s) > -i_2.$$

Continuing in this manner we can conclude

$$= \frac{1}{\Gamma(s)}\left[-\frac{1}{(s+i_0)(s+i_1)\dots(s+i_n)}\int_0^\infty t^{s+i_n}\frac{d}{dt}\left(t^{i_{n-1}-i_n+1}\frac{d}{dt}\cdots\frac{d}{dt}\left(t^{-i_0}K_P(t)\right)\right)\,dt\right]$$

for $\mathrm{Re}(s) > -(i_n+1)$, from which we can compute property i) and property ii) follows immediately. □

Remark 2.38. This result is to be expected because formally we see

$$\begin{aligned}\zeta_P(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_P(t)\,dt &= \frac{1}{\Gamma(s)}\left[\int_0^1 t^{s-1}K_P(t)\,dt + \int_1^\infty t^{s-1}K_P(t)\,dt\right]\\ &\cong \frac{1}{\Gamma(s)}\left[\int_0^1 t^{s-1}\sum_{n\in\mathbb{N}_0} a_{i_n}t^{i_n}\,dt + \int_1^\infty t^{s-1}K_P(t)\,dt\right]\\ &= \frac{1}{\Gamma(s)}\left[\sum_{n\in\mathbb{N}_0}\frac{a_{i_n}}{s+i_n} + \int_1^\infty t^{s-1}K_P(t)\,dt\right],\end{aligned}$$

where we notice only the first part gives contribution to the residue. ⋄

Definition 2.39. *The (local) heat kernel associated with the operator P is defined as the solution to the following problem on M:*

$$\left(\frac{\partial}{\partial t} + P\right)K(t,x,y) = 0,$$

$$\mathcal{B}K(t,x,y)\Big|_{x\in\partial M} = 0,$$

$$\lim_{t\to 0} K(t,x,y) = \delta(x,y).$$

Remark 2.40. We need to understand the definition above as follows:

i) $\mathcal{B}$ stand for a boundary operator, in particular, the Dirichlet boundary operator for us.

ii) The last notation means

$$\lim_{t\to 0}\int_M K(t,x,y)f(y)\,dy = f(x)\ \text{for}\ \int_M \delta(x,y)f(y)\,dy = f(x).$$

This means that $\int_M K(t,x,y)f(y)\,dy$ solves the heat equation with boundary conditions $\mathcal{B}$ imposed and with initial temperature distribution $f(x)$. ⋄

Theorem 2.41. *There is at most one solution of*

$$u_t - u_{xx} = f(x,t),\ 0 < x < L,\ t > 0;$$

$$u(x,0) = \pi(x);\ u(0,t) = g(t),\ u(L,t) = h(t).$$

Proof. We begin by assuming there are two solutions, say u_1 and u_2. We define $w = u_1 - u_2$. We write

$$0 = 0w = (w_t - w_{xx})w) = w_t w - w_{xx} w = \frac{1}{2}(w^2)_t - (w_x w)_x + w_x^2$$

and integrate form $x = 0$ to $x = L$ to see

$$0 = \int_0^L \left(\frac{1}{2}(w^2)_t - (w_x w)_x + w_x^2\right)\ dx = \int_0^L \frac{1}{2}(w^2)_t\ dt - w_x w\Big|_0^L + \int_0^L w_x^2\ dx$$

so that we see

$$\frac{d}{dt}\int_0^L \frac{1}{2} w^2\ dx = -\int_0^L w_x^2\ dx \leq 0.$$

Now integrating from $t = 0$ to $t = t$ we see

$$\int_0^L w(x,t)^2\ dx \leq \int_0^L w(x,0)^2\ dx = 0$$

by initial conditions. Hence it must be that $w(x,t) \equiv 0$ and so $u_1 = u_2$ for all $t \geq 0$. □

Remark 2.42. Let $\phi_l(x)$ be normalized eigenfunctions of P with eigenvalues λ_l; that is let

$$P\phi_l(x) = \lambda_l \phi_l(x),\ \phi_l\Big|_{x\in\partial M} = 0.$$

Then

$$K_P(t,x,x) = \sum_{l=1}^{\infty} e^{-\lambda_l t}\bar{\phi}_l(x)\phi_l(x).$$

Connection to the global heat kernel is established by integration, that is

$$\int_M K_P(t,x,x)\ dx = \sum_{l=1}^{\infty} e^{-\lambda_l t} = K_P(t).$$

◇

2.4.1 Small-t Asymptotics for Sturm-Liouville Operators

Now we move our attention to understanding what the small-t asymptotics look like for Sturm-Liouville operators. Let us first consider manifolds without boundary.

Example 2.43. *$P = -\Delta$ (Laplacian) on S^1 (manifold without boundary) with periodic BC*

$$-\frac{\partial^2}{\partial x^2}\phi_n(x) = \lambda_n \phi_n(x),\ \ \phi_n(0) = \phi(L).$$

Solutions are of the form

$$\phi_n(x) = e^{i\sqrt{\lambda_n}x},\ \ \phi(0) = 1 = \phi(L) = e^{i\sqrt{\lambda_n}L}$$

so that we have eigenvalues given by

$$\sqrt{\lambda_n}L = 2\pi n,\ \ n \in \mathbb{Z} \Longleftrightarrow \lambda_n = \left(\frac{2\pi n}{L}\right)^2.$$

Hence we find the heat kernel utilizing the Poisson Resummation formula with $x = \frac{4\pi}{L^2}t$ and $\nu = 0$ (see exercise 1.12)

$$K_P(t) = \sum_{n\in\mathbb{Z}} e^{-\left(\frac{2\pi n}{L}\right)^2 t} = \frac{L}{\sqrt{4\pi t}}\sum_{n\in\mathbb{Z}} e^{-\frac{n^2L^2}{4t}} = \frac{L}{\sqrt{4\pi L}}(1 + \textit{exponentially damped})$$

$$= (4\pi t)^{-\frac{1}{2}}\, Vol(S^1) + \textit{exponentially damped},$$

where in this case we have the length of the curve S^1.

Example 2.44. *Let $P = -\Delta$ on $M = S^1 \times \cdots \times S^1$, d-times (Taurus, manifold without boundary) with periodic BC. We find eigenvalues*

$$\lambda_{n_1\ldots n_d} = \left(\frac{2\pi}{L_1}\right)n_1^2 + \cdots + \left(\frac{2\pi}{L_d}\right)n_d^2,$$

where L_i is the perimeter of S_i^1. Then we find the heat kernel

$$K_P(t) = \sum_{n_1,\ldots n_d\in\mathbb{Z}} e^{-\lambda_{n_1\cdots n_d}t} = \prod_{i=1}^{d}\sum_{n_i\in\mathbb{Z}} e^{-\left(\frac{2\pi}{L_i}\right)^2 n_i^2} = (4\pi t)^{-\frac{d}{2}}\, Vol(M) + \textit{exponentially damped.}$$

Example 2.45. *Let M be as in example 2.44 and $P = -\Delta + m^2$. Note this simply shifts eigenvalues and keeps eigenfunctions if m is constant, as clearly*

$$P\phi_l = \lambda_l\phi_l \Longleftrightarrow (P + m^2)\phi_l = (\lambda_l + m^2)\phi_l.$$

Further, this addition to the operator only adds a factor to the previous heat kernel, namely

$$\begin{aligned} K_P(t) &= (4\pi t)^{-\frac{d}{2}}\, Vol(M) e^{-m^2 t} + \textit{exponentially damped} \\ &= \sum_{k=0}^{\infty} a_k t^{-\frac{d}{2}+k} + \textit{exponentially damped}, \end{aligned}$$

where a_k is the heat kernel coefficient containing values from the mass (4π) and factorials.

Theorem 2.46. *Let M be a d-dimensional smooth, compact Riemannian manifold without boundary and $P = -\Delta + V(x)$. Then*

$$K_P(t) \underset{t\to 0}{\sim} \sum_{k=0}^{\infty} a_k t^{k-\frac{d}{2}}.$$

Proof. Outside the scope of this manuscript; motivated by example 2.45. □

Now let us consider manifolds with boundary.

Example 2.47. *Let $M = [0, L]$ and consider the problem*

$$-\frac{d^2}{dx^2}\phi_n(x) = \lambda_n \phi_n(x),\ \ \phi_n(0) = \phi_n(L) = 0$$

whose solutions are

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right),\ \ n \in \mathbb{N},\ \ \lambda_n = \left(\frac{\pi n}{L}\right)^2.$$

We utilize the Poisson Resummation formula again to find the Local Heat Kernel,

$$\begin{aligned} K_P(t,x,y) &= \frac{2}{L} \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) \\ &= -\frac{1}{2L} \sum_{n\in\mathbb{Z}} \left[e^{i\frac{\pi n}{L}(x-y)} - e^{i\frac{\pi n}{L}(x+y)} \right] e^{-\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

which by Poisson Resummation formula gives

$$= \frac{1}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} \left[e^{-\frac{L^2}{t}\left(n-\frac{x-y}{2L}\right)^2} - e^{-\frac{L^2}{t}\left(n-\frac{x+y}{2L}\right)^2} \right]$$

Notice that $x = y$ gives that the first term is exactly what we had previously in example 2.4! Further, if $\frac{x}{L} \notin \mathbb{Z}$, then the first term is exponentially damped, meaning when x is not on the boundary as $\frac{x}{L} \in \mathbb{Z}$ means $x = 0$ or L.

Next we find the Global Heat Kernel

$$K_P(t) = \sum_{n\in\mathbb{Z}} e^{-\left(\frac{n\pi}{L}\right)^2 t} = \int_0^L K_P(t,x,x)\, dx,$$

where we find from the local heat kernel,

$$K_P(t,x,x) = \frac{1}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} \left[e^{-\frac{L^2 n^2}{t}} - e^{-\frac{L^2}{t}\left(n-\frac{x}{L}\right)^2} \right].$$

As $t \to 0$, in the first term $n = 0$ gives contribution and in the second term $n = 0, 1$ give contributions when $x = 0, L$ respectively (for $x \in \partial M$). Hence we find

$$K_P(t) = \frac{L}{\sqrt{4\pi t}} - \frac{1}{2} + \textit{exponentially damped.}$$

Notice now that the powers of t increase by $\frac{1}{2}$ instead of 1 as before! This should make one think that the corresponding theorem for manifolds with boundary will need to be adjusted slightly from theorem 2.46*.*

Example 2.48. *Let us consider the d-dimensional manifold $M = [0, L] \times S^1 \times \cdots \times S^1$ (boundary is the Taurus) with the same operator as in example* 2.47*. Proceeding as before we find*

$$\begin{aligned} K_P(t) &= (4\pi t)^{-\frac{d}{2}}\, Vol(M) + (4\pi t)^{-\frac{d-1}{2}}\, Vol(S^1 \times \cdots \times S^1) \left(-\frac{1}{2}\right) + \textit{exponentially damped} \\ &= (4\pi t)^{-\frac{d}{2}}\, Vol(M) + (4\pi t)^{-\frac{d-1}{2}} \left(-\frac{1}{4}\right) Vol(\partial M) + \textit{exponentially damped.} \end{aligned}$$

Remark 2.49. This shows that the spectrum of an operator gives information of the volume of a Riemannian manifold and its boundary! ⋄

Theorem 2.50. *Let M be a d-dimensional smooth, compact Riemannian manifold without boundary and $P = -\Delta + V(x)$ with Dirichlet boundary conditions. Then*

$$K_P(t) \underset{t\to 0}{\sim} \sum_{k=0,\frac{1}{2},1,\ldots}^{\infty} a_k t^{k-\frac{d}{2}}.$$

Proof. Outside the scope of this manuscript; motivated by example 2.48. □

Theorem 2.51. *Let M and P be as in theorem* 2.46*. Then $\zeta_P(s)$ has poles at*

$$\frac{d}{2}, \frac{d}{2} - 1, \ldots, 1 \textit{ for } d \textit{ even}$$

and

$$\frac{d}{2}, \frac{d}{2} - 1, \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots \textit{ for } d \textit{ odd.}$$

Proof. Direct consequence of theorem 2.37. □

Remark 2.52. This gives that the rightmost pole of the spectrum of Δ gives the dimension of the manifold. ⋄

Theorem 2.53. *Let M and P be as in theorem 2.50. Then $\zeta_P(s)$ has poles at*

$$\frac{d}{2}, \frac{d-1}{2}, \frac{d}{2}-1, \dots, 1, \frac{1}{2},$$

and

$$-\frac{2n+1}{2} \text{ for } n \in \mathbb{N}_0.$$

Proof. Direct consequence of theorem 2.37. □

2.5 Zeta Functions and Heat Kernel on the Two-Ball (Disk)

We want to solve the following BVP on the disk, $M = \{(x,y) \in \mathbb{R}^2; |(x,y)| \leq 1\}$:

$$-\Delta\phi(x,y) = \lambda\phi(x,y), \; \phi\Big|_{(x,y)\in\partial M} = 0.$$

We will move to polar coordinates, i.e. let $x = r\cos\theta$ and $y = r\sin\theta$. Then we rewrite the Laplacian in polar coordinates as

$$\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi_{m,n}(r,\theta) = \lambda_{m,n}\phi_{m,n}(r,\theta)$$

$$\phi_{m,n}(1,\theta) = 0, \; \phi_{m,n}(r,\theta) = \phi_{m,n}(r,\theta+2\pi).$$

Next we look for solutions by isolating the radial and spherical portions of this expression, i.e. use seperation of variables to write

$$\psi_{m,n}(r,\theta) = R(r)\Theta(\theta)$$

so that we see

$$-\frac{1}{r}\frac{(\partial_r r\partial_r)R}{R} - \frac{1}{r^2}\frac{\partial_\theta^2\Theta}{\Theta} = \lambda_{m,n}.$$

From this we find for the angular portion

$$\Rightarrow -\partial_\theta^2\Theta = m^2\Theta, \text{ where } m^2 \text{ is a constant.}$$

Imposing BC, we conclude

$$\Theta(\theta) = e^{im\theta}, \; m \in \mathbb{Z}.$$

Now for the radial differential equation we have

$$-\frac{1}{r}(\partial_r r \partial_r)R + \frac{m^2}{r^2}R = \lambda_{m,n}R$$

$$-\frac{1}{r}\partial_r R - \partial_r^2 R + \frac{m^2}{r^2}R = \lambda_{m,n}R$$

$$r^2\partial_r^2 R + r\partial_r R + (\lambda_{m,n}r^2 - m^2)R = 0.$$

In particular we find this is Bessel's Differential Equation of Order m, i.e.

$$r^2R''(r) + rR'(r) + (\lambda_{m,n}r^2 - m^2)R(r) = 0,$$

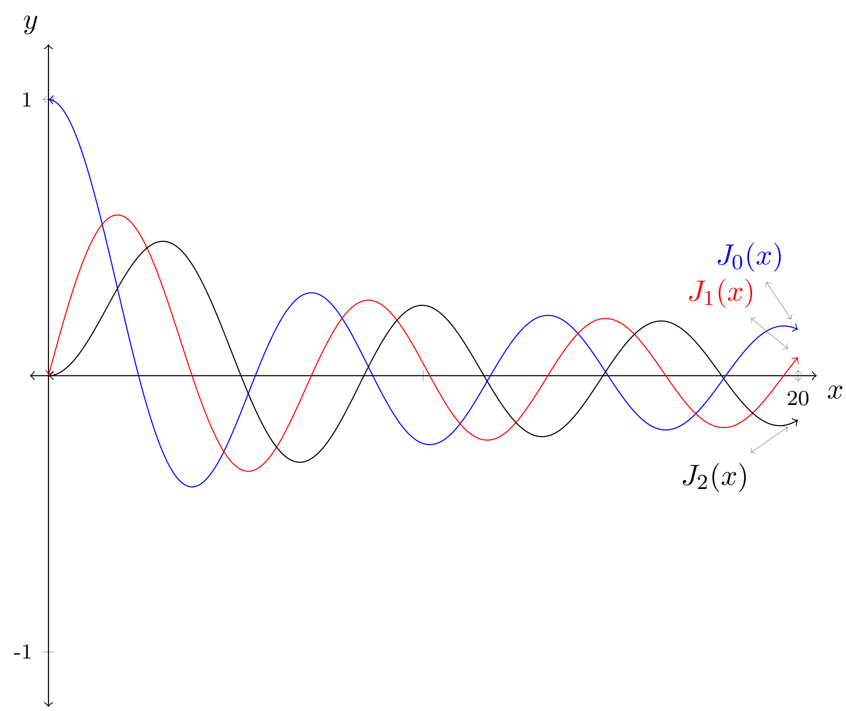
so we denote

$$R(r) = J_{|m|}(\sqrt{\lambda_{m,n}}r),\ m \in \mathbb{Z},\ n \in \mathbb{N},$$

where each n enumerates the zeros for given m. These are known as the Bessel Functions of the first kind, i.e. solutions of Bessel's differential equation that are finite at the origin. We will learn more about these shortly. Further, imposing BC we find

$$R(1) = 0 \iff J_{|m|}(\sqrt{\lambda_{m,n}}) = 0.$$

Remark 2.54. Let us take a look at a few of the Bessel functions of the first kind, in particular $J_0(x)$, $J_1(x)$, and $J_2(x)$. Note they are oscillatory functions with decaying amplitude as pictured below.



◇

Now we calculate the zeta function where γ will be the appropriate complex contour for our operator throughout this section. We have

$$\zeta(s) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{N}} \lambda_{m,n}^{-s} = \sum_{m\in\mathbb{Z}} \oint_\gamma z^{-2s} \frac{d}{dz} \ln\left(J_{|m|}(z)\right)\, dz, \text{ for } \mathrm{Re}(s) > 1.$$

Hence, we need to understand Bessel functions if we are to understand the zeta function and heat kernel for this problem!

2.5.1 Excursion on Bessel Functions

We now work to understand some properties of the Bessel functions. If we substitute $x = \sqrt{\lambda_{m,n}}$ into Bessel's Differential Equation above, using $\frac{d}{dr} = \frac{dx}{dr}\frac{d}{dx} = \sqrt{\lambda_{m,n}}\frac{d}{dx}$, we see it becomes

$$\left[x^2\frac{d^2}{dx^2} + \frac{d}{dx} + (x^2 - m^2)\right] J_m(x) = 0. \tag{2.11}$$

Ansatz for solution: (Frobenius Method)
Consider $y'' + \frac{a(x)}{x}y' + \frac{b(x)}{x^2}y = 0$ with $a(x),\ b(x)$ analytic near $x = 0$, that is we can express

$$a(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and } a(x) = \sum_{m=0}^{\infty} b_m x^m.$$

For $x \approx 0$, the leading terms of the differential equation are

$$x^2 y'' + a_0 x y' + b_0 y = 0.$$

Try: $(y = x^w)$

$y = x^w \Rightarrow w(w-1)x^w + a_0 w x^w + b_0 x^w = 0 \Rightarrow w(w-1) + a_0 w + b_0 = 0$ (initial or characteristic eq.).

Example of Bessel Equation: $(a_0 = 1,\ b_0 = -m^2)$
This gives an initial equation of

$$w(w-1) + w - m^2 = 0 \Rightarrow w^2 - m^2 = 0 \Rightarrow w = \pm m.$$

We choose $+m$ as we want finite at origin in vibrating drum problem. Hence we will try a substitution of

$$J_m(x) = x^m \sum_{n=0}^{\infty} c_n x^n,$$

where we note if $m \notin \mathbb{Z}$, we must factor out the leading behavior to write a proper power series representation and c_n is to be determined. We continue by substituting this expression into (2.11), for which we note

$$J'_m(x) = \sum_{n=0}^{\infty} c_n(n+m)x^{n+m-1}, \; J''_m(x) = \sum_{n=0}^{\infty} c_n(n+m)(n+m-1)x^{n+m-2},$$

to attain

$$\sum_{n=0}^{\infty} \left[c_n(n+m)(n+m-1)x^{n+m} + c_n(n+m)x^{n+m} + c_n x^{n+m+2} - m^2 c_n x^{n+m}\right] = 0.$$

From this we find

$$[m(m-1)c_0 + mc_0 - m^2c_0]x^m + [(m+1)mc_1 + (m+1)c_1 - m^2c_1]x^{m+1} +$$
$$+ \sum_{n=2}^{\infty} \left[c_n(n+m)(n+m-1) + c_n(n+m) + c_{n-2} - m^2 c_n\right] x^{n+m} = 0.$$

This implies that each coefficient must be zero, so we now attempt to find the coefficients, c_i.
$\underline{x^m}$: $m(m-1)c_0 + mc_0 - m^2c_0 = m^2c_0 - mc_0 + mc_0 - m^2c_0 = 0$, so we do not learn anything about c_0.
$\underline{x^{m+1}}$: $(m+1)mc_1 + (m+1)c_1 - m^2c_1 = m^2c_1 + mc_1 + mc_1 + c_1 - m^2c_1 = 2mc_1 + c_1 = (2m+1)c_1$ $\Rightarrow (2m+1)c_1 = 0 \Rightarrow c_1 = 0$.
$\underline{x^{n+m}}$: $[(n+m)(n+m-1) + (n+m) - m^2]c_n = -c_{n-2} \Rightarrow n(2m+n)c_n = -c_{n-2}$ $\Rightarrow c_n = -\dfrac{c_{n-2}}{n(2m+n)}$.
Hence, we are left with c_0 undetermined, however we see that for all odd coefficients, $c_{k+1} = 0, \; k \in \mathbb{N}_0$, and for even coefficients we have $c_{2k} = -\dfrac{c_{2k-2}}{4k(m+k)}$. Further, by induction we conclude that

$$c_{2k} = (-1)^k \frac{c_0 m!}{4^k k!(k+m)!}.$$

Thus we have found

$$J_m(x) = c_0 x^m \sum_{k=0}^{\infty} (-1)^k \frac{m!}{4^k k!(k+m)!} x^{2k} = c_0 x^m \sum_{k=0}^{\infty} (-1)^k \frac{m!}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k}$$
$$= c_0 2^m m! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m},$$

and now notice that the best choice for c_0 is

$$c_0 = \frac{1}{2^m m!},$$

so that we can summarize that for this choice of c_0, we have

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m}. \tag{2.12}$$

We will also need $J_m(\pm ix)$:

$$J_m(\pm ix) = (\pm i)^m \sum_{k=0}^{\infty} \frac{1}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m} = e^{\pm i\frac{\pi}{2}m} I_m(x),$$

where

$$I_m(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m} \tag{2.13}$$

is the modified Bessel function of the first kind. Substituting $\pm ix$ for x in (2.11) shows

$$\left[x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} - (x^2 + m^2)\right] I_m(x) = 0. \tag{2.14}$$

We will need asymptotics of $I_m(z)$ for large $|z|$, but we will use the differential equation (2.14) rather than the series (2.13).

2.5.2 Asymptotic Expansion of Bessel Functions

In general we will consider the following differential equation

$$(\partial_z^2 + u\partial_z + v)\phi(z) = 0. \tag{2.15}$$

Ansatz: Look for solution of the form $\phi(z) = \exp\left\{\int^z p(t)\ dt\right\}\psi(z)$. Then we have

$$\phi'(z) = p(z)\phi(z) + \exp\left\{\int^z p(t)\ dt\right\}\psi'(z),$$

$$\phi''(z) = p'(z)\phi(z) + p^2(z)\phi(z) + p(z)\exp\left\{\int^z p(t)\ dt\right\}\psi'(z)+$$

$$+2p(z)\exp\left\{\int^z p(t)\ dt\right\}\psi'(z) + \exp\left\{\int^z p(t)\ dt\right\}\psi''(z).$$

Substituting into (2.15) and dividing out $\exp\{\int^z p(t)\ dt\} \neq 0$ gives

$$\psi''(z) + w\psi'(z) + q\psi(z) = 0$$

with

$$w = u + 2p(z),\ q = p'(z) + p^2(z) + up(z) + v.$$

Now we pick $p(z) = -\frac{1}{2}u$ so that the first derivative is gone and we obtain the following expression for q,

$$q = -\frac{u'}{2} - \frac{u^2}{4} + v. \tag{2.16}$$

Summarizing, we have that for q given by (2.16), ψ satisfies

$$\psi'' + q\psi = 0.$$

Now we define $s = \partial_z \ln\psi$, so that $\partial_z \ln\phi = p + s$ and

$$\phi(z) = (\text{constant}) \exp\left\{\int^z p(t)\ dz\right\} \exp\left\{\int^z s(t)\ dz\right\}.$$

Thus we find utilizing the simplified differential equation found above,

$$s = \frac{\psi'}{\psi},\ s' = \frac{\psi''}{\psi} - \frac{(\psi')^2}{\psi^2} = -q - s^2.$$

We assume the differential equation contains a parameter m and that we are interested in the large-m asymptotic behavior of s. We assume further that as $m \to \infty$, the function q has the asymptotic expansion

$$q = \sum_{i=-2}^{\infty} m^{-i} q_i.$$

Substituting into the differential equation for s shows we must then have as $m \to \infty$,

$$s = \sum_{i=-1}^{\infty} m^{-i} s_i.$$

We substitute both of these forms into the differential equation for s and obtain

$$\sum_{i=-1}^{\infty} m^{-i} s_i' = -\sum_{i=-2}^{\infty} m^{-i} q_i - \left(\sum_{j=-1}^{\infty} m^{-j} s_j\right)\left(\sum_{k=-1}^{\infty} m^{-k} s_k\right)$$

which shows

$$ms_{-1}' + \sum_{i=0}^{\infty} m^{-i} s_i' = -m^2 q_{-2} - mq_{-1} - \sum_{i=0}^{\infty} m^{-i} q_i - m^2 s_{-1}^2 - 2ms_{-1}s_0 - \sum_{i=0}^{\infty} m^{-i} \sum_{j=-1}^{\infty} s_j s_{i-j}.$$

Comparing powers as before, but this time for m we find:
$\underline{m^2}$: $0 = -q_{-2} - s_{-1}^2$ so that we see

$$s_{-1} = \pm\sqrt{-q_{-2}}. \tag{2.17}$$

$\underline{m}$: $s'_{-1} = -q_{-1} - 2s_{-1}s_0$ so that we see

$$s_0 = -\frac{s'_{-1} + q_{-1}}{2s_{-1}} = -\frac{1}{2}\partial_z \ln s_{-1} - \frac{q_{-1}}{2s_{-1}}. \tag{2.18}$$

$\underline{m^{-i},\ i \in \mathbb{N}_0}$: $s'_i = -q_i - \sum_{j=-1}^{i+1} s_j s_{i-j} = -q_i - 2s_{i+1}s_{-i} - \sum_{j=0}^{i} s_j s_{i-j}$ so that we see

$$s_{i+1} = -\frac{1}{2s_{-1}}\left(s'_i + q_i + \sum_{j=0}^{i} s_j s_{i-j}\right). \tag{2.19}$$

Remark 2.55. Motivation for the next step is as follows. We need to do a substitution as we must consider a sum of integrals and its convergence, namely

$$\zeta(s) = \sum_{m\in\mathbb{Z}} \oint_\gamma z^{-2s} \frac{d}{dz} \ln J_{|m|}(z)\ dz \text{ for } \mathrm{Re}(s) > 1.$$

Let $z = y|m|$, then

$$\zeta(s) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \oint_\gamma y^{-2s} \frac{d}{dy} \ln J_{|m|}(y|m|)\ dy + \oint_\gamma y^{-2s} \frac{d}{dy} \ln J_0(y)\ dy.$$

$\diamond$

We now apply the asymptotic analysis we have achieved to the Bessel function. The most suitable form for our analysis is obtained by substituting $x = ym$ into (2.14) to find

$$\left[y^2 \frac{d^2}{dy^2} + y\frac{d}{dy} - m^2(y^2+1)\right] I_m(ym) = 0$$

or

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} - m^2\left(1 + \frac{1}{y^2}\right)\right] I_m(ym) = 0.$$

For this example, from (2.15) and (2.16), we have

$$u = \frac{1}{y}, \quad v = -m^2\left(1 + \frac{1}{y^2}\right), \quad q = -m^2\left(1 + \frac{1}{y^2}\right) + \frac{1}{4y^2}$$

$$\Rightarrow q_{-2} = -\left(1 + \frac{1}{y^2}\right), \quad q_{-1} = 0, \quad q_0 = \frac{1}{4y^2}, \quad q_i = 0 \text{ for } i \in \mathbb{N}.$$

Further, from (2.17), (2.18), and (2.19), we have

$$s_{-1} = \pm\sqrt{\frac{1+y^2}{y^2}}, \quad s_0 = -\frac{1}{2}d_z \ln s_{-1} = \frac{1}{2y(1+y^2)}, \quad s_1 = -\frac{q_0 + s_0' + s_0^2}{2s_{-1}} = \mp\frac{y(y^2-4)}{8(1+y^2)^{5/2}}.$$

We use this information to find the following for $I_m(mz)$:

$$\int^z s_{-1}\ dy = \pm\left[\sqrt{1+z^2} + \ln\frac{z}{1+\sqrt{1+z^2}}\right] + C_1,$$

$$\int^z (s_0 - (2y)^{-1}\ dy = -\frac{1}{4}\ln(1 = z^2) + C_2,$$

$$\int^z s_1\ dy = \mp\frac{2-3z^2}{24(1+z^2)^{3/2}} + C_3.$$

It is customary to use $t = \sqrt{1+z^2}^{-1}$ and conclude

$$I_m(mz) \sim \text{constant} \cdot e^{\pm m\beta} t^{\frac{1}{2}} e^{\pm\frac{1}{m}\left(\frac{t}{8} - \frac{5t^3}{24}\right)} e^{\mathcal{O}(m^{-2})} \tag{2.20}$$

where

$$\beta = \sqrt{1+z^2} + \ln\frac{z}{1+\sqrt{1+z^2}} \text{ for large } m,$$

and each of the factors on the right-hand side corresponds to $s_{-1},\ s_0,\ s_1$, and all other s_i respectively from left to right.

We can now finally finish the asymptotics of $I_0(x)$ as $x \to \infty$! The differential equation reads

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - 1\right] I_0(x) = 0.$$

<u>Ansatz</u>: $I_0(x) = \exp\left[\int^x p(t)\ dt\right]\psi(x)$. Choose $p(t) = -\frac{1}{2x}$ as before to consider the simpler

$$\psi'' + q\psi = 0 \Rightarrow q = -1 + \frac{1}{4x^2} = \frac{1-4x^2}{4x^2}.$$

Further, only considering the leading order we have

$$\psi'' - \psi = 0 \Rightarrow \psi(x) = e^{\pm x},$$

and hence we find

$$I_0(x) = \exp\left[\int^x -\frac{1}{2t}\ dt\right] e^{\pm x}(1 + \mathcal{O}(x^{-1})) = \frac{\text{constant}}{\sqrt{x}} e^{\pm x}(1 + \mathcal{O}(x^{-1})).$$

Thus we have found the asymptotics we will need for computing the heat kernel we were after.

Remark 2.56. As we shift back to the zeta function we were considering, let us remember what the goal was. We wished to see that for the heat kernel,

$$K(t) = \frac{a_0}{t} + \frac{a_{1/2}}{t^{1/2}} + a_1 + \mathcal{O}(t^{1/2})$$

we have $a_0 =$ volume of manifold, $a_{1/2} =$ surface area, and $a_1 =$ topological information. ⋄

2.5.3 Return to the Zeta Function

We now continue where we were before we started our excursion on Bessel functions in subsection 2.5.1. We begin by writing

$$\zeta(s) = 2\sum_{m=1}^{\infty} \frac{1}{2\pi i}\oint_\gamma z^{-2s}\frac{d}{dz}\ln\left[J_m(z)z^{-m}\right]\ dz + \frac{1}{2\pi i}\oint_\gamma z^{-2s}\frac{d}{dz}\ln J_0(z)\ dz$$

$$= 2\frac{\sin(\pi s)}{\pi}\sum_{m=1}^{\infty} m^{-2s}\int_0^\infty k^{-2s}\frac{d}{dk}\ln\left[I_m(mk)k^{-m}\right]\ dk + \frac{\sin(\pi s)}{\pi}\int_0^\infty k^{-2s}\frac{d}{dk}\ln I_0(k)\ dk$$

$$=: \zeta_1(s) + \zeta_2(s)$$

where we multiplied by z^{-m} to control the $z \to 0$ behavior. Next we set out to find analytic continuations of both as we did for the zeta function of Riemann.

Analytic Continuation of ζ_2:

We start with ζ_2 as it is the easier of the two integrals to analyze. We have

$$\zeta_2(s) = \frac{\sin(\pi s)}{\pi}\int_0^1 k^{-2s}\frac{d}{dk}\ln I_0(k)\ dk + \frac{\sin(\pi s)}{\pi}\int_1^\infty k^{-2s}\frac{d}{dk}\ln I_0(k)\ dk,$$

where we note that the inside is an analytic function of k, with 0 and ∞ providing the meromorphic structure. For the first integral we note from (2.13), we see for small k, $I_0(k) = \mathcal{O}(k^{-2})$, and hence the integral behaves as k^{-2s+1}, hence we find $\mathrm{Re}(s) < 1$. Similarly, we find that the second integral behaves as K^{-2s} and $\mathrm{Re}(s) > \frac{1}{2}$. Hence, ζ_2 is defined for $\frac{1}{2} < \mathrm{Re}(s) < 1$. Continuing our calculation, we add and subtract the small-k behavior

$$\zeta_2(s) = \frac{\sin(\pi s)}{\pi}\int_0^1 k^{-2s}\frac{d}{dk}\left[\ln I_0(k) - \frac{k^2}{4}\right]\ dk - \frac{\sin(\pi s)}{\pi}\int_0^1 k^{-2s}\frac{d}{dk}\left[\frac{k^2}{4}\right]\ dk$$

$$+\frac{\sin(\pi s)}{\pi}\int_1^\infty k^{-2s}\frac{d}{dk}\left[\ln(I_0(k)\sqrt{k}e^{-k})\right]\ dk - \frac{\sin(\pi s)}{\pi}\int_1^\infty k^{-2s}\frac{d}{dk}\left[\ln(\sqrt{k}e^{-k})\right]\ dk$$

$$=: \zeta_{21} + \zeta_{22} + \zeta_{23} + \zeta_{24}.$$

ζ_{21}: Integral is analytic for $\text{Re}(s) < 2$, so no contribution to $a_0 = \text{Res}[\zeta(s), 1]$, $a_{1/2} = \text{Res}[\zeta(s), 1/2]$, and $\zeta(0)$ because $\sin(0) = 0$.

ζ_{23}: Integral is analytic for $\text{Re}(s) > -\dfrac{1}{2}$, so no contribution by same reasoning as above.

ζ_{22}:

$$\zeta_{22}(s) = \frac{\sin(\pi s)}{2\pi} \int_0^1 k^{-2s+1} \; dk = \frac{\sin(\pi s)}{4\pi(1-s)},$$

so no contribution as well by calculation.

ζ_{24}:

$$\zeta_{24}(s) = -\frac{\sin(\pi s)}{\pi} \int_1^\infty k^{-2s} \frac{d}{dk} [\ln\sqrt{k} - k] \; dk$$

$$= -\frac{\sin(\pi s)}{\pi} \int_1^\infty k^{-2s} \left[\frac{1}{2k} - 1\right] \; dk = -\frac{\sin(\pi s)}{\pi} \left[\frac{1}{4s} + \frac{1}{1-2s}\right].$$

$$\text{Res}[\zeta_{24}, 1/2] = \frac{1}{2\pi}, \quad \zeta_{24}(0) = -\frac{1}{4}.$$

ζ_2: Hence we found that

$$\text{Res}[\zeta_2(s), 1/2] = \frac{1}{2\pi}, \quad \zeta_2(0) = -\frac{1}{4}.$$

Notice as this is really a simplification to one-dimension, we should not expect residues at 1!

Analytic Continuation of ζ_1:

Now we will look at ζ_1, noting that it will be considerably more difficult to analyze. We keep β and t defined as in the asymptotics found for $I_m(mz)$ in (2.20), which we will use presently. Similarly to before, we will add and subtract up to the order of m^{-1} (found from (2.20) for s_{-1}, s_0, and s_1) so that what remains will be of order m^{-2}, remembering we have already multiplied by an appropriate factor of k^{-m} to control the zero behavior. Hence we have

$$\zeta_1(s) = 2\frac{\sin(\pi s)}{\pi} \sum_{m=1}^\infty m^{-2s} \int_0^\infty k^{-2s} \frac{d}{dk} \ln\left[I_m(mk)k^{-m}\right] \; dk$$

$$= 2\frac{\sin(\pi s)}{\pi} \sum_{m=1}^\infty m^{-2s} \int_0^\infty k^{-2s} \frac{d}{dk} \left\{\ln\left[I_m(mk)k^{-m}\right] - \ln\left[e^{m\beta} t^{\frac{1}{2}} k^{-m}\right] - \frac{1}{m}\left(\frac{t}{8} - \frac{5t^3}{24}\right)\right\} \; dk$$

$$+2\frac{\sin(\pi s)}{\pi} \sum_{m=1}^\infty m^{-2s} \int_0^\infty k^{-2s} \frac{d}{dk} \left\{\ln\left[k^{-m} e^{m\beta}\right] + \ln[1+k^2]^{-\frac{1}{4}} + \frac{1}{m}\left(\frac{t}{8} - \frac{5t^3}{24}\right)\right\} \; dk$$

$$=: \zeta_{11} + \zeta_{12} + \zeta_{13} + \zeta_{14}.$$

ζ_{11}: Can be shown not to contribute to residue at $s = 1$, $\dfrac{1}{2}$, and value at $s = 0$.

ζ_{12}: We calculate

$$\zeta_{12} = 2\frac{\sin(\pi s)}{\pi}\sum_{m=1}^{\infty} m^{-2s}\int_0^\infty k^{-2s}\frac{d}{dk}\ln\left[k^{-m}e^{m\beta}\right]\,dk$$
$$= 2\frac{\sin(\pi s)}{\pi}\sum_{m=1}^{\infty} m^{-2s+1}\int_0^\infty k^{-2s}\frac{\sqrt{1+k^2}-1}{k}\,dk$$
$$= 2\frac{\sin(\pi s)}{\pi}\zeta_R(2s-1)\int_0^\infty k^{-2s-1}(\sqrt{1+k^2}-1)\,dk.$$

Analyzing the asymptotic behavior, we find:
As $k \to \infty$: The integrand behaves like k^{-2s}, so $\mathrm{Re}(s) > \frac{1}{2}$.
As $k \to 0$: Expanding $\sqrt{1+k^2}$ shows that $(\sqrt{1+k^2}-1) = \mathcal{O}(k^2)$, so the integrand behaves like k^{-2s+1}, so $\mathrm{Re}(s) < 1$.

We now look to understand the integral we found above for ζ_{12} where $\frac{1}{2} < \mathrm{Re}(s) < 1$.

Lemma 2.57.

$$g(s) = \int_0^\infty z^{-2s-1}(\sqrt{1+z^2}-1)\,dz = -\frac{\Gamma(-s)\Gamma(s-1/2)}{4\sqrt{\pi}} \text{ for } \frac{1}{2} < \mathrm{Re}(s) < 1.$$

Proof.

$$g(s) = \int_0^\infty z^{-2s-1}((1+z^2)^{-\alpha} - 1^{-\alpha})\,dz\Big|_{\alpha=-\frac{1}{2}}$$
$$= \int_0^\infty z^{-2s-1}\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(e^{-(1+z^2)t} - e^{-t})\,dt\,dz\Big|_{\alpha=-\frac{1}{2}}$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\int_0^\infty z^{-2s-1}(e^{-z^2t}-1)\,dz\,dt\Big|_{\alpha=-\frac{1}{2}}.$$

Next we introduce the substitution $u = z^2t$, $\frac{dz}{du} = \frac{1}{2\sqrt{ut}}$. Substituting gives

$$g(s) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\int_0^\infty \frac{1}{2\sqrt{ut}}\left(\frac{u}{t}\right)^{\frac{-2s-1}{2}}(e^{-u}-1)\,du\,dt\Big|_{\alpha=-\frac{1}{2}}$$
$$= \frac{1}{2\Gamma(\alpha)}\int_0^\infty t^{\alpha-1-\frac{1}{2}+\frac{2s+1}{2}}e^{-t}\int_0^\infty u^{-\frac{1}{2}-\frac{2s+1}{2}}(e^{-u}-1)\,du\,dt\Big|_{\alpha=-\frac{1}{2}}$$

$$= \frac{1}{2\Gamma(\alpha)} \int_0^\infty t^{\alpha+s-1} e^{-t}\, dt \int_0^\infty u^{-s-1}(e^{-u}-1)\, du \Big|_{\alpha=-\frac{1}{2}}$$

$$= \frac{1}{2\Gamma(\alpha)} \Gamma(\alpha+s)\Big|_{\alpha=-\frac{1}{2}} \int_0^\infty u^{-s-1}(e^{-u}-1)\, du$$

where we note the integral is valid for $0 < \mathrm{Re}(s) < 1$, so that we continue by introducing the complex parameter, β with $\mathrm{Re}(\beta) \leq 0$, noting the integrand is a continuous function of β,

$$= -\frac{\Gamma(s-1/2)}{4\sqrt{\pi}} \lim_{\beta\to 0} \int_0^\infty u^{-s-1}(e^{-u}-e^{\beta u})\, du = -\frac{\Gamma(-s)\Gamma(s-1/2)}{4\sqrt{\pi}}.$$

□

Notice the above result can also be seen as a special case of exercise C.2 if we initially introduce a complex parameter as we did in the last step. In this case, we see that the contribution comes directly from the first integral. This is a common way of treating integrals that are valid in some region under a single integral but where directly breaking apart the integrals will result in at least one divergent integral.

So we have surmised that

$$\zeta_{12}(s) = -\frac{1}{2}\frac{\sin(\pi s)}{\pi^{3/2}} \zeta_R(2s-1)\Gamma(-s)\Gamma(s-1/2).$$

At $s = 1$, we note the above has 2 poles and 1 zero, so we find (applying lemma 1.28, theorem 1.32, and remark 1.34)

$$\mathrm{Res}[\zeta_{12}(s), 1] = -\frac{1}{2}\frac{1}{\pi^{3/2}}\pi\left(\frac{1}{2}\right)(-1)\Gamma(1/2) = \frac{1}{4} = (4\pi)^{-d/2}\cdot\pi\Big|_{d=2}.$$

At $s = \frac{1}{2}$, we note the above has 1 pole, so we find

$$\mathrm{Res}[\zeta_{12}(s), 1/2] = -\frac{1}{2}\frac{1}{\pi^{3/2}}\zeta_R(0)\Gamma(-1/2)(1) = -\frac{1}{2\pi}.$$

Further, at $s = 0$ we have one pole and one zero, so we find

$$\zeta_{12}(0) = -\frac{1}{2}\frac{1}{\pi^{3/2}}\pi(-1)\zeta_R(-1)\Gamma(-1/2) = \frac{1}{12}.$$

ζ_{13}: We calculate

$$\zeta_{13}(s) = 2\frac{\sin(\pi s)}{\pi} \sum_{m=1}^{\infty} m^{-2s} \int_0^\infty k^{-2s} \frac{d}{dk} \ln(1+k^2)^{-\frac{1}{4}}\, dk$$

$$= -\frac{\sin(\pi s)}{\pi} \sum_{m=1}^{\infty} m^{-2s} \int_0^{\infty} k^{-2s+1}(1+k^2)^{-1}\ dk\ (0 < \text{Re}(s) < 1)$$

$$= -\frac{\sin(\pi s)}{2\pi} \zeta_R(2s)\Gamma(1-s)\Gamma(s),$$

with the last equality a direct result of exercise C.2 with $\mu = -2s+2,\ \beta = 1,\ p = 2$, and $\nu = 1$. Hence, we find the following:
At $s = 1$, we note the above has 1 pole and 1 zero, so we find

$$\text{Res}[\zeta_{13}(s), 1] = 0.$$

At $s = \frac{1}{2}$, we note the above has 1 pole, so we find

$$\text{Res}[\zeta_{13}(s), 1/2] = -\frac{1}{2\pi}\frac{1}{2}\Gamma(1/2)\Gamma(1/2) = -\frac{1}{4}.$$

Further, at $s = 0$ we have one pole and one zero, so we find

$$\zeta_{13}(0) = -\frac{1}{2\pi}\pi\zeta_R(0)\Gamma(1)(1) = \frac{1}{4}.$$

ζ_{14}: We calculate

$$\zeta_{14}(s) = 2\frac{\sin(\pi s)}{\pi} \sum_{m=1}^{\infty} m^{-2s} \int_0^{\infty} k^{-2s} \frac{d}{dk}\left[\frac{1}{m}\left(\frac{t}{8} - \frac{5t^3}{24}\right)\right]\ dk$$

$$= \frac{1}{4}\frac{\sin(\pi s)}{\pi}\zeta_R(2s+1) \int_0^{\infty} k^{-2s}\frac{d}{dk}\left(t - \frac{5t^3}{3}\right)\ dk.$$

$$= -\frac{1}{4}\frac{\sin(\pi s)}{\pi}\zeta_R(2s+1)\Gamma(1-s)\left[\frac{\Gamma(s+1/2)}{\Gamma(1/2)} - \frac{5\Gamma(s+3/2)}{3\Gamma(3/2)}\right],$$

where we have used exercise C.2 once again with $\mu = -2s+2,\ \beta = 1,\ p = 2$, and $\nu = \frac{3}{2}$ for the first integral and $\mu = -2s+2,\ \beta = 1,\ p = 2$, and $\nu = \frac{5}{2}$ for the second integral. Hence, we find the following:
At $s = 1$, we note the above has 1 pole and 1 zero, so we find

$$\text{Res}[\zeta_{14}(s), 1] = 0.$$

At $s = 1/2$, we note the above has 1 pole and 1 zero, so we find

$$\text{Res}[\zeta_{14}(s), 1/2] = 0.$$

Further, at $s = 0$ we have one pole and one zero, so we find

$$\zeta_{14}(0) = -\frac{1}{4}\frac{\pi}{\pi}\left(\frac{1}{2}\right)\Gamma(1)\left[\frac{\Gamma(1/2)}{\Gamma(1/2)} - \frac{5\Gamma(3/2)}{3\Gamma(3/2)}\right] = \frac{1}{12}.$$

ζ_1: Hence we found that

$$\text{Res}[\zeta_1(s), 1] = \frac{1}{4}, \quad \text{Res}[\zeta_1(s), 1/2] = -\frac{1}{2\pi} - \frac{1}{4}, \quad \zeta_1(0) = \frac{5}{12}.$$

ζ: Combining all contributions, we find

$$\text{Res}[\zeta(s), 1] = \frac{1}{4} = (4\pi)^{-d/2} \cdot \pi\Big|_{d=2} \qquad (\text{Res}[\zeta(s), d/2] = (4\pi)^{-d/2} \cdot \text{vol}(M)),$$

$$\text{Res}[\zeta(s), 1/2] = -\frac{1}{4} = -\frac{1}{4}(4\pi)^{-(d-1)/2} 2\pi \cdot \frac{1}{\sqrt{\pi}}\Big|_{d=2} \qquad \text{(where } 2\pi \text{ was the area of boundary)},$$

$$\zeta(0) = \frac{1}{6} = (4\pi)^{-d/2} \cdot \frac{1}{6}\int_{S^1} 2k \, d\theta$$

(Integral over boundary of manifold where $2k$ is extrinsic curvature).

2.5.4 Relation with Topology of a Manifold

For the two disk we found

$$\zeta(0) = \frac{1}{6} = \frac{1}{2\pi}\int_{S^1} \frac{1}{r}\, d\phi = \frac{1}{12\pi} \cdot \frac{1}{r} \cdot 2\pi r.$$

Definition 2.58. *Let w be a differentiable vector field on some open set $U \subset S$, and $p \in U$. Let α be a parametrized curve $\alpha : (-\epsilon, \epsilon) \to V$ with $\alpha(0) = p$ and $\alpha'(0) = y$. The projection of $\left(\frac{dw}{dt}\right)(0)$ into the tangential plane $T_p(s)$ is called the covariant derivative of w at p with respect to y. We notate this as*

$$\left(\frac{Dw}{dt}\right)(0) \text{ or } (D_y w)(p).$$

Definition 2.59. *Let w be a differentiable unit vector field along a parametrized curve $\alpha : I \to S$. We define the algebraic value λ of the covariant derivative of w with respect to t by*

$$\frac{Dw}{dt} = \lambda(N \times W),$$

the curl where N is the normal vector.

Definition 2.60. *Let $\alpha(s)$ be a parametrization of a curve C in $U \subset S$ with $p \in U$ and s the arc length. The algebraic value of the covariant derivative*

$$\frac{D\alpha'}{ds} = \kappa g$$

is called the geodesic curvature of C at p.

**Intuitively this measures how fast the curve changes.*

Definition 2.61. *Let $S \subset \mathbb{R}^3$ be a surface and $N: S \to S^2 \subset \mathbb{R}^3$ be the map that maps $p \in S$ to the normal at p on S^2. Then N is called the Gauss map.*

Definition 2.62. *Let $p \in S$ and $dN_p: T_p(S) \to T_p(S)$. Then $\det(dN_p)$ is called the Gaussian curvature k of S at p.*

Definition 2.63 (Euler Characteristic of a Two Surface)**.**

$$\chi = \frac{1}{2\pi}\int_M k\ dx + \frac{1}{2\pi}\int_{\partial M} \kappa g\ dy + \frac{1}{2\pi}\sum_i (\pi - \theta_i) \text{ where } \theta_i \text{ is interior angle of vertex.}$$

Remark 2.64. Here we notice that the second term is what we found when evaluating $\zeta(0)$! ⋄

Special Cases:

Square- $k = 0,\ \kappa g = 0,\ N = \text{constant},\ \alpha'' = 0 \Rightarrow \chi = \frac{1}{2\pi}\sum_i^4 (\pi - \pi/2) = 1.$

Circle- $k = 0,\ \kappa g = 1,\ N = \text{constant} \Rightarrow \frac{1}{2\pi}\int_{\partial M} 1\ dy = 1.$

Hemisphere- $k = 1,\ \kappa g = 0 \Rightarrow \frac{1}{2\pi}\int_M k\ dx = 1.$

2.6 Zeta Functions for General Sturm-Liouville Operators on Bounded Intervals

We consider

$$L = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + V(x)\right], \quad x \in (a,b). \tag{2.21}$$

Note these types of operators naturally come from polar coordinates as we developed at the beginning of section 2.5.

Definition 2.65. *We define the first quasi derivative as*

$$y^{[1]}(x) = p(x)y'(x).$$

2.6.1 Separated Boundary Conditions

Let $\alpha, \beta \in [0, \pi)$. Then we consider the separated boundary conditions

$$\begin{aligned} y(a)\cos\alpha + y^{[1]}(a)\sin\alpha = 0, \\ y(b)\cos\beta + y^{[1]}(b)\sin\beta = 0. \end{aligned} \tag{2.22}$$

Next we choose two linearly independent solutions $\theta(\lambda, x, a)$ and $\phi(\lambda, x, a)$ such that for $Ly = \lambda y$ we have

$$\begin{aligned} \theta(\lambda, a, a) = \phi^{[1]}(\lambda, a, a) = 1, \\ \theta^{[1]}(\lambda, a, a) = \phi(\lambda, a, a) = 0. \end{aligned} \tag{2.23}$$

These are indeed linearly independent, which can be seen by looking at the Wronskian:

$$\tilde{W}(\theta, \phi) = \theta\phi^{[1]} - \theta^{[1]}\phi = 1.$$

Writing general solutions as a linear combination we find

$$y(x) = A\theta(\lambda, x, a) + B\phi(\lambda, x, a). \tag{2.24}$$

This function will be an eigenfunction as long as (2.22) are satisfied. We introduce the following,

$$\begin{aligned} U_{\alpha,\beta,1}(y) = y(a)\cos\alpha + y^{[1]}(a)\sin\alpha \\ U_{\alpha,\beta,2}(y) = y(b)\cos\beta + y^{[1]}(b)\sin\beta. \end{aligned} \tag{2.25}$$

Then (2.24) is an eigenfunction if

$$AU_{\alpha,\beta,1}(\theta) + BU_{\alpha,\beta,1}(\phi) = 0,$$

$$AU_{\alpha,\beta,2}(\theta) + BU_{\alpha,\beta,2}(\phi) = 0$$

or

$$\begin{pmatrix} U_{\alpha,\beta,1}(\theta) & U_{\alpha,\beta,1}(\phi) \\ U_{\alpha,\beta,2}(\theta) & U_{\alpha,\beta,2}(\phi) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \tag{2.26}$$

We look at nontrivial solutions, thus eigenvalues, which occur if

$$0 = \det \begin{pmatrix} U_{\alpha,\beta,1}(\theta) & U_{\alpha,\beta,1}(\phi) \\ U_{\alpha,\beta,2}(\theta) & U_{\alpha,\beta,2}(\phi) \end{pmatrix} = \det \begin{pmatrix} \cos\alpha & \sin\alpha \\ U_{\alpha,\beta,2}(\theta) & U_{\alpha,\beta,2}(\phi) \end{pmatrix} =: F_{\alpha,\beta}(\lambda).$$

Explicitly we have

$$F_{\alpha,\beta}(\lambda) = \begin{cases} \phi(\lambda,b,a) & \alpha=\beta=0 \\ -\sin(\beta)\ \phi^{[1]}(\lambda,b,a)+\cos(\beta)\ \phi(\lambda,b,a) & \alpha=0,\ \beta\in(0,\pi) \\ \cos(\alpha)\ \phi(\lambda,b,a)-\sin(\alpha)\ \theta(\lambda,b,a) & \alpha\in(0,\pi),\ \beta=0 \\ \cos(\alpha)[-\sin(\beta)\ \phi^{[1]}(\lambda,b,a)+\cos(\beta)\ \phi(\lambda,b,a)] & \\ \ -\sin(\alpha)[-\sin(\beta)\ \theta^{[1]}(\lambda,b,a)+\cos(\beta)\ \theta(\lambda,b,a)] & \alpha,\beta\in(0,\pi) \end{cases} \tag{2.27}$$

and thus

$$\zeta_L(s) = \frac{1}{2\pi i}\oint_\gamma z^{-s}\frac{d}{dz}\ln F_{\alpha,\beta}(z)\ dz = \sum_l \lambda_l^{-s} \text{ for } \mathrm{Re}(s)>1,\ \alpha,\beta\in[0,\pi).$$

2.6.2 Coupled Boundary Conditions

We will first consider the following weighted Hilbert space inner product

$$\langle\phi,\psi\rangle = \int_a^b \phi(x)\phi^*(x)\ r(x)dx.$$

Now let us consider for the operator L as in (2.21),

$$\langle Lf,g\rangle - \langle f,Lg\rangle = 0.$$

We calculate

$$\langle Lf,g\rangle - \langle f,Lg\rangle = 0$$

$$\int_a^b (Lf)g^*\ r(x)dx - \int_a^b f(Lg^*)\ r(x)dx$$

$$\int_a^b -\left(\frac{d}{dx}f^{[1]}(x)\right)g^*(x)\ dx + \int_a^b f(x)\left(\frac{d}{dx}g^{*[1]}(x)\right)\ dx = 0$$

$$\left[-f^{[1]}(x)g^*(x)+f(x)g^{*[1]}(x)\right|_a^b + \int_a^b f^{[1]}(x)\left(\frac{d}{dx}g^*(x)\right)\ dx - \int_a^b \left(\frac{d}{dx}f(x)\right)g^{*[1]}(x)\ dx = 0$$

$$\left[-f^{[1]}(x)g^*(x)+f(x)g^{*[1]}(x)\right|_a^b = 0$$

$$-f^{[1]}(b)g^*(b)+f(b)g^{*[1]}(b) = -f^{[1]}(a)g^*(a)+f(a)g^{*[1]}(a)$$

$$\det\begin{pmatrix} f(b) & g^*(b) \\ f^{[1]}(b) & g^{*[1]}(b)\end{pmatrix} = \det\begin{pmatrix} f(a) & g^*(a) \\ f^{[1]}(a) & g^{*[1]}(a)\end{pmatrix}.$$

Natural guess for boundary conditions: $\varphi \in [0, 2\pi),\ R \in SL_2(\mathbb{R})$

$$\begin{pmatrix} y(b) \\ y^{[1]}(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} y(a) \\ y^{[1]}(a) \end{pmatrix}. \tag{2.28}$$

As before, we choose two linearly independent solutions $\theta(\lambda, x, a)$ and $\phi(\lambda, x, a)$ such that for $Ly = \lambda y$ we have

$$\begin{aligned} \theta(\lambda, a, a) = \phi^{[1]}(\lambda, a, a) = 1, \\ \theta^{[1]}(\lambda, a, a) = \phi(\lambda, a, a) = 0. \end{aligned} \tag{2.29}$$

Writing general solutions as a linear combination we find

$$y(x) = A\theta(\lambda, x, a) + B\phi(\lambda, x, a). \tag{2.30}$$

This function will be an eigenfunction as long as (2.28) are satisfied. We introduce the following,

$$V_{\varphi,R,1}(y) = y(b) - e^{i\varphi} R_{11} y(a) - e^{i\varphi} R_{12} y^{[1]}(a),$$

$$V_{\varphi,R,2}(y) = y^{[1]}(b) - e^{i\varphi} R_{21} y(b) - e^{i\varphi} R_{22} y^{[1]}(a).$$

Then (2.30) is an eigenfunction if

$$AV_{\varphi,R,1}(\theta) + BV_{\varphi,R,1}(\phi) = 0,$$

$$AV_{\varphi,R,2}(\theta) + BV_{\varphi,R,2}(\phi) = 0$$

or

$$\begin{pmatrix} V_{\varphi,R,1}(\theta) & V_{\varphi,R,1}(\phi) \\ V_{\varphi,R,2}(\theta) & V_{\varphi,R,2}(\phi) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \tag{2.31}$$

We find nontrivial solutions if

$$F_{\varphi,R}(\lambda) := \det \begin{pmatrix} V_{\varphi,R,1}(\theta) & V_{\varphi,R,1}(\phi) \\ V_{\varphi,R,2}(\theta) & V_{\varphi,R,2}(\phi) \end{pmatrix} = 0.$$

Explicitly we have

$$\begin{aligned} F_{\varphi,R}(\lambda) = {} & e^{2i\varphi} R_{11} R_{21} \phi(\lambda, b, a) - e^{i\varphi} R_{11} \phi^{[1]}(\lambda, b, a) \\ & - [e^{i\varphi} R_{22} + e^{2i\varphi} R_{12} R_{21}] \theta(\lambda, b, a) + e^{i\varphi} R_{12} \theta^{[1]}(\lambda, b, a) + e^{2i\varphi} R_{11} R_{22} + 1. \end{aligned} \tag{2.32}$$

Example 2.66 (See exercise 2.3). *We start by considering*

$$-\frac{d^2}{dx^2}y(x) = \lambda y(x), \quad y(0) = e^{-2\pi ia}y(2\pi), \quad y'(0) = e^{-2\pi ia}y'(2\pi)$$

so that following (2.28)*, we find*

$$\begin{pmatrix} y(b) \\ y^{[1]}(b) \end{pmatrix} = e^{2\pi ia}\begin{pmatrix} y(a) \\ y^{[1]}(a) \end{pmatrix}, \text{ with } R = I_{2\times 2} \text{ and } \varphi = 2\pi a.$$

We find two linearly independent solutions as in (2.29)*,*

$$\theta(\lambda, x, 0) = \cos(\sqrt{\lambda}x), \quad \theta'(\lambda, x, 0) = -\sqrt{\lambda}\sin(\sqrt{\lambda}x);$$

$$\phi(\lambda, x, 0) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}, \quad \phi'(\lambda, x, 0) = \cos(\sqrt{\lambda}x)$$

and thus

$$V_{\varphi,R,1}(\theta) = \cos(\sqrt{\lambda}2\pi) - e^{2\pi ia},$$

$$V_{\varphi,R,1}(\phi) = \frac{\sin(\sqrt{\lambda}2\pi)}{\sqrt{\lambda}},$$

$$V_{\varphi,R,2}(\theta) = -\sqrt{\lambda}\sin(\sqrt{\lambda}2\pi),$$

$$V_{\varphi,R,2}(\phi) = \cos(\sqrt{\lambda}2\pi) - e^{2\pi ia}.$$

Hence we conclude

$$F_{\varphi,R}(\lambda) = (\cos(\sqrt{\lambda}2\pi) - e^{2\pi ia})^2 + \sin^2(\sqrt{\lambda}2\pi) = 2e^{2\pi ia}(\cos(2\pi a) - \cos(2\pi\sqrt{\lambda})).$$

2.7 Traces of Resolvents for Sturm-Liouville Operators

As before, we consider

$$L = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right], \quad x \in (a, b).$$

We are interested in understanding the trace of the resolvent, that is

$$Tr[(L-\lambda)^{-1}], \text{ for } \lambda \text{ not in the spectrum of } L.$$

We will calculate this value in the following examples.

Example 2.67 (Simple Case)**.** *We begin by considering the simplest case, that is letting* $r(x) = 1 = p(x)$ *and* $q(x) = 0$ *for* $x \in (0, \pi)$*, so that*

$$L = -\frac{d^2}{dx^2}, \quad L\phi_n = \lambda_n \phi_n, \quad \phi(0) = 0 = \phi(\pi).$$

Then

$$\phi_n(x) = \sin(nx) \text{ and } \lambda_n = n^2, \quad n \in \mathbb{N}.$$

Calculating the Resolvent Trace:

$$Tr[(L-\lambda)^{-1}] = \sum_{n=1}^{\infty} (n^2 - \lambda)^{-1} = -\sum_{n=1}^{\infty} \frac{d}{d\lambda} \ln(n^2 - \lambda)$$

$$= -\sum_{n=1}^{\infty} \frac{d}{d\lambda} \left[\ln n^2 + \ln\left(1 - \frac{\lambda}{n^2}\right)\right] = -\sum_{n=1}^{\infty} \frac{d}{d\lambda} \ln\left(1 - \frac{\lambda}{n^2}\right) = -\frac{d}{d\lambda} \sum_{n=1}^{\infty} \ln\left(1 - \frac{\lambda}{n^2}\right)$$

$$= -\frac{d}{d\lambda} \ln \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{n^2}\right).$$

We want $f(\lambda)$ *such that* $f(n^2) = 0$, $n \in \mathbb{N}$*, so by the Hadamard Factorization theorem (theorem* A.8*), we find*

$$f(\lambda) = \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}\pi}.$$

This suggests that

$$\prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{n^2}\right) = \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}\pi}$$

and we continue the above calculation

$$Tr[(L-\lambda)^{-1}] = -\frac{d}{d\lambda} \ln \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}\pi} = \frac{1 - \pi\sqrt{\lambda}\cot(\sqrt{\lambda}\pi)}{2\lambda}.$$

Now let us conduct a little check to see if this makes sense.

$$\lim_{\lambda \to 0} Tr[(L-\lambda)^{-1}] = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

so it checks out.

Example 2.68 (General Case)**.**

$$Tr[(L-\lambda)^{-1}] = \sum_{l=1}^{\infty} \frac{1}{\lambda_l - \lambda} = -\sum_{l=1}^{\infty} \frac{d}{d\lambda} \ln(\lambda_l - \lambda)$$

$$= -\sum_{l=1}^{\infty} \frac{d}{d\lambda} \ln\left(1 - \frac{\lambda}{\lambda_l}\right) = -\frac{d}{d\lambda} \sum_{l=1}^{\infty} \ln\left(1 - \frac{\lambda}{\lambda_l}\right) = -\frac{d}{d\lambda} \ln \prod_{l=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_l}\right)$$

where we note that we have from previous work in section 2.6*,*

$$\prod_{l=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_l}\right) \approx F(\lambda)$$

up to a constant. Hence we find for general Sturm-Liouville operators,

$$Tr[(L-\lambda)^{-1}] = -\frac{d}{d\lambda} \ln F(\lambda).$$

2.8 Exercises

Exercise 2.1. Show that

$$\cot z = \frac{\sin(2x) - i\sinh(2y)}{\cosh(2y) - \cos(2x)}.$$

Exercise 2.2. Show the following is well defined in the strip $\frac{1}{2} < \text{Re}(s) < 1$,

$$\int_0^{\infty} x^{-2s} \frac{d}{dx} \ln\left[\frac{\sin(ix\pi)}{ix}\right] dx.$$

Exercise 2.3. Determine the functional determinant associated with the BVP

$$P\phi_l(x) := -\frac{\partial^2}{\partial x^2}\phi_l(x) = \lambda_l \phi_l(x); \qquad \phi_l(0) = e^{-2\pi i a}\phi_l(2\pi); \qquad \phi_l'(0) = e^{-2\pi i a}\phi_l'(2\pi).$$

Hint: Derive the starting point

$$\zeta_P(s) = \frac{1}{2\pi i} \oint_{\gamma} x^{-2s} \frac{d}{dx} \ln(\cos(2\pi x) - \cos(2\pi a))\, dx,$$

where γ is enclosing all eigenvalues counterclockwise and proceed as we did for the Zeta function of Riemann.

Exercise 2.4. Using separation of variables, determine the eigenfunction and eigenvalues for the following problem,

$$\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}\right)\phi_\lambda(x,y,z)=\lambda\phi_\lambda(x,y,z),$$
$$\phi_\lambda(0,y,z)=\phi_\lambda(a,y,z)=0;$$
$$\phi_\lambda(x,0,z)=\phi_\lambda(x,L,z);\qquad \phi'_\lambda(x,0,z)=\phi'_\lambda(x,L,z);$$
$$\phi_\lambda(x,y,0)=\phi_\lambda(x,y,L);\qquad \phi'_\lambda(x,y,0)=\phi'_\lambda(x,y,L).$$

Exercise 2.5. Show that the general second-order equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+l(x)y=\lambda r(x)y,\qquad a\le x\le b,$$

with separated BC, where $p(x)$ and $r(x)$ are positive in $[a,b]$, can be reduced to the following

$$-\frac{d^2u}{dz^2}+V(x)u=\mu u$$

with separated boundary conditions.
Hint: Let

$$c=\frac{1}{\pi}\int_a^b\left(\frac{r(x)}{p(x)}\right)^{1/2}dx.$$

Substitute

$$z=\frac{1}{c}\int_a^x\left(\frac{r(\tau)}{p(\tau)}\right)^{1/2}d\tau,\quad u=(r(x)p(x))^{1/4}y,$$

to see

$$\mu=c^2\lambda,\quad V(x)=\frac{\theta''(z)}{\theta(z)}+c^2\frac{l(x)}{r(x)}\text{ with }\theta(z)=(r(x)p(x))^{1/4}.$$

Exercise 2.6. Improve the asymptotic estimates found in chapter 2 by completing the following.

i) Continue the process in the proof of lemma 2.21 to find the next to leading order behavior of $\psi(x,s^2)$ as $|s|\to\infty$ for $V\in C^1([0,\pi])$.

ii) Continue the process from i) to find the next order behavior of $\psi(x,s^2)$ as $|s|\to\infty$ for $V\in C^2([0,\pi])$.

Exercise 2.7. Improve the asymptotic estimates as needed to find an improved version of theorem 2.23 with $V\in C^3([0,\pi])$ and $s_n^2>0$. In detail, show that

$$s_n=n+\frac{\alpha_1}{n}+\frac{\alpha_3}{n^3}+\mathcal{O}\left(\frac{1}{n^4}\right),\text{ where }\alpha_1=\frac{1}{2\pi}\int_0^\pi V(\tau)\,d\tau.$$

Determine α_3 explicitly.

For exercises 2.8 and 2.9, we will consider the eigenvalue problem with mixed boundary conditions. Let $I = [0, \pi]$ and let $V \in C^\infty([0, \pi])$ be real. Eigenvalues λ_n and eigenfunctions $y_n(x)$ for this problem are defined by

$$-\frac{d^2}{dx^2} y_n(x) + V(x) y_n(x) = \lambda_n y_n(x), \quad y_n'(x) = 0, \quad y_n(\pi) = 0.$$

References for these exercises are [GK19], [KL08], [KM03], and [KM04].

Exercise 2.8. **A**s a warm-up, assume $V(x) = 0$ for $x \in I$. Find the zeta function associated with this problem and evaluate $\zeta_P'(0)$.
Hint: The zeta function you are supposed to find can be expressed in terms of $\zeta_R(s)$. Finding $\zeta_P'(0)$ therefore reduces to knowing $\zeta_R'(0)$.

Exercise 2.9. For general $V \in C^\infty([0, \pi])$, find $\zeta'(0)$ following the steps described below.

i) Relate the BVP to a suitable IVP following the ideas presented for Dirichlet boundary conditions.

ii) Find an integral equation for the solution of the relevant IVP. It should read

$$\psi(x, \lambda) = \cos(sx) + \frac{1}{s} \int_0^x \sin[s(x - \tau)] V(\tau) \psi(\tau, \lambda)\, d\tau.$$

iii) Let $\lambda = s^2$ and $s = \sigma + it$. Show there exists $s_0 > 0$ such that for $|s| > s_0$ one has the estimate

$$\psi(x, \lambda) = \cos(sx) + \mathcal{O}\left(\frac{e^{|t|x}}{|s|}\right).$$

iv) Show that the large-n asymptotic of the eigenvalues is given by

$$s_n = n + \frac{1}{2} + \frac{\alpha}{n + \frac{1}{2}} + \mathcal{O}\left(\frac{1}{n^2}\right), \text{ where } \alpha = \frac{1}{2\pi} \int_0^\pi V(\tau)\, d\tau.$$

v) Use the contour integral formalism to show

$$\zeta_P'(0) = -\ln(2\psi(\pi, 0)).$$

Appendix A

Basics of Complex Analysis

In this appendix, we will review some of the basic results from complex analysis used within this text.
Notation: Given any closed, simple (i.e. not self-intersecting) curve Γ, we will denote the interior of the region enclosed by Γ as (Γ). The union of the curve (the boundary of the interior) and the interior will be denoted $[\Gamma]$.
Notation: We let $H(\cdot)$ denote the set of holomorphic functions on $\cdot$.

A.1 Morera's Theorem

Theorem A.1 (Morera's theorem)**.** *If $f(z)$ is continuous in an open set Ω and*

$$\int_\gamma f(z)\ dz = 0 \text{ for every piecewise, close, simple curve } \gamma \in \Omega,$$

then $f \in H(\Omega)$.

A.2 Cauchy Residue Theorem

Theorem A.2 (Cauchy Residue theorem)**.** *Suppose C is a circle and f is holomorphic in an open disk containing C except at points $z_1, \ldots z_m$, all inside C. Suppose further that f has a pole at each z_i, $1 \leq i \leq m$, each with residue α_i. Then*

$$\frac{1}{2\pi i}\int_C f(z)\ dz = \sum_{j=1}^{m} \alpha_j.$$

A.3 Argument Principle and Applications

Let Γ be a closed, simple curve in the complex plane and let $f(z)$ be analytic in $[\Gamma]$ with a root of order m at $z = a \in (\Gamma)$. Then

$$f(z) = (z-a)^m \phi(z), \text{ where } \phi(a) \neq 0 \text{ and } \phi \in H([\Gamma]).$$

Hence,

$$f'(z) = m(z-a)^{m-1}\phi(z) + (z-a)^m \phi'(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{\phi'(z)}{\phi(z)}.$$

Thus we find the following lemma:

Lemma A.3. *If $f(z) \in H([\Gamma])$, where Γ is a closed, simple curve and $f(z)$ has no roots on Γ, then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = N_0,$$

where $N_0 =$ the number of roots of f in Γ, counting multiplicity.

Now, let Γ be a closed, simple curve in the complex plane and let $f(z)$ have a pole of order m at $z = a \in (\Gamma)$, and be otherwise analytic in $[\Gamma]$. Then

$$f(z) = (z-a)^{-m} \phi(z), \text{ where } \phi(a) \neq 0 \text{ and } \phi \in H([\Gamma]).$$

Hence,

$$f'(z) = -m(z-a)^{-m-1}\phi(z) + (z-a)^{-m} \phi'(z)$$

and

$$\frac{f'(z)}{f(z)} = -\frac{m}{z-a} + \frac{\phi'(z)}{\phi(z)}.$$

We combine this with the previous lemma to find the argument principle:

Theorem A.4 (Argument Principle)**.** *If $f(z)$ is analytic except possibly for some poles in (Γ), where Γ is a closed, simple curve on which $f(z)$ has neither roots nor poles, then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = N_0 - N_\infty,$$

where N_0 and N_∞ are, respectively, the number of roots and poles of f in Γ, counting multiplicity.

We can extend this result further as follows:

Theorem A.5 (General Formulation of the Argument Principle)**.** *Let $f(z)$ be meromorphic and $g(z)$ analytic on $[\Gamma]$, where Γ is a closed, simple curve. Then*

$$\frac{1}{2\pi i}\int_\Gamma g(z)\frac{f'(z)}{f(z)}\ dz = \sum_{j=1}^{n} g(a_j) - \sum_{k=1}^{m} g(b_k)$$

where $a_1, \ldots, a_n$ and $b_1, \cdots_m$ are the roots and poles, respectively, of $f(z)$ in (Γ) (but none on Γ) and they occur as often as their multiplicity.

Now let us consider a simple application of this principle.

Theorem A.6. *Let $f(z)$ be a meromorphic function with a finite number of poles located at $z_1, \ldots z_m$, none of which are integers, and take C_N to be the circle $|z| = N + \dfrac{1}{2}$ with $N \in \mathbb{N}$. Then*

$$\sum_{n\in\mathbb{Z}} f(n) = -\pi \sum_{k=1}^{m} \operatorname{Res}[f(z)\cot(\pi z), z_k],$$

provided

$$\lim_{N\to\infty}\int_{C_N} f(z)\cot(\pi z)\ dz = 0.$$

Proof. Note,

$$\operatorname{Res}[\cot(\pi z), n] = \frac{1}{\pi}.$$

Choose N large enough such that all poles of $f(z)$ lie inside of C_N. We calculate,

$$\frac{1}{2\pi i}\int_{C_N} f(z)\cot(\pi z)dz = \sum_{n=-N}^{N}\frac{1}{\pi}f(n) + \sum_{k=1}^{m}\operatorname{Res}[f(z)\cot(\pi z), z_k],$$

from which we attain the assertion by sending $N \to \infty$. □

A.4 Rouché's Theorem

Theorem A.7 (Rouché's theorem)**.** *Suppose f, g are holomorphic on $[\Gamma]$, where Γ is a closed, simple curve, and $|f| > |g|$ for all z on Γ. Then the functions f and $f + g$ have the same number of roots inside Γ.*

Proof. Let

$$N(t) = \frac{1}{2\pi i}\int_\Gamma \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\ dz, \quad 0 \le t \le 1.$$

We note first that $N(t)$ counts the number of zeros of $f(z)+tg(z)$ inside Γ as $f(z)+tg(z)\neq 0$ on Γ by the assumption $|f(z)|>|g(z)|$ on Γ. Now we notice the desired result is equivalent to $N(0)=N(1)$. Further, as $N(t)$ is integer-valued, it is sufficient to show that $N(t)$ is continuous.

Hence our goal will be to see that $N(t)$ is continuous, so given $t_1\neq t_2$, $0\leq t_1,t_2\leq 1$, we calculate

$$\begin{aligned}N(t_1)-N(t_2) &= \frac{1}{2\pi i}\int_\Gamma \left[\frac{f'(z)+t_1g'(z)}{f(z)+t_1g(z)}-\frac{f'(z)+t_2g'(z)}{f(z)+t_2g(z)}\right]\,dz\\ &= \frac{1}{2\pi i}\int_\Gamma \frac{(t_1-t_2)(f(z)g'(z)-f'(z)g(z))}{(f(z)+t_1g(z))(f(z)+t_2g(z))}\,dz.\end{aligned}$$

Hence we have that

$$|N(t_1)-n(t_2)|\leq \frac{|t_1-t_2|}{2\pi}\frac{ML}{m^2},$$

where

$$\begin{aligned}M &= \max\{|f(z)g'(z)-f'(z)g(z)|,\ z\in\Gamma\},\\ m &= \min\{|f(z)|-|g(z)|,\ z\in\Gamma\}\ (>0 \text{ by assumption } |f(z)|>|g(z)|),\\ L &= \text{ length of } \Gamma.\end{aligned}$$

This implies that $N(t)$ is continuous by choosing t_1 close to t_2, making the right hand side arbitrarily small, so that the left hand side must be identically zero as $N(t)$ assumes only integer values. □

A.5 Hadamard Factorization Theorem

Theorem A.8 (Hadamard Factorization Theorem)**.** *Suppose $f(z)$ is an entire function (not identically zero) of finite growth order ρ, and $a_1,a_2,\ldots,a_k,\ldots$ are its zeros (other than the origin) repeated as often as their multiplicities. Then there exists a polynomial, $g(z)$, of degree $\leq\rho$ and an integer $\leq\rho$ such that $f(z)$ can be expressed as*

$$f(z)=e^{g(z)}z^m\prod_{k=1}^{\infty}E_m\left(\frac{z}{a_k}\right),$$

where n is the multiplicity of the root at the origin, with $n=0$ if $f(0)\neq 0$, and

$$E_m=(1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^m}{m}\right).$$

Appendix B

Solutions to Exercises

In this appendix, we will offer solutions to the given exercises in the text.

B.1 Solutions to Chapter 1 Exercises

Exercise 1.1 i).

Proof. Notice first that the term $f(x)(\lfloor x \rfloor - x) = \mathcal{O}(f(x))$ for all $f(x) \geq 0$ in Euler's Summation formula as $(\lfloor x \rfloor - x) \in (-1, 0]$ so that

$$|f(x)(\lfloor x \rfloor - x)| \leq 1 \cdot f(x) \text{ for } f(x) \geq 0.$$

Now let $f(x) = \frac{1}{x}$, $y = 1$, noting $f'(x) = -\frac{1}{x^2}$ is continuous on $[1, x]$ for $x > 1$. Then we have

$$\sum_{1<n\leq x} \frac{1}{n} = \int_1^x \frac{1}{t}\,dt - \int_1^x (t - \lfloor t \rfloor)\frac{1}{t^2}\,dt + \frac{1}{x}(\lfloor x \rfloor - x) - \frac{1}{1}(1 - 1)$$

so that

$$1 + \sum_{1<n\leq x} \frac{1}{n} = 1 + \ln x - \int_1^\infty (t - \lfloor t \rfloor)\frac{1}{t^2}\,dt + \int_x^\infty (t - \lfloor t \rfloor)\frac{1}{t^2}\,dt + \frac{1}{x}(\lfloor x \rfloor - x).$$

Notice $f(x) = \frac{1}{x} > 0$ for $x > 1$ so that by the above, $\frac{1}{x}(\lfloor x \rfloor - x) = \mathcal{O}\left(\frac{1}{x}\right)$. Further,

$$\left|\int_x^\infty (t - \lfloor t \rfloor)\frac{1}{t^2}\,dt\right| \leq \int_x^\infty \left|(t - \lfloor t \rfloor)\frac{1}{t^2}\right|\,dt \leq \int_x^\infty \frac{1}{t^2}\,dt = \frac{1}{x} = \mathcal{O}\left(\frac{1}{x}\right).$$

Hence, we calculate

$$\sum_{n\le x}\frac{1}{n} = 1+\ln x-\lim_{n\to\infty}\int_1^n \frac{t-\lfloor t\rfloor}{t^2}\ dt+\mathcal{O}\left(\frac{1}{x}\right)$$

$$= 1+\ln x-\lim_{n\to\infty}\left(\sum_{m=1}^{n-1}\int_m^{m+1}\frac{t-m}{t^2}\ dt\right)+\mathcal{O}\left(\frac{1}{x}\right)$$

$$= 1+\ln x-\lim_{n\to\infty}\left(\sum_{m=1}^{n-1}\ln t+\frac{m}{t}\Big|_m^{m+1}\right)+\mathcal{O}\left(\frac{1}{x}\right)$$

$$= 1+\ln x-\lim_{n\to\infty}\left(\sum_{m=1}^{n-1}\ln(m+1)+\frac{m}{m+1}-\ln m-1\right)+\mathcal{O}\left(\frac{1}{x}\right)$$

$$= 1+\ln x-\lim_{n\to\infty}\left(\ln n+\sum_{m=1}^{n-1}\frac{m}{m+1}-1\right)+\mathcal{O}\left(\frac{1}{x}\right)$$

$$= \ln x+\lim_{n\to\infty}\left(1-\ln n+\sum_{m=1}^{n-1}\frac{1}{m+1}\right)+\mathcal{O}\left(\frac{1}{x}\right)$$

$$= \ln x+\lim_{n\to\infty}\left(-\ln n+\sum_{m=0}^{n-1}\frac{1}{m+1}\right)+\mathcal{O}\left(\frac{1}{x}\right)$$

$$= \ln x+\gamma+\mathcal{O}\left(\frac{1}{x}\right).$$

□

Exercise 1.1 ii).

Proof. Let $f(x)=\frac{1}{x^s}$ for $s>1,\ y=1$ noting $f'(x)=-\frac{s}{x^{s+1}}$ is continuous on $[1,x]$ for $x>1$. Then we have

$$\sum_{1<n\le x}\frac{1}{n^s}=\int_1^x t^{-s}\ dt-\int_1^x(t-\lfloor t\rfloor)\frac{s}{t^{s+1}}\ dt+\frac{1}{x^s}(\lfloor x\rfloor-x)-\frac{1}{1}(1-1)$$

so that

$$1+\sum_{1<n\le x}\frac{1}{n}=1+\frac{x^{1-s}}{1-s}-\frac{1}{1-s}-\int_1^\infty(t-\lfloor t\rfloor)\frac{s}{t^{s+1}}\ dt+\int_x^\infty(t-\lfloor t\rfloor)\frac{s}{t^{s+1}}\ dt+\frac{1}{x^s}(\lfloor x\rfloor-x).$$

Further, as in 1.1 i), we see

$$\frac{1}{x^s}(\lfloor x \rfloor - x) = \mathcal{O}(s^{-s}) \text{ and } \left| \int_x^\infty (t - \lfloor t \rfloor) \frac{s}{t^{s+1}}\, dt \right| = \mathcal{O}(x^{-s}).$$

Hence, we calculate

$$\begin{aligned}
\sum_{1 \le x} \frac{1}{n^s} &= 1 + \frac{x^{1-s}}{1-s} - \frac{1}{1-s} - \int_1^\infty \frac{s}{t^s}\, dt + \lim_{n\to\infty} \left(\sum_{m=1}^{n-1} \int_m^{m+1} \frac{sm}{t^{s+1}}\, dt \right) + \mathcal{O}(x^{-s}) \\
&= 1 + \frac{x^{1-s}}{1-s} - \frac{1}{1-s} + \frac{s}{1-s} + \lim_{n\to\infty} \left(\sum_{m=1}^{n-1} -\frac{m}{t^s}\Big|_m^{m+1} \right) + \mathcal{O}(x^{-s}) \\
&= \frac{x^{1-s}}{1-s} + \lim_{n\to\infty} \left(\sum_{m=1}^{n-1} -\frac{m}{(m+1)^s} + \frac{m}{m^s} \right) + \mathcal{O}(x^{-s}) \\
&= \frac{x^{1-s}}{1-s} + \sum_{m=1}^{\infty} \frac{1}{m^s} + \mathcal{O}(x^{-s}) = \frac{x^{1-s}}{1-s} + \zeta_R(s) + \mathcal{O}(x^{-s}) \text{ for } s > 1,
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\sum_{m=1}^{n-1} -\frac{m}{(m+1)^s} + \frac{m}{m^s} &= -\frac{1}{2^s} + 1 - \frac{2}{3^s} + \frac{2}{2^s} - \frac{3}{4^s} + \frac{3}{3^s} - \cdots - \frac{n-1}{n^s} + \frac{n-1}{(n-1)^s} \\
&= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{(n-1)^s} = \sum_{m=1}^{n-1} \frac{1}{m^s}.
\end{aligned}$$

□

Exercise 1.1 iii).

Proof. Consider the result from 1.1 ii) and notice for $s > 1,\ x > 1$ we have

$$\sum_{n>x} \frac{1}{n^s} = \lim_{y\to\infty} \left(\sum_{x<n\le y} \frac{1}{n^s} \right) = \lim_{y\to\infty} \left(\sum_{n\le y} \frac{1}{n^s} - \sum_{n \le x} \frac{1}{n^s} \right)$$

$$= \lim_{y\to\infty} \left(\frac{y^{1-s}}{1-s} + \zeta_R(s) + \mathcal{O}(y^{-s}) - \frac{x^{1-s}}{1-s} - \zeta_R(s) + \mathcal{O}(x^{-s}) \right) = \mathcal{O}(x^{1-s}).$$

□

Exercise 1.1 iv).

Proof. Let $f(x) = x^\alpha,\ \alpha \geq 0,\ y = 0$ noting $f'(x) = \alpha x^{\alpha-1}$ is continuous on $[0, x]$ for $x > 0$. We calculate for $\alpha = 0$

$$\sum_{n \leq x} n^\alpha = \sum_{n \leq x} 1 = \lfloor x \rfloor = x + (\lfloor x \rfloor) = x + \mathcal{O}(1).$$

For $\alpha > 0$. we calculate as in previous exercises to find

$$\sum_{n \leq x} n^\alpha = \int_0^x t^\alpha \, dt + \alpha \int_0^x (t - \lfloor t \rfloor) t^{\alpha-1} \, dt + x^\alpha(\lfloor x \rfloor - x) - 0^\alpha(0 - 0)$$

$$= \frac{x^{\alpha+1}}{\alpha + 1} + \mathcal{O}(x^\alpha).$$

□

Exercise 1.2 i).

Proof. Let $x \geq 2,\ f(x) = \dfrac{\ln x}{x},\ y = 1$ noting $f'(x) = \dfrac{1 - \ln x}{x^2}$ is continuous on $[1, x]$ for $x > 1$. Then we have as before

$$\sum_{2 \leq n \leq x} \frac{\ln n}{n} = \int_1^x \frac{\ln t}{t} \, dt + \int_1^x (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} \, dt + \frac{\ln x}{x}(\lfloor x \rfloor - x)$$

$$= \frac{1}{2} \ln^2 x + \int_1^\infty (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} \, dt - \int_x^\infty (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} \, dt + \mathcal{O}\left(\frac{\ln x}{x}\right).$$

Further,

$$\left| \int_x^\infty (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} \, dt \right| \leq \left| \frac{1 - \ln t}{t^2} \right| \, dt \leq \int_x^\infty \frac{2 \ln t}{t^2} \, dt$$

$$= 2 \frac{\ln x + 1}{x} = 2 \left(1 + \frac{1}{\ln x}\right) \frac{\ln x}{x} = \mathcal{O}\left(\frac{\ln x}{x}\right).$$

Hence the assertion holds with

$$A = \int_1^\infty (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} \, dt.$$

□

Exercise 1.2 ii).

Proof. Let $x \geq 2$, $f(x) = \frac{1}{x\ln x}$, $y = \frac{3}{2}$ noting $f'(x) = \frac{-\ln x - 1}{(x\ln x)^2}$ is continuous on $[3/2, x]$ for $x > 3/2$. Then we have as before

$$\sum_{2\leq n\leq x} \frac{1}{n\ln n} = \int_{3/2}^{x} \frac{1}{t\ln t}\, dt - \int_{3/2}^{x} (t - \lfloor t\rfloor)\frac{\ln t + 1}{(t\ln t)^2}\, dt + \frac{1}{x\ln x}(\lfloor x\rfloor - x) - \frac{1}{3/2\ln 3/2}\left(-\frac{1}{2}\right)$$

$$= \ln(\ln x) - \ln(\ln 3/2) - \int_{3/2}^{\infty} (t - \lfloor t\rfloor)\frac{\ln t + 1}{(t\ln t)^2}\, dt + \int_{x}^{\infty} (t - \lfloor t\rfloor)\frac{\ln t + 1}{(t\ln t)^2}\, dt + \mathcal{O}\left(\frac{1}{x\ln x}\right) + \frac{1}{3\ln 3/2}$$

Further,

$$\left|\int_{x}^{\infty} (t - \lfloor t\rfloor)\frac{\ln t + 1}{(t\ln t)^2}\, dt\right| \leq \left|\frac{1}{x\ln x}\right| = \frac{1}{x\ln x} = \mathcal{O}\left(\frac{1}{x\ln x}\right).$$

Hence the assertion holds with

$$B = -\ln(\ln 3/2) + \frac{1}{3\ln 3/2} - \int_{3/2}^{\infty} (t - \lfloor t\rfloor)\frac{\ln t + 1}{(t\ln t)^2}\, dt.$$

□

Exercise 1.3 i).

Proof. Let $x \geq 2$, noting that $\frac{d}{dt}\frac{t}{\ln t} = \frac{1}{\ln t} - \frac{1}{\ln^2 t}$. Then we have

$$Li(x) = \int_{2}^{x} \frac{1}{\ln t}\, dt = \int_{2}^{x} \frac{d}{dt}\frac{t}{\ln t}\, dt + \int_{2}^{x} \frac{1}{\ln^2 t}\, dt = \frac{x}{\ln x} + \int_{2}^{x} \frac{1}{\ln^2 t}\, dt - \frac{2}{\ln 2}.$$

Further, as $\frac{d}{dt}\frac{t}{\ln^2 t} = \frac{1}{\ln^2 t} - \frac{2}{\ln^3 t}$, we see

$$Li(x) = \frac{x}{\ln x} + \int_{2}^{x} \frac{d}{dt}\frac{t}{\ln^2 t}\, dt + \int_{2}^{x} \frac{2}{\ln^3 t}\, dt - \frac{2}{\ln 2} = \frac{x}{\ln x}\left(1 + \frac{1}{\ln x}\right) + 2!\int_{2}^{x} \frac{dt}{\ln^3 t} + \left(\frac{-2}{\ln 2} + \frac{-2}{\ln^2 2}\right).$$

Thus the $n = 2$ case is establish. Now assume $n = m$ holds, and we shall show $n = m + 1$ holds, thus showing the result through induction. Assume we have

$$Li(x) = \frac{x}{\ln x}\left(1 + \sum_{k=1}^{m-1} \frac{k!}{\ln^k x}\right) + m!\int_{2}^{x} \frac{dt}{\ln^{m+1} t} + \sum_{k=1}^{m} -\frac{2(k-1)!}{\ln^k 2}$$

noting that

$$\frac{d}{dt}\frac{t}{\ln^{m+1} t} = \frac{1}{\ln^{m+1} t} - \frac{m+1}{\ln^{m+2} t}.$$

Hence we see

$$Li(x) = \frac{x}{\ln x}\left(1+\sum_{k=1}^{m-1}\frac{k!}{\ln^k x}\right) + m!\int_2^x \frac{d}{dt}\frac{t}{\ln^{m+1}t}\ dt + m!\int_2^x \frac{m+1}{\ln^{m+2}t}\ dt + \sum_{k=1}^{m} -\frac{2(k-1)!}{\ln^k 2}$$

$$= \frac{x}{\ln x}\left(1+\sum_{k=1}^{m-1}\frac{k!}{\ln^k x}\right) + m!\frac{x}{\ln^{m+1}x} - m!\frac{2}{\ln^{m+1}2} + m!\int_2^x \frac{m+1}{\ln^{m+2}t}\ dt + \sum_{k=1}^{m} -\frac{2(k-1)!}{\ln^k 2}$$

$$= \frac{x}{\ln x}\left(1+\sum_{k=1}^{m}\frac{k!}{\ln^k x}\right) + (m+1)!\int_2^x \frac{dt}{\ln^{m+2}t} + \sum_{k=1}^{m+1} -\frac{2(k-1)!}{\ln^k 2}.$$

Hence the assertion holds with

$$C_m = \sum_{k=1}^{m} -\frac{2(k-1)!}{\ln^k 2}.$$

□

Exercise 1.3 ii).

Proof. Let $x \geq 2$. Note that we have

$$\lim_{x\to\infty}\frac{x}{\ln^n x} = \infty$$

with the only discontinuity outside of $x \geq 2$, so by L'Hospital's rule we see

$$\lim_{x\to\infty}\left(\frac{\int_2^x \frac{dt}{\ln^n t}}{\frac{x}{\ln^n x}}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{\ln^n x}}{\frac{1}{\ln^n x} - \frac{n}{\ln^{n+1}x}}\right) = \lim_{x\to\infty}\frac{1}{1-\frac{n}{\ln x}} = \frac{1}{1} = 1.$$

Hence, we have shown that

$$\int_2^x \frac{dt}{\ln^n t} = \mathcal{O}\left(\frac{x}{\ln^n x}\right).$$

□

Exercise 1.4.

Proof. We first show i) implies ii). Suppose

$$\pi(x) = \frac{x}{\ln x} + \mathcal{O}\left(\frac{x}{\ln^2 x}\right).$$

Then by theorem 1.12 with $x \geq 2$, we see

$$\theta(x) = \pi(x)\ln x - \int_2^x \frac{\pi(t)}{t}\ dt = x + \ln x\,\mathcal{O}\left(\frac{x}{\ln^2 x}\right) - \int_2^x \frac{dt}{\ln t} - \mathcal{O}\left(\int_2^x \frac{dt}{\ln^2 t}\right)$$

$$= x + \mathcal{O}\left(\frac{x}{\ln x}\right) - \int_2^x \frac{dt}{\ln t} - \mathcal{O}\left(\int_2^x \frac{dt}{\ln^2 t}\right) = x + \mathcal{O}\left(\frac{x}{\ln x}\right),$$

where we have utilized results found in 1.3 ii).

Now we show ii) implies i). Suppose

$$\theta(x) = x + \mathcal{O}\left(\frac{x}{\ln x}\right).$$

Similarly, by theorem 1.12 with $x \geq 2$, we see

$$\pi(x) = \frac{\theta(x)}{\ln x} + \int_2^x \frac{\theta(t)}{t\ln^2 t} dt = \frac{x}{\ln x} + \frac{1}{\ln x}\mathcal{O}\left(\frac{x}{\ln x}\right) + \int_2^x \frac{dt}{\ln^2 t} + \mathcal{O}\left(\int_2^x \frac{dt}{\ln^3 t}\right)$$

$$= \frac{x}{\ln x} + \mathcal{O}\left(\frac{x}{\ln^2 x}\right) + \int_2^x \frac{dt}{\ln^2 t} + \mathcal{O}\left(\int_2^x \frac{dt}{\ln^3 t}\right) = \frac{x}{\ln x} + \mathcal{O}\left(\frac{x}{\ln^2 x}\right).$$

□

Exercise 1.5.

Proof. First note that we have the following Taylor series,

$$e^{xz} = \sum_{j=0}^{\infty} \frac{x^j z^j}{j!}$$

Then by definition of the Bernoulli polynomials and numbers we see,

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = (e^{xz}) \frac{z}{e^z - 1} = \left(\sum_{n=0}^{\infty} \frac{x^n z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{x^{n-k} z^{n-k}}{(n-k)!} \frac{B_k}{k!} z^k\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{1}{(n-k)!k!} B_k x^{n-k} z^n\right)$$

By comparing both sides, multiplying by $n!$, and dividing by z^n, we see the desired result,

$$B_n(x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} B_k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$

□

Exercise 1.6.

Proof. By definition of the Bernoulli polynomials we see,

$$\sum_{n=0}^{\infty} \frac{B_n(x+1)}{n!} z^n - \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{ze^{(x+1)z}}{e^z-1} - \frac{ze^{xz}}{e^z-1} = \left(\frac{ze^{xz}}{e^z-1}\right)(e^z-1)$$

$$= ze^{xz} = z\sum_{j=0}^{\infty} \frac{x^j z^j}{j!} = \sum_{j=0}^{\infty} \frac{x^j z^{j+1}}{j!}$$

By letting $j = n-1$ we see,

$$\frac{B_n(x+1)}{n!} z^n - \frac{B_n(x)}{n!} z^n = \frac{x^{n-1}}{(n-1)!} z^n$$

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

Further, for $n \geq 2$ with letting $x = 0$, we have $B_n(1) - B_n(0) = 0$ as desired. □

Exercise 1.7.

Proof. We let $x = 1$ to notice that with exercises 1.5 and 1.6, for $n \geq 2$, we have,

$$B_n = B_n(0) = B_n(1) = \sum_{k=0}^{n} \binom{n}{k} B_k$$

□

Exercise 1.8.

Proof. To see the first relation, note the following,

$$\sum_{n=0}^{\infty} \frac{B_n(1-x)}{n!} z^n = \frac{ze^{(1-x)z}}{e^z-1} = \frac{ze^{-xz}}{1-e^{-z}} = \frac{(-z)e^{x(-z)}}{e^{-z}-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!}(-z)^n = \sum_{n=0}^{\infty} (-1)^n \frac{B_n(x)}{n!} z^n$$

From which we see,

$$B_n(1-x) = (-1)^n B_n(x)$$

To see the second relation, we replace x with $-x$ and apply the first relation, then exercise 1.6 to see,

$$(-1)^n B_n(-x) = B_n(1+x) = B_n(x) + nx^{n-1}$$

□

Exercise 1.9.

Proof. We define the Hurwitz zeta function as above, noting $(n+a) > 0$, so that by lemma 1.22 with $\text{Re}(s) > 0$ we see,

$$\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} \sum_{n=0}^{\infty} e^{-nt} dt$$

Which by the geometric series gives us,

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} \frac{1}{1-e^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2} \frac{(-t)e^{a(-t)}}{e^{-t}-1} dt$$

$$= \frac{1}{\Gamma(s)} \int_0^{1} t^{s-2} \frac{(-t)e^{a(-t)}}{e^{-t}-1} dt + \frac{1}{\Gamma(s)} \int_1^{\infty} t^{s-2} \frac{(-t)e^{a(-t)}}{e^{-t}-1} dt$$

The second term does not contribute to any of the properties we are considering because it is analytic everywhere and $[\Gamma(-n)]^{-1} = 0$ for nonnegative integers.
We analyze the first term by applying the definition of Bernoulli polynomials to see,

$$\frac{1}{\Gamma(s)} \int_0^{1} t^{s-2} \frac{(-t)e^{a(-t)}}{e^{-t}-1} dt = \frac{1}{\Gamma(s)} \sum \int_0^1 t^{s-2} \sum_{n=0}^{\infty} \frac{B_n(a)}{n!} (-t)^n dt$$

$$= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{B_n(a)}{n!} (-1)^n \int_0^1 t^{s-2+n} dt$$

$$= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{B_n(a)}{n!} \frac{(-1)^n}{s-1+n}$$

Now we see that at $s = 1$ we get a first order pole at $n = 0$, so that,

$$\text{Res}[\zeta_H(s,a), 1] = \frac{1}{\Gamma(1)} \frac{B_0(a)}{0!} (-1)^0 = B_0(a) = 1$$

Further, we calculate the second property by evaluating at $-n$ as follows,

$$\zeta_H(-n,a) = \lim_{s \to -n} \left[\frac{1}{\Gamma(s)} \frac{B_{n+1}(a)}{(n+1)!} \frac{(-1)^{n+1}}{s+n} \right]$$

By lemma 1.21, we use the singular points of $\Gamma(s)$ to find the desired result below,

$$= \lim_{s \to -n} \left[\frac{n!(s+n)(-1)^{n+1}}{(-1)^n (n+1)!(s+n)} B_{n+1}(a) \right] = -\frac{B_{n+1}(a)}{n+1}$$

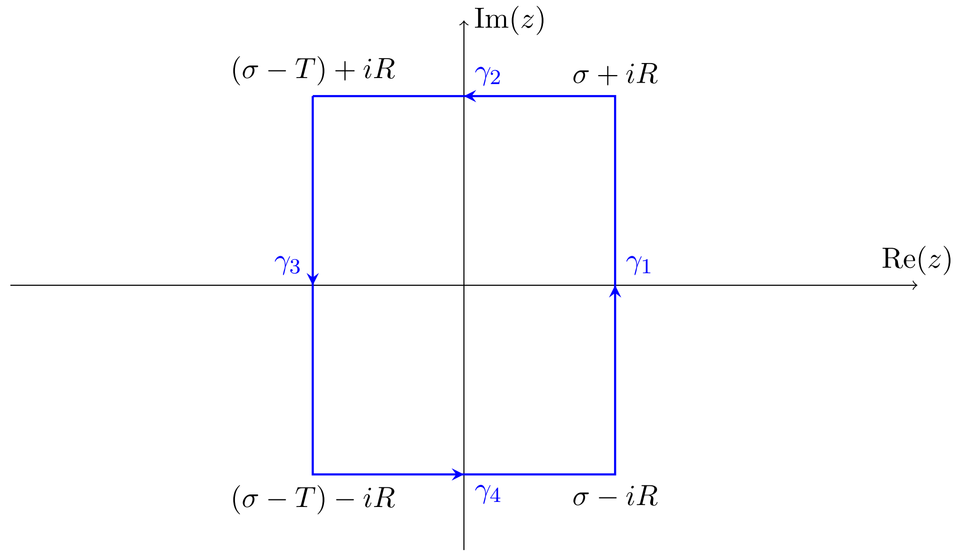
□

Exercise 1.10.

Proof. First note that we have the following Maclaurin series,

$$e^{-a} = \sum_{n=0}^{\infty} \frac{(-1)^n a^n}{n!}$$

Next, define $f(z) := a^{-z}\Gamma(z)$ and consider $\oint_\gamma f(z)dz$ with the path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ defined as below with $R, T > 0$,



Notice the only poles of $f(z)$ are exactly the poles of $\Gamma(z)$, so that by lemma 1.21 and the Cauchy Residue Theorem we have as $R \to \infty$ and $T \to \infty$,

$$\oint_\gamma f(z)dz = 2\pi i \sum_{n=0}^{\infty} \text{Res}[f(z), -n] = 2\pi i \sum_{n=0}^{\infty} \frac{(-1)^n a^n}{n!} = 2\pi i e^{-a}$$

Next we calculate each of the integrals for γ_1, γ_2, γ_3, and γ_4.
γ_1: We use $z = t$ with $\sigma - i\infty \leq t \leq \sigma + i\infty$ to see,

$$\oint_{\gamma_1} f(z)dz = \int_{\sigma - i\infty}^{\sigma + i\infty} a^{-t}\Gamma(t)dt$$

which we note is the integral we are seeking to solve for.

γ_2: We use $z = (\sigma - t) + iR$ with $0 \le t \le T$ to see,

$$\oint_{\gamma_2} f(z)dz = -\int_0^T a^{-(\sigma-t)-iR}\Gamma((\sigma-t)+iR)dt$$

Now we estimate using the first given identity from the problem as follows,

$$\left|-\int_0^T a^{-(\sigma-t)-iR}\Gamma((\sigma-t)+iR)dt\right| = \left|-\int_0^T a^{-(\sigma-t)-iR}\Gamma((\sigma-t)+iR)\frac{e^{R\pi/2}R^{1/2-\sigma+t}}{e^{R\pi/2}R^{1/2-\sigma+t}}dt\right|$$

$$\le T\left|e^{(-\sigma+T-iR)(\ln|a|+i\arg(a))}\right| |\Gamma((\sigma-T)+iR)| \frac{e^{R\pi/2}R^{1/2-\sigma+T}}{e^{R\pi/2}R^{1/2-\sigma+T}}$$

$$= Te^{(-\sigma+T)\ln|a|+R\arg(a)} |\Gamma((\sigma-T)+iR)| \frac{e^{R\pi/2}R^{1/2-\sigma+T}}{e^{R\pi/2}R^{1/2-\sigma+T}}$$

Now taking $R \to \infty$ we have,

$$\lim_{R\to\infty}\left|-\int_0^T a^{-(\sigma-t)-iR}\Gamma((\sigma-t)+iR)dt\right| \le \sqrt{2\pi}Te^{(-\sigma+T)\ln|a|}\lim_{R\to\infty} e^{R(\arg(a)-\pi/2)}R^{-1/2+\sigma-T}$$

$$\le \sqrt{2\pi}T|a|^{-\sigma+T}\lim_{R\to\infty} e^{-\delta R}R^{-1/2+\sigma-T}$$

as $|arg(a)| < \frac{\pi}{2} - \delta$, which we see goes to 0 as $R \to \infty$. Thus we have,

$$\oint_{\gamma_2} f(z)dz \to 0 \text{ as } R \to \infty \text{ and } T \to \infty$$

γ_4: We use $z = (\sigma - t) - iR$ with $0 \le t \le T$ to see,

$$\oint_{\gamma_4} f(z)dz = \int_0^T a^{-(\sigma-t)+iR}\Gamma((\sigma-t)-iR)dt$$

Now we again estimate using the first given identity from the problem as follows,

$$\left|\int_0^T a^{-(\sigma-t)+iR}\Gamma((\sigma-t)-iR)dt\right| = \left|\int_0^T a^{-(\sigma-t)+iR}\Gamma((\sigma-t)-iR)\frac{e^{R\pi/2}R^{1/2-\sigma+t}}{e^{R\pi/2}R^{1/2-\sigma+t}}dt\right|$$

$$\le T\left|e^{(-\sigma+T+iR)(\ln|a|+i\arg(a))}\right| |\Gamma((\sigma-T)-iR)| \frac{e^{R\pi/2}R^{1/2-\sigma+T}}{e^{R\pi/2}R^{1/2-\sigma+T}}$$

$$= Te^{(-\sigma+T)\ln|a| - R\arg(a)} \left|\Gamma((\sigma - T) - iR)\right| \frac{e^{R\pi/2}R^{1/2-\sigma+T}}{e^{R\pi/2}R^{1/2-\sigma+T}}$$

Now taking $R \to \infty$ we have,

$$\lim_{R\to\infty} \left| \int_0^T a^{-(\sigma-t)+iR}\Gamma((\sigma - t) - iR)dt \right| \leq \sqrt{2\pi}Te^{(-\sigma+T)log|a|} \lim_{R\to\infty} e^{-R(\arg(a)+\pi/2)}R^{-1/2+\sigma-T}$$

$$\leq \sqrt{2\pi}T|a|^{-\sigma+T} \lim_{R\to\infty} e^{-\delta R}R^{-1/2+\sigma-T}$$

as $|arg(a)| < \frac{\pi}{2} - \delta$, which we see goes to 0 as $R \to \infty$. Thus we have,

$$\oint_{\gamma_4} f(z)dz \to 0 \text{ as } R \to \infty \text{ and } T \to \infty$$

γ_3: We use $z = (\sigma - T) + i(R - 2t)$ with $0 \leq t \leq R$ to see,

$$\oint_{\gamma_3} f(z)dz = -2i \int_0^R a^{-(\sigma-T)-i(R-2t)}\Gamma((\sigma - T) + i(R - 2t))dt$$

Now we estimate using the second given identity from the problem as follows,

$$\left| -2i \int_0^R a^{-(\sigma-T)-i(R-2t)}\Gamma((\sigma - T) + i(R - 2t))dt \right|$$

$$\leq 2R \left| e^{[(T-\sigma)-i(R-2R)](\ln|a|+i\arg(a))} \right| \left|\Gamma((\sigma - T) + i(R - 2R))\right|$$

$$= 2Re^{(T-\sigma)\ln|a|-R\arg(a)}|\Gamma(\sigma - T - iR)|$$

Now taking $T \to \infty$ we have,

$$\lim_{T\to\infty} \left| -2i \int_0^R a^{-(\sigma-T)-i(R-2t)}\Gamma((\sigma - T) + i(R - 2t))dt \right|$$

$$\leq 2\sqrt{2\pi}R \lim_{T\to\infty} e^{(T-\sigma)\ln|a|-R\arg(a)} \left| e^{(\sigma-T-iR-1/2)\ln(\sigma-T-iR)+\sigma-T+iR}(1 + o(1)) \right|$$

$$\leq 2\sqrt{2\pi}Re^{-R\arg(a)} \lim_{T\to\infty} |a|^{T-\sigma} \sqrt{(\sigma - T)^2 + R^2}^{\sigma-T-1/2} e^{R\arg(\sigma-T-iR)}e^{\sigma-T}$$

Which we see goes to 0 as $T \to \infty$ and $R \to \infty$. Thus we have,

$$\oint_{\gamma_3} f(z)dz \to 0 \text{ as } T \to \infty \text{ and } R \to \infty$$

Hence we obtain the desired result from comparing the results from the Cauchy Residue Theorem to the sum of the integrals above after sending R and T to infinity,

$$\int_{\sigma-\infty}^{\sigma+\infty} a^{-t}\Gamma(t)dt = 2\pi i e^{-a}$$

□

Exercise 1.11.

Proof. First we define $G(v) := \sum_{n\in\mathbb{Z}} F(n+v)$, noting that $G(v)$ is a function of period 1. Further, as $G \in L^2([0,1])$, we write the Fourier Series for $G(v)$ as follows,

$$G(v) = \sum_{m\in\mathbb{Z}} c_m e^{2\pi imv}, \text{ where } c_m = \int_0^1 G(v)e^{-2\pi imv}dv = \int_0^1 \sum_{n\in\mathbb{Z}} F(n+v)e^{-2\pi imv}dv$$

Since $\sum_{n\in\mathbb{Z}} F(n+v)$ converges uniformly in v, we interchange and substitute $x = n+v$,

$$\int_0^1 \sum_{n\in\mathbb{Z}} F(n+v)e^{-2\pi imv}dv = \sum_{n\in\mathbb{Z}} \int_0^1 F(n+v)e^{-2\pi imv}dv = \sum_{n\in\mathbb{Z}} \int_n^{n+1} F(x)e^{-2\pi im(x-n)}dx$$

$$= \sum_{n\in\mathbb{Z}} e^{2\pi imn} \int_n^{n+1} F(x)e^{-2\pi imx}dx = \sum_{n\in\mathbb{Z}} \int_n^{n+1} F(x)e^{-2\pi imx}dx = \int_{-\infty}^{\infty} F(x)e^{-2\pi imx}dx = \hat{F}(m)$$

Hence we have the desired result,

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \hat{F}(n)e^{2\pi inv}$$

□

Exercise 1.12.

Proof. First we let $F(u) = e^{-u^2\pi/x}$ and calculate the Fourier Transform as follows,

$$\hat{F}(n) = \int_{-\infty}^{\infty} F(u)e^{-2\pi iun}du = \int_{-\infty}^{\infty} e^{-u^2\pi/x}e^{-2\pi iun}du$$

$$= \int_{-\infty}^{\infty} e^{-u^2\pi/x-2\pi iun}du = \int_{-\infty}^{\infty} e^{-\pi/x(u^2+2xiun)}du$$

$$= e^{-n^2\pi x} \int_{-\infty}^{\infty} e^{-\pi/x(u^2+2ixun+(ixn)^2)}du$$

$$= e^{-n^2\pi x}\int_{-\infty}^{\infty} e^{-\pi/x(u+ixn)^2}du$$
$$= e^{-n^2\pi x}\sqrt{\frac{\pi}{\pi x^{-1}}} = e^{-n^2\pi x}\sqrt{x}$$

Now letting $u = n + v$ and applying the previous problem, we see the desired result,

$$\sum_{n\in\mathbb{Z}} e^{-(n+v)^2\pi/x} = \sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \hat{F}(n)e^{2\pi inv} = \sqrt{x}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x+2\pi inv}$$

□

Exercise 1.13.

Proof. First we note the following,

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi x} = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi x}$$

which by 1.10 gives,

$$= 1 + \frac{1}{\pi i}\sum_{n=1}^{\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}(n^2\pi x)^{-t}\Gamma(t)dt = 1 + \frac{1}{\pi i}\sum_{n=1}^{\infty}\int_{\sigma-i\infty}^{\sigma+i\infty} n^{-2t}(\pi x)^{-t}\Gamma(t)dt$$

Now we see that to interchange the summation and integration we must consider when,

$$\sum_{n=1}^{\infty}\left|n^{-2t}\right| \leq \infty \Rightarrow \sum_{n=1}^{\infty} n^{-2\sigma} < \infty \Rightarrow \sigma > \frac{1}{2}$$

Hence, if $\sigma > \frac{1}{2}$, we have that,

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi x} = 1 + \frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\zeta_R(2t)(\pi x)^{-t}\Gamma(t)dt$$

Now we evaluate the integral as in 1.10 using the same contour with $T > \sigma$. Notice the function $f(z) := \zeta_R(2z)(\pi x)^{-z}\Gamma(z)$ has simple poles at 0 and $\frac{1}{2}$ since $\Gamma(z)$ has poles at negative integers, but $\zeta_R(2z)$ has zeros at even negative integers (or all integers z). Next we calculate the residue at each pole,

$$\mathrm{Res}[f(z), 0] = \lim_{z\to 0}[(z)\zeta_R(2z)(\pi x)^{-z}\Gamma(z)] = \lim_{z\to 0}\left[(z)\zeta_R(2z)(\pi x)^{-z}\frac{\Gamma(z+1)}{z}\right]$$
$$= \zeta_R(0)(\pi x)^0\Gamma(1) = -\frac{1}{2}$$

and $\text{Res}[f(z), 1/2] = \lim_{z\to 1/2}[(z-1/2)\zeta_R(2z)(\pi x)^{-z}\Gamma(z)] = (1/2)\pi^{-1/2}x^{-1/2}\Gamma(1/2) = 1/2x^{-1/2}$

Where the second equality holds by theorem 1.25.

Thus we see the following where C is the contribution from the two horizontal lines and the right vertical line,

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi x} = 1 + \frac{1}{\pi i}\left[2\pi i\left(-\frac{1}{2}+\frac{1}{2\sqrt{x}}\right)+C\right] = \frac{1}{\sqrt{x}}+\frac{C}{\pi i}$$

Finally, we compare this result to 1.12 with $v=0$ to see our answer holds,

$$\sqrt{x}\left(\sum_{n\in\mathbb{Z}} e^{-n^2\pi x}\right) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi/x}$$

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x}}\sum_{n=1}^{\infty} e^{-n^2\pi/x}$$

Thus we find the leading term in the small-x behavior acts like the polynomial $\dfrac{1}{\sqrt{x}}$. □

Exercise 1.14.

Proof. By 1.10 we have,

$$\sum_{n=1}^{\infty} e^{-\beta n^\alpha} = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}(\beta n^\alpha)^{-t}\,\Gamma(t)dt = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\sigma-i\infty}^{\sigma+i\infty} n^{-\alpha t}\beta^{-t}\Gamma(t)dt$$

As in 1.13, we can interchange the summation and integration for $\sigma > \frac{1}{\alpha}$ to see,

$$\sum_{n=1}^{\infty} e^{-\beta n^\alpha} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\zeta_R(\alpha t)\beta^{-t}\Gamma(t)dt$$

Now we evaluate the integral as in1.10 using the same contour. Notice the function $f(z) := \zeta_R(\alpha z)\beta^{-z}\Gamma(z)$ has simple poles at $\frac{1}{\alpha}$ and 0. If α is an even integer, these are the only two poles. If not, $f(z)$ also has simple poles at the negative integers. Next we calculate the residue at the first three poles (assuming α is not an even integer),

$$\text{Res}[f(z),0] = \lim_{z\to 0}[(z)\zeta_R(\alpha z)\beta^{-z}\Gamma(z)] = \lim_{z\to 0}\left[(z)\zeta_R(\alpha z)\beta^{-z}\frac{\Gamma(z+1)}{z}\right] = \zeta_R(0)\beta^0\Gamma(1) = -\frac{1}{2}$$

$$\text{Res}[f(z),1/\alpha] = \lim_{z\to 1/\alpha}[(z-1/\alpha)\zeta_R(\alpha z)\beta^{-z}\Gamma(z)] = 1/\alpha\beta^{-1/\alpha}\Gamma(1/\alpha)$$

$$\text{Res}[f(z),-1] = \lim_{z\to -1}\left[(z+1)\zeta_R(\alpha z)\beta^{-z}\frac{\Gamma(z+2)}{z(z+1)}\right] = -\zeta_R(-\alpha)\beta\Gamma(1) = -\beta\zeta_R(-\alpha)$$

Hence, the three leading terms are $-\frac{1}{2}$, $\beta^{-1/\alpha}\Gamma(1/\alpha)$, and $-\beta\zeta_R(-\alpha)$. □

Exercise 1.15.

Proof. First we rewrite the summation with the follow results from geometric series,

$$\frac{d}{dx}\left[\ln\left(1-x\right)\right] = -\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n \Rightarrow \ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

So that,

$$\sum_{n=1}^{\infty} \ln\left(1-e^{-\beta n}\right) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{e^{-\beta mn}}{m}$$

By 1.10 we have,

$$-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{e^{-\beta mn}}{m} = -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \int_{\sigma-i\infty}^{\sigma+i\infty} \beta^{-t} n^{-t} m^{-(t+1)} \Gamma(t) dt$$

As in the previous problems, we notice for $\sigma > 1$ we have,

$$\sum_{n=1}^{\infty} \ln\left(1-e^{-\beta n}\right) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \beta^{-t} \zeta_R(t)\zeta_R(t+1)\Gamma(t) dt$$

Now we evaluate the integral as in 1.10 using the same contour. Notice the function $f(z) := \beta^{-z}\zeta_R(z)\zeta_R(z+1)\Gamma(z)$ has simple poles at -1 and 1, and a second order pole at 0. We see this by noting that $\zeta_R(t)$ has a pole at 1, and both $\Gamma(t)$ and $\zeta_R(t+1)$ have poles at 0. Further, we know $\Gamma(t)$ has poles at the negative integers, however $\zeta_R(t)$ has zeros for each even negative integer t and $\zeta_R(t+1)$ has zeros at each odd negative integer $t \leq -3$. Thus all the negative integer poles of $\Gamma(t)$ are cancelled by a zero except at -1.
Next we calculate the residue at each pole,

$$\text{Res}[f(z),1] = \lim_{z\to 1}[(z-1)\beta^{-z}\zeta_R(z)\zeta_R(z+1)\Gamma(z)] = \beta^{-1}\zeta_R(2)\Gamma(1) = \frac{\pi^2}{6}\beta^{-1}$$

$$\text{Res}[f(z),-1] = \lim_{z\to -1}\left[(z+1)\beta^{-z}\zeta_R(z)\zeta_R(z+1)\frac{\Gamma(z+2)}{z(z+1)}\right] = -\beta\zeta_R(-1)\zeta_R(0) = -\frac{1}{24}\beta$$

For the residue at 0, we follow the technique of theorem 1.25 to deal with only the relevant portion of the integral representation of $\zeta_R(z+1)$, noting this portion can be represented as $\frac{1}{\Gamma(z+1)}\sum_{n=0}^{\infty}\frac{B_n}{n!}\frac{1}{z+n}$. Further, at $z=0$ we have a pole at $n=0$. We now calculate the residue,

$$\text{Res}[f(z),0] = \lim_{z\to 0}\frac{d}{dz}\left[z^2\beta^{-z}\zeta_R(z)\frac{1}{z\Gamma(z+1)}\frac{\Gamma(z+1)}{z}\right] = \lim_{z\to 0}\frac{d}{dz}[\beta^{-z}\zeta_R(z)]$$

$$= \lim_{z\to 0}[-\ln\beta e^{-z\ln\beta}\zeta_R(z) + \beta^{-z}\zeta_R'(z)] = -\ln\beta\zeta_R(0) + \zeta_R'(0) = \frac{1}{2}\ln\beta - \frac{1}{2}\ln(2\pi)$$

Hence, the three leading terms are $\frac{\pi^2}{6}\beta^{-1}$, $-\frac{1}{24}\beta$, and $= \frac{1}{2}\ln\beta - \frac{1}{2}\ln(2\pi)$. ☐

B.2 Solutions to Chapter 2 Exercises

Exercise 2.1.

Proof. Notice we have the following for $z = x + iy$,

$$\cot z = \frac{\cos z}{\sin z} = \frac{2i(e^{ix-y}+e^{-ix+y})}{2(e^{ix-y}-e^{-ix+y})} = -i\frac{e^{ix-y}+e^{-ix+y}}{e^{-ix+y}-e^{ix-y}} = -i\frac{e^{ix-y}+e^{-ix+y}}{e^{-ix+y}-e^{ix-y}} \cdot \frac{e^{ix+y}-e^{-ix-y}}{e^{ix+y}-e^{-ix-y}}$$

$$= -i\frac{e^{2ix}-e^{-2y}+e^{2y}-e^{-2ix}}{e^{2y}-e^{-2ix}-e^{2ix}+e^{-2y}} = -i\frac{e^{2ix}-e^{-2ix}+e^{2y}-e^{-2y}}{e^{2y}+e^{-2y}-(e^{2ix}+e^{-2ix})}$$

$$\frac{\frac{e^{2ix}-e^{-2ix}}{2i} - i\frac{e^{2y}-e^{-2y}}{2}}{\frac{e^{2y}+e^{-2y}}{2} - \frac{e^{2ix}+e^{-2ix}}{2}} = \frac{\sin(2x) - i\sinh(2y)}{\cosh(2y)-\cos(2x)}$$

□

Exercise 2.2.

Proof. We first analyze the behavior of the derivative portion of the integrand. Note by series expansion we have,

$$\frac{\sin(ix\pi)}{ix} = \frac{1}{ix}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(ix\pi)^{2n-1}}{(2n-1)!} = \frac{1}{x}\sum_{n=1}^{\infty}\frac{\pi^{2n-1}x^{2n-1}}{(2n-1)!} = \pi + \frac{\pi^3}{3!}x^2 + \frac{\pi^5}{5!}x^4 + \dots.$$

From which we see for small x we have that,

$$\frac{\sin(ix\pi)}{ix} = \pi + \mathcal{O}(x^2) \Rightarrow \ln\left[\frac{\sin(ix\pi)}{ix}\right] = O(x^2) \Rightarrow \frac{d}{dx}\ln\left[\frac{\sin(ix\pi)}{ix}\right] = O(x)$$

Next we consider what happens for $x > 1$,

$$\ln\left[\frac{\sin(ix\pi)}{ix}\right] = \ln\left[\frac{e^{\pi x}}{2x}(1-e^{-2\pi x})\right] = \pi x - \ln(2x) + \ln(1-e^{-2\pi x})$$

$$\Rightarrow \frac{d}{dx}\ln\left[\frac{\sin(ix\pi)}{ix}\right] = \pi - \frac{1}{x} + \frac{2\pi e^{-2\pi x}}{1-e^{-2\pi x}} = \pi - \frac{1}{x} + \frac{2\pi}{e^{2\pi x}-1} < \pi - 1 + \frac{2\pi}{e^{2\pi}-1} \text{ for } x > 1$$

so that we see it is bounded by a constant for $x > 1$.
Now we consider the integral,

$$\int_0^{\infty} x^{-2s}\frac{d}{dx}\ln\left[\frac{\sin(ix\pi)}{ix}\right]dx = \int_0^1 x^{-2s}\frac{d}{dx}\ln\left[\frac{\sin(ix\pi)}{ix}\right]dx + \int_1^{\infty} x^{-2s}\frac{d}{dx}\ln\left[\frac{\sin(ix\pi)}{ix}\right]dx$$

This gives for the first integral on the right that the integrand is equal to $O(x^{-2s+1})$. This gives that for convergence we need $2\text{Re}(s) - 1 < 1$, or $\text{Re}(s) < 1$.

Furthermore, for the second integral on the right we have that the integrand is equal to $O(x^{-2s})$, so that we need $2\mathrm{Re}(s) > 1$, or $\mathrm{Re}(s) > \frac{1}{2}$.

Hence the integral is well defined in the strip $\frac{1}{2} < \mathrm{Re}(s) < 1$. □

Exercise 2.3.

Proof. We first note that $\lambda = 0$ and $\lambda = -k^2$ do not give a non-trivial solution to the given problem. Next with $a \in (0,1)$, we let $\lambda = k^2$ from which we see solutions are of the form $\phi(x) = Ae^{ikx} + Be^{-ikx}$. Now imposing BC,

$$\phi(0) = A + B = Ae^{2\pi i(k-a)} + Be^{-2\pi i(k+a)}$$

$$\phi'(0) = ik(A - B) = ik\left[Ae^{2\pi i(k-a)} - Be^{-2\pi i(k+a)}\right]$$

Now dividing $\phi'(0)$ by ik, then adding and subtracting the equations gives,

$$A = Ae^{2\pi i(k-a)}$$

$$B = Be^{-2\pi i(k+a)}$$

As $A = 0 = B$ gives a trivial solution, we consider the following two cases (noting we omit when both $A \neq 0$ and $B \neq 0$),

Case 1: If $A \neq 0,\ B = 0$ then $k = l + a,\ l \in \mathbb{Z} \Rightarrow \phi_l(x) = Ae^{i(l+a)x}$

Case 2: If $A = 0,\ B \neq 0$ then $k = l - a,\ l \in \mathbb{Z} \Rightarrow \phi_l(x) = Be^{-i(l-a)x}$

Note that not all of these eigenfunctions are linearly independent, in particular, $l \geq 0$ in Case 1 is equal to $l \leq 0$ in Case 2, and $l < 0$ in Case 1 is equal to $l > 0$ in Case 2. So we choose Case 1 with $l \geq 0$ and Case 2 with $l > 0$. Thus we have,

$$0 = (e^{2\pi i(k-a)} - 1)(e^{-2\pi i(k+a)} - 1) = e^{-4\pi ia} - e^{2\pi ik}e^{-2\pi ia} - e^{-2\pi ik}e^{-2\pi ia} + 1$$

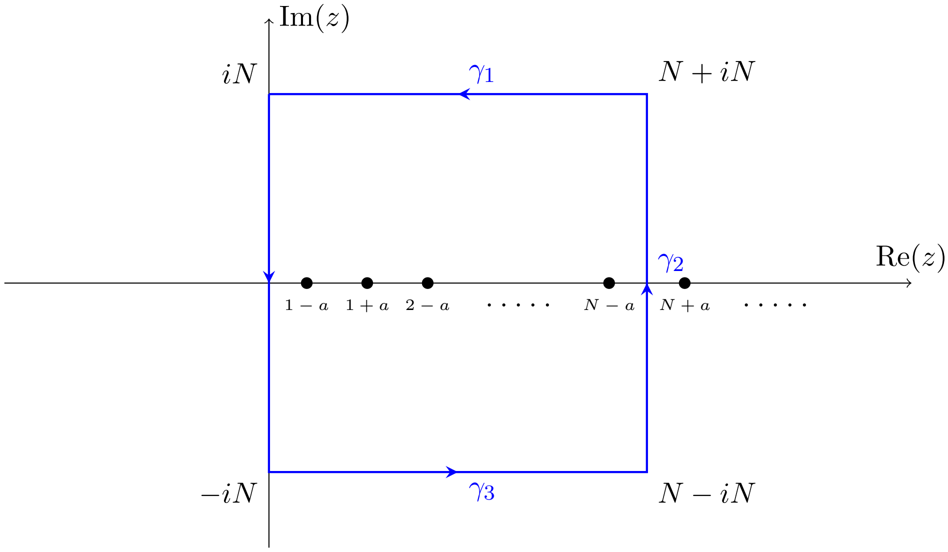
$$= e^{-2\pi ia}(e^{-2\pi ia} - e^{2\pi ik} - e^{-2\pi ik} + e^{2\pi ia}) = -2e^{-2\pi ia}(\cos(2\pi k) - \cos(2\pi a))$$

From which we conclude the set of solutions is of the form $\cos(2\pi k) - \cos(2\pi a)$. Now we consider the following integral with the branch cut $(-\infty, 0]$,

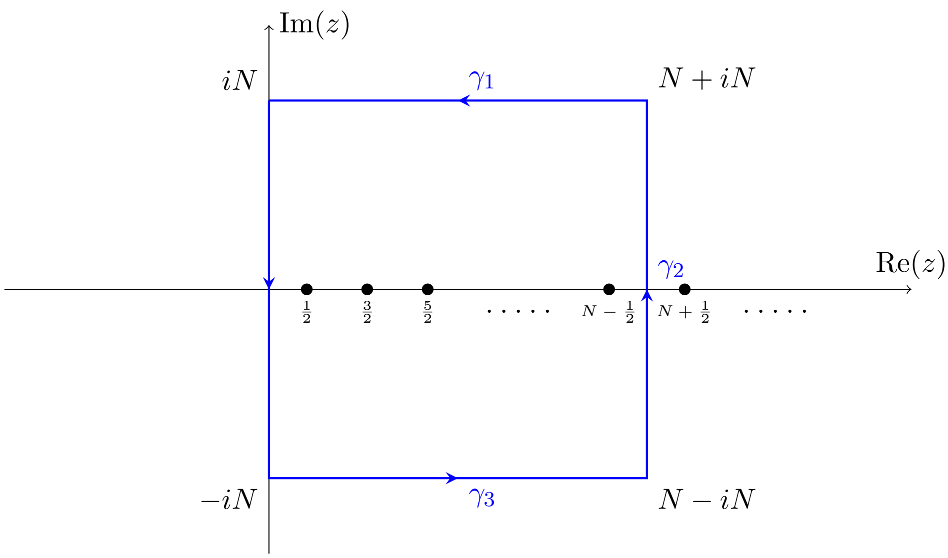
$$\zeta_p(s) = \frac{1}{2\pi i}\oint_\gamma z^{-2s}\frac{d}{dz}\ln(\cos(2\pi z) - \cos(2\pi a))dz$$

where γ is enclosing all eigenvalues counterclockwise for $N \in \mathbb{N}$. Below are the three cases where the location of the eigenvalues relative to each other differ depending on values of a.

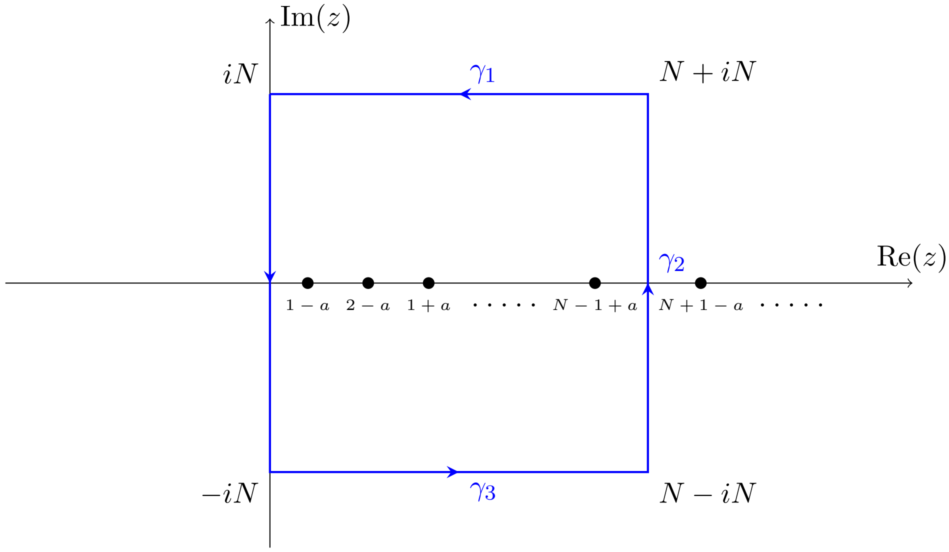
Case 1: When $a \in (0, \frac{1}{2})$ we have distinct eigenvalues producing simple poles as below,



Case 2: When $a = \frac{1}{2}$ we have a single eigenvalue producing a simple pole at $\frac{1}{2}$ and all other eigenvalues are repeated twice producing residue equal to 2 at each pole as below,



Case 3: When $a \in (\frac{1}{2}, 1)$ we have distinct eigenvalues producing simple poles as below,



Next we calculate each of the integrals for γ_1, γ_2, γ_3, and the imaginary axis, noting,

$$\frac{d}{dz}\ln(\cos(2\pi z)-\cos(2\pi a)) = -2\pi\frac{\sin(2\pi z)}{\cos(2\pi z)-\cos(2\pi a)} = 2\pi i\frac{e^{2\pi iz}-e^{-2\pi iz}}{e^{2\pi iz}+e^{-2\pi iz}-e^{2\pi ia}-e^{-2\pi ia}}$$

Note the following calculations hold for $a \in (0,1)$ (more precisely for $a \notin \mathbb{Z}$).
γ_1: We let $z = t + iN$ for $0 \le t \le N$, where t runs from N to 0 to see,

$$\left|\frac{e^{2\pi iz}-e^{-2\pi iz}}{e^{2\pi iz}+e^{-2\pi iz}-e^{2\pi ia}-e^{-2\pi ia}}\right| = \left|\frac{1-e^{4\pi iz}}{1+e^{4\pi iz}-e^{2\pi iz+2\pi ia}-e^{2\pi iz-2\pi ia}}\right|$$

$$\le \frac{1-e^{-4\pi N}}{1+e^{-4\pi N}-2e^{-2\pi N}}$$

γ_3: We let $z = t - iN$ for $0 \le t \le N$, where t runs from 0 to N to see as above,

$$\left|\frac{e^{2\pi iz}-e^{-2\pi iz}}{e^{2\pi iz}+e^{-2\pi iz}-e^{2\pi ia}-e^{-2\pi ia}}\right| = \left|\frac{1-e^{-4\pi iz}}{1+e^{-4\pi iz}-e^{-2\pi iz+2\pi ia}-e^{-2\pi iz-2\pi ia}}\right|$$

$$\le \frac{1-e^{-4\pi N}}{1+e^{-4\pi N}-2e^{-2\pi N}}$$

γ_2: We let $z = N + it$ for $-N \le t \le N$, where t runs from $-N$ to N to see as above,

$$\left|\frac{e^{2\pi iz}-e^{-2\pi iz}}{e^{2\pi iz}+e^{-2\pi iz}-e^{2\pi ia}-e^{-2\pi ia}}\right| = \left|-\frac{e^{4\pi iz}-1}{1+e^{4\pi iz}-e^{2\pi iz+2\pi ia}-e^{2\pi iz-2\pi ia}}\right|$$

$$\leq \frac{e^{4\pi N}-1}{1+e^{4\pi N}-2e^{2\pi N}} = \frac{1-e^{-4\pi N}}{1+e^{-4\pi N}-2e^{-2\pi N}}$$

So that we have,

$$\left|\oint_{\gamma_1\cup\gamma_2\cup\gamma_3} z^{-2s}\frac{d}{dz}\ln(\cos(2\pi z)-\cos(2\pi a))dz\right| \leq 6\pi N^{-2s+1}\frac{1-e^{-4\pi N}}{1+e^{-4\pi N}-2e^{-2\pi N}}$$

Which we see goes to 0 as N goes to infinity for $\text{Re}(s) > \frac{1}{2}$.

On the imaginary axis for $\text{Re}(s) > \frac{1}{2}$ we let $z = iy$ to see as in class,

$$\zeta_p(s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} z^{-2s}\frac{d}{dz}\ln(\cos(2\pi z)-\cos(2\pi a))dz$$

$$= \frac{\sin(\pi s)}{\pi}\left[\int_0^1 y^{-2s}\frac{d}{dy}\ln(\cos(2\pi iy)-\cos(2\pi a))dy + \int_1^\infty y^{-2s}\frac{d}{dy}\ln(\cos(2\pi iy)-\cos(2\pi a))dy\right]$$

Now we consider the second integral on the right,

$$\int_1^\infty y^{-2s}\frac{d}{dy}\ln(\cos(2\pi iy)-\cos(2\pi a))dy = \int_1^\infty y^{-2s}\frac{d}{dy}\ln(2^{-1}(e^{-2\pi y}+e^{2\pi y})-\cos(2\pi a))dy$$

$$= \int_1^\infty y^{-2s}\frac{d}{dy}\ln(e^{2\pi y}[2^{-1}(1+e^{-4\pi y})-e^{-2\pi y}\cos(2\pi a)])dy$$

$$= 2\pi\int_1^\infty y^{-2s}dy + \int_1^\infty y^{-2s}\frac{d}{dy}\ln(2^{-1}(1+e^{-4\pi y})-e^{-2\pi y}\cos(2\pi a))dy$$

$$= \frac{2\pi}{2s-1} + \int_1^\infty y^{-2s}\frac{d}{dy}\ln(2^{-1}(1+e^{-4\pi y})-e^{-2\pi y}\cos(2\pi a))dy$$

In summary we have,

$$\zeta_p(s) = \frac{\sin(\pi s)}{\pi}\int_0^1 y^{-2s}\frac{d}{dy}\ln(\cos(2\pi iy)-\cos(2\pi a))dy + \frac{2\sin(\pi s)}{2s-1}$$
$$+\frac{\sin(\pi s)}{\pi}\int_1^\infty y^{-2s}\frac{d}{dy}\ln(2^{-1}(1+e^{-4\pi y})-e^{-2\pi y}\cos(2\pi a))dy$$

Which gives the following,

$$\zeta_p'(0) = \ln(\cosh(2\pi)-\cos(2\pi a)) - \ln(1-\cos(2\pi a)) - 2\pi - \ln(2) - \ln(2^{-1}(1+e^{-4\pi})-e^{-2\pi}\cos(2\pi a))$$

$$= \ln\left[\frac{\cosh(2\pi)-\cos(2\pi a)}{2e^{2\pi}(1-\cos(2\pi a))(2^{-1}(1+e^{-4\pi})-e^{-2\pi}\cos(2\pi a))}\right]$$

$$= \ln\left[\frac{e^{2\pi}+e^{-2\pi}-2\cos(2\pi a)}{2(1-\cos(2\pi a))(e^{2\pi}+e^{-2\pi}-2\cos(2\pi a))}\right]$$

$$= -\ln(4\sin^2(\pi a)) = -2\ln(2\sin(\pi a))$$

□

Exercise 2.4.

Proof. We first search for solutions by separating variables by defining $\phi_\lambda(x,y,z) = X(x)Y(y)Z(z)$ and substituting into the differential equation to see,

$$-X''(x)Y(y)Z(z) - X(x)Y''(y)Z(z) - X(x)Y(y)Z''(z) = \lambda X(x)Y(y)Z(z)$$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\lambda$$

It follows that each term on the left must be a constant, since the right hand side does not depend on any of the variables. If those constants are $-\lambda_x$, $-\lambda_y$, and $-\lambda_z$ respectively, we have $\lambda = \lambda_x+\lambda_y+\lambda_z$, and the following,

$$X''(x) + \lambda_x X(x) = 0, \quad Y''(y) + \lambda_y Y(y) = 0, \quad Z''(z) + \lambda_Z Z(z) = 0$$

First we let $\lambda_x = \lambda_y = \lambda_z = 0$ noting solutions will be of the form of linear equations,

$$X(x) = Ax + B \quad Y(y) = Cy + D \quad Z(z) = Ez + F$$

Imposing BC for x gives the trivial solution below,

$$B = 0 \text{ and } Aa = 0 \Rightarrow A = 0$$

Imposing BC for y (and similarly for z) gives,

$$D = CL + D \text{ and } C = C \Rightarrow C = 0$$

from which we see we get the constant function as a solution for $Y(y)$ (and similarly $Z(z)$).

Next we let $\lambda_x = -p_x^2$, $\lambda_y = -p_y^2$, and $\lambda_z = -p_z^2$ from which we see solutions are of the form

$$X(x) = A\cosh(p_x x) + B\sinh(p_x x) \quad Y(y) = C\cosh(p_y y) + D\sinh(p_y y)$$

$$Z(z) = E\cosh(p_z z) + F\sinh(p_z z)$$

Imposing BC for x gives,

$$A = 0 \text{ and } \sinh(p_x a) = 0 \Rightarrow p_x = 0$$

so that we are restricted back to the previous case above.

Imposing BC for y (and similarly for z) gives,

$$C = C\cosh(p_y L) + D\sinh(p_y L) \text{ and } D = -C\sinh(p_y L) + D\cosh(p_y L)$$

If $\cosh(p_y L) \neq 1$, we solve for C and substitute as follows,

$$C = \frac{D\sinh(p_y L)}{1-\cosh(p_y L)} \Rightarrow D = -\frac{D\sinh^2(p_y L)}{1-\cosh(p_y L)} + D\cosh(p_y L) = D\cosh(p_y L) + D + D\cosh(p_y L)$$

$$\Rightarrow 2D\cosh(p_yL) = 0 \Rightarrow D = 0 \Rightarrow C = 0$$

Thus we consider when $\cosh(p_yL) = 1$ which implies $p_y = 0$, so that we are once again restricted back to the previous case above.

Finally we let $\lambda_x = p_x^2$, $\lambda_y = p_y^2$, and $\lambda_z = p_z^2$ from which we see solutions are of the form

$$X(x) = A\cos(p_xx)+B\sin(p_xx) \quad Y(y) = C\cos(p_yy)+D\sin(p_yy) \quad Z(z) = E\cos(p_zz)+F\sin(p_zz)$$

Imposing BC for x gives,

$$A = 0 \text{ and } \sin(p_xa) = 0 \Rightarrow p_x = \frac{n\pi}{a},\ n \in \mathbb{Z}$$

$$\Rightarrow \lambda_{x,n} = \left(\frac{n\pi}{a}\right)^2 \text{ and } X_n(x) = \sin\left(\frac{n\pi x}{a}\right),\ n \in \mathbb{N}$$

Imposing BC for y gives,

$$C = C\cos(p_yL) + D\sin(p_yL) \text{ and } D = -C\sin(p_yL) + D\cos(p_yL)$$

If $\cos(p_yL) \neq 1$, we solve for C and substitute as follows,

$$C = \frac{D\sin(p_yL)}{1-\cos(p_yL)} \Rightarrow D = -\frac{D\sin^2(p_yL)}{1-\cos(p_yL)} + D\cos(p_yL) = -D(1+\cos(p_yL)) + D\cos(p_yL) = -D$$

$$\Rightarrow D = 0 \Rightarrow C = 0$$

Thus we consider when $\cos(p_yL) = 1$ which implies $p_y = \frac{2n\pi}{L},\ n \in \mathbb{Z}$.

$$C = C + D\sin(p_yL) \Rightarrow \sin(p_yL) = 0 \Rightarrow p_y = \frac{n\pi}{L},\ n \in \mathbb{Z}$$

$$\Rightarrow \lambda_{y,n} = \left(\frac{2n\pi}{L}\right)^2 \text{ and } Y_n(y) = C\cos\left(\frac{2n\pi y}{L}\right) + D\sin\left(\frac{2n\pi y}{L}\right),\ n \in \mathbb{N}$$

As the BC for z are the same as y, the same computation gives,

$$\lambda_{z,n} = \left(\frac{2n\pi}{L}\right)^2 \text{ and } Z_n(z) = E\cos\left(\frac{2n\pi z}{L}\right) + F\sin\left(\frac{2n\pi z}{L}\right),\ n \in \mathbb{N}$$

Thus we have the following eigenvalues and eigenfunctions,

$$\lambda_{n,m,j} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{2m\pi}{L}\right)^2 + \left(\frac{2j\pi}{L}\right)^2 \text{ and}$$

$$\phi_{n,m,j}(x,y,z) = \sin\left(\frac{n\pi x}{a}\right)\left(C\cos\left(\frac{2m\pi y}{L}\right) + D\sin\left(\frac{2m\pi y}{L}\right)\right)\left(E\cos\left(\frac{2j\pi z}{L}\right) + F\sin\left(\frac{2j\pi z}{L}\right)\right)$$

where $n \in \mathbb{N}$ and $m, j \in \mathbb{N}_0$ □

Exercise 2.5.

Proof. First, we let $z(x) = \frac{1}{c}\int_a^x \left(\frac{r(\tau)}{p(\tau)}\right)^{1/2} d\tau$ where $c = \frac{1}{\pi}\int_a^b \left(\frac{r(x)}{p(x)}\right)^{1/2} dx$. Notice that we then have,

$$\frac{dz}{dx} = \frac{1}{c}\left(\frac{r(x)}{p(x)}\right)^{1/2}$$

Further, we let θ be a function such that $\theta(z(x)) = (r(x)p(x))^{1/4}$, and g be a function such that $g(z) = y$. We calculate as follows,

$$p(x)\frac{dy}{dx} = p(x)\frac{dz}{dx}\frac{dg}{dz} = p(x)\frac{1}{c}\left(\frac{r(x)}{p(x)}\right)^{1/2} g'(z) = \frac{1}{c}\left(r(x)p(x)\right)^{1/2} g'(z) = \frac{1}{c}\theta^2(z)g'(z)$$

Now we let $u(z) = \theta(z)g(z)$ and note $u'(z) = \theta'(z)g(z) + \theta(z)g'(z)$. Substituting above and continuing the calculation shows,

$$p(x)\frac{dy}{dx} = \frac{1}{c}\theta^2(z)g'(z) = \frac{1}{c}\theta(z)(u'(z) - \theta'(z)g(z)) = \frac{1}{c}(\theta(z)u'(z) - \theta'(z)u(z))$$

Thus we find that,

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] = -\frac{d}{dx}\left[\frac{1}{c}(\theta(z)u'(z) - \theta'(z)u(z))\right] = -\frac{1}{c}\frac{dz}{dx}\frac{d}{dz}\left[\theta(z)u'(z) - \theta'(z)u(z)\right]$$

$$= -\frac{1}{c}\left[\frac{1}{c}\left(\frac{r(x)}{p(x)}\right)^{1/2}\right]\left[\theta(z)u''(z) - \theta''(z)u(z)\right] = \frac{1}{c^2}\left(\frac{r(x)}{p(x)}\right)^{1/2}\left[\theta''(z)u(z) - \theta(z)u''(z)\right]$$

Now we reduce the general second-order equation we started with as follows,

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + l(x)y = \lambda r(x)y$$

$$\frac{1}{c^2}\left(\frac{r(x)}{p(x)}\right)^{1/2}\left[\theta''(z)u(z) - \theta(z)u''(z)\right] + l(x)g(z) = \lambda r(x)g(z)$$

$$\frac{1}{c^2}\left(\frac{r(x)}{p(x)}\right)^{1/2}\left[\theta''(z)u(z) - \theta(z)u''(z)\right] + l(x)\frac{u(z)}{\theta(z)} = \lambda r(x)\frac{u(z)}{\theta(z)}$$

$$(p(x)r(x))^{-1/2}\theta(z)\left[\theta''(z)u(z) - \theta(z)u''(z)\right] + c^2 l(x)u(z)(r(x))^{-1} = c^2\lambda u(z)$$

$$(\theta(z))^{-1}\left[\theta''(z)u(z) - \theta(z)u''(z)\right] + c^2 l(x)u(z)(r(x))^{-1} = c^2\lambda u(z)$$

$$(\theta(z))^{-1}\theta''(z)u(z) - u''(z) + c^2 l(x)u(z)(r(x))^{-1} = c^2\lambda u(z)$$

$$-u''(z) + [(\theta(z))^{-1}\theta''(z) + c^2 l(x)(r(x))^{-1}]u(z) = c^2\lambda u(z)$$

Letting $\mu = c^2\lambda$ and $V(x) = (\theta(z))^{-1}\theta''(z) + c^2 l(x)(r(x))^{-1}$ gives the desired result,

$$-u''(z) + V(x)u(z) = \mu u(z)$$

Now we show this satisfies separated boundary conditions. For the initial equation, we have solutions of the form,

$$y(a)\cos(\alpha) + y'(a)\sin(\alpha) = 0$$

We can then show this is equivalent to,

$$u(0)\left[c\sqrt{\frac{p(0)}{r(0)}}\cos(\alpha) - \frac{\theta'(0)}{\theta(0)}\sin(\alpha)\right] + u'(0)\sin(\alpha)$$

Now we ask, given a BC of the form $Au(0) + Bu'(0)$ how can we get to the original form? Division by $C = \pm\sqrt{A^2+B^2}$ shows the resulting coefficients satisfy $\left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 = 1$. □

Exercise 2.6 i).

Proof. Note that we have the following results from lemmas 2.4 and 2.5 for $\lambda = s^2$ and $s = \sigma + it$,

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda)d\tau \text{ and } \psi(x,\lambda) = O(|s|^{-1}e^{|t|x})$$

We continue the process that was done in lemma 2.5 by substituting as follows,

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda)d\tau$$

$$= \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)\left[\frac{\sin(s\tau)}{s} + \frac{1}{s}\int_0^\tau \sin[s(\tau-u)]V(u)\psi(u,\lambda)du\right]d\tau$$

So that we have,

$$\begin{aligned}\psi(x,\lambda) = \frac{\sin(sx)}{s} &+ \frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)\sin(s\tau)d\tau \\ &+\frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\psi(u,\lambda)du\ d\tau\end{aligned} \tag{B.1}$$

Now we analyze the first integral on the right of (B.1) using trig identities,

$$\frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)\sin(s\tau)d\tau = \frac{1}{2s^2}\int_0^x [\cos(sx-2s\tau)-\cos(sx)]V(\tau)d\tau$$

$$= -\frac{\cos(sx)}{2s^2}\int_0^x V(\tau)d\tau + \frac{1}{2s^2}\int_0^x \cos(sx-2s\tau)V(\tau)d\tau$$

$$= -\frac{\cos(sx)}{2s^2}\int_0^x V(\tau)d\tau - \frac{1}{4s^3}\int_0^x [-2s\cos(sx-2s\tau)]V(\tau)d\tau$$

Which gives through integration by parts,

$$= -\frac{\cos(sx)}{2s^2}\int_0^x V(\tau)d\tau - \frac{1}{4s^3}V(\tau)\sin(sx-2s\tau)\Big|_0^x + \frac{1}{4s^3}\int_0^x \sin(sx-2s\tau)V'(\tau)d\tau$$

$$= -\frac{\cos(sx)}{2s^2}\int_0^x V(\tau)d\tau + \frac{\sin(sx)}{4s^3}(V(x)+V(0)) + \frac{1}{4s^3}\int_0^x \sin(sx-2s\tau)V'(\tau)d\tau \qquad \text{(B.2)}$$

Now we consider the last two pieces of (B.2),

$$\left|\frac{\sin(sx)}{4s^3}(V(x)+V(0))\right| \le \frac{|V(x)+V(0)|}{4|s|^3}\left|\frac{e^{isx}-e^{-isx}}{2i}\right| \le \frac{|V(x)+V(0)|}{8|s|^3}(e^{|t|x}-e^{-|t|x})$$

$$= \frac{|V(x)+V(0)|}{8}(1-e^{-2|t|x})|s|^{-3}e^{|t|x} = O(|s|^{-3}e^{|t|x})$$

and,

$$\left|\frac{1}{4s^3}\int_0^x \sin(sx-2s\tau)V'(\tau)d\tau\right| = \left|\frac{1}{4s^3}\int_0^x \frac{e^{is(x-2\tau)}-e^{-is(x-2\tau)}}{2i}V'(\tau)d\tau\right|$$

$$\le \frac{x}{8|s|^3}\max_{0\le\tau\le x} V'(\tau)(e^{|t|x}-e^{-|t|x}) = \frac{x}{8}\max_{0\le\tau\le x} V'(\tau)(1-e^{-2|t|x})|s|^{-3}e^{|t|x}$$

$$= O(|s|^{-3}e^{|t|x})$$

for large $|s|$, so that we have,

$$\frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)\sin(s\tau)d\tau = -\frac{\cos(sx)}{2s^2}\int_0^x V(\tau)d\tau + \mathcal{O}(|s|^{-3}e^{|t|x})$$

Next we analyze the second integral on the right of (B.1) by noting for some constant M,

$$|\sin[s(x-u)]\psi(u,\lambda)| \le M\left|\frac{e^{is(x-u)}-e^{-is(x-u)}}{2i}|s|^{-1}e^{|t|u}\right|$$

$$\leq \frac{M}{2}|s|^{-1}e^{|t|u}(e^{|t|(x-u)} - e^{-|t|(x-u)}) = \frac{M}{2}|s|^{-1}(e^{|t|x} - e^{|t|(2u-x)})$$
$$= \frac{M}{2}(1 - e^{|t|(2u-2x)})|s|^{-1}e^{|t|x} \leq \frac{M}{2}|s|^{-1}e^{|t|x} = O(|s|^{-1}e^{|t|x})$$

for large $|s|$ as $0 \leq u \leq x$. So we see that,

$$\frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\psi(u,\lambda)du\ d\tau$$
$$= \frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau O(|s|^{-1}e^{|t|\tau})V(u)du\ d\tau$$
$$= \frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)O(|s|^{-1}e^{|t|x})d\tau = \frac{1}{s^2}\int_0^x O(|s|^{-1}e^{|t|x})V(\tau)d\tau$$
$$= \frac{1}{s^2}O(|s|^{-1}e^{|t|x}) = O(|s|^{-3}e^{|t|x})$$

Hence, we conclude that,

$$\psi(x,\lambda) = \frac{sin(sx)}{s} - \frac{\cos(sx)}{2s^2}\int_0^x V(\tau)d\tau + \mathcal{O}(|s|^{-3}e^{|t|x})$$

□

Exercise 2.6 ii).

Proof. Note that we have the following results from lemmas 2.4 and 2.5 for $\lambda = s^2$ and $s = \sigma + it$,

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda)d\tau \text{ and } \psi(x,\lambda) = O(|s|^{-1}e^{|t|x})$$

We continue the process noting that we have from (B.1) and (B.2) above,

$$\begin{aligned}\psi(x,\lambda) = \frac{\sin(sx)}{s} &- \frac{\cos(sx)}{2s^2}\int_0^x V(\tau)d\tau + \frac{\sin(sx)}{4s^3}(V(x)+V(0)) \\ &+ \frac{1}{4s^3}\int_0^x \sin(sx-2s\tau)V'(\tau)d\tau \\ &+ \frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\psi(u,\lambda)du\ d\tau\end{aligned} \tag{B.3}$$

Notice that with integration by parts as in exercise 2.6 i), we see that for the second to last integral in (B.3) above,

$$\frac{1}{4s^3}\int_0^x \sin(sx-2s\tau)V'(\tau)d\tau = O(|s|^{-4}e^{|t|x})$$

Now we analyze the last integral on the right of (B.3),

$$\frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\psi(u,\lambda)du\ d\tau$$

$$= \frac{1}{s^2}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\left[\frac{\sin(su)}{s}+\frac{1}{s}\int_0^u \sin[s(u-v)]V(v)\psi(v,\lambda)dv\right]du\ d\tau$$

$$\begin{aligned} &= \frac{1}{s^3}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\sin(su)du\ d\tau \\ &+ \frac{1}{s^3}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\int_0^u \sin[s(u-v)]V(v)\psi(v,\lambda)dv\ du\ d\tau \end{aligned} \tag{B.4}$$

Note, as in exercise 2.6 i), we see the last integral in (B.4) is equal to $O(|s|^{-4}e^{|t|x})$ by substitution of $\psi(v,\lambda)$. Now we analyze the first integral on the right of (B.4) using trig identities as before to find,

$$\begin{aligned} &\frac{1}{s^3}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\sin(su)du\ d\tau \\ &= \frac{1}{2s^3}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau [\cos(s\tau-2su)-\cos(s\tau)]V(u)du\ d\tau \end{aligned}$$

$$\begin{aligned} = -\frac{1}{2s^3}\int_0^x \sin[s(x-\tau)]\cos(s\tau)V(\tau)\int_0^\tau V(u)du\ d\tau \\ + \frac{1}{2s^3}\int_0^x \sin[s(x-\tau)]V(\tau)\int_0^\tau \cos(s\tau-2su)V(u)du\ d\tau \end{aligned} \tag{B.5}$$

Which we see similarly to exercise 2.6 i) that by integration by parts, the second integral of (B.5) is equal to $O(|s|^{-4}e^{|t|x})$. Further, the first integral simplifies using trig identities as follows,

$$-\frac{1}{2s^3}\int_0^x \sin[s(x-\tau)]\cos(s\tau)V(\tau)\int_0^\tau V(u)du\ d\tau$$

$$= -\frac{\sin(sx)}{4s^3}\int_0^x V(\tau)\int_0^\tau V(u)du\ d\tau - \frac{1}{4s^3}\int_0^x \sin(sx-2s\tau)V(\tau)\int_0^\tau V(u)du\ d\tau \tag{B.6}$$

Noting once again that integration by parts gives the second integral in (B.6) is equal to $O(|s|^{-4}e^{|t|x})$. Finally we analyze the first integral on the right of (B.6) by noting the following from integration by parts,

$$\int_0^x V(\tau)\int_0^\tau V(u)du\ d\tau = \left[\int_0^\tau V(u)du\int_0^\tau V(u)du\right]_0^x - \int_0^x V(\tau)\int_0^\tau V(u)du\ d\tau$$

So that we see,

$$\int_0^x V(\tau)\int_0^\tau V(u)du\ d\tau = \frac{1}{2}\left[\int_0^x V(\tau)d\tau\right]^2$$

Hence we obtain the next order behavior of $\psi(x,\lambda)$ as,

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} - \frac{\cos(sx)}{2s^2}\int_0^x V(\tau)d\tau + \frac{\sin(sx)}{4s^3}(V(x)+V(0)) - \frac{\sin(sx)}{8s^3}\left[\int_0^x V(\tau)d\tau\right]^2 + \mathcal{O}(|s|^{-4}e^{|t|x})$$

□

Exercise 2.7.

Proof. Note that we have the following results from lemmas 2.4 and 2.5 for $\lambda = s^2$ and $s \in \mathbb{R}$,

$$\psi(x,\lambda) = \frac{\sin(sx)}{s} + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda)d\tau \text{ and } \psi(x,\lambda) = O(|s|^{-1})$$

We improve the asymptotic estimates by looking for λ such that $\psi(\pi,\lambda) = 0$ and using what we found in exercise 2.6 i) and ii) for $\psi(x,\lambda)$, namely combining equations (B.3), (B.4), (B.5), and (B.6) with the solution for part ii), we see,

$$\begin{aligned}
0 = s\psi(\pi,\lambda) &= \sin(s\pi) - \frac{\cos(s\pi)}{2s}\int_0^\pi V(\tau)d\tau + \frac{\sin(s\pi)}{4s^2}(V(\pi)+V(0)) \\
&+ \frac{1}{4s^2}\int_0^\pi \sin(s\pi - 2s\tau)V'(\tau)d\tau - \frac{1}{4s^2}\int_0^\pi \sin(s\pi-2s\tau)V(\tau)\int_0^\tau V(u)du\ d\tau \\
&- \frac{\sin(s\pi)}{8s^2}\left[\int_0^\pi V(\tau)d\tau\right]^2 + \frac{1}{2s^2}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau \cos(s\tau - 2su)V(u)du\ d\tau \\
&+ \frac{1}{s^2}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\int_0^u \sin[s(u-v)]V(v)\psi(v,\lambda)dv\ du\ d\tau
\end{aligned} \tag{B.7}$$

Substitution of $\psi(v,\lambda)$ and integration by parts gives for the last integral of (B.7),

$$\begin{aligned}
&\frac{1}{s^2}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\int_0^u \sin[s(u-v)]V(v)\psi(v,\lambda)dv\ du\ d\tau \\
&= \frac{1}{s^2}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u) \\
&\int_0^u \sin[s(u-v)]V(v)\left[\frac{\sin(sv)}{s} + \frac{1}{s}\int_0^v \sin[s(v-\mu)]V(\mu)\psi(\mu,\lambda)d\mu\right]dv\ du\ d\tau \\
&= \frac{1}{s^3}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\int_0^u \sin[s(u-v)]V(v)\sin(sv)dv\ du\ d\tau + \mathcal{O}(|s|^{-4})
\end{aligned}$$

Now by the trig identity $2\sin[s(u-v)]\sin(sv) = \cos(su-2sv)-\cos(su)$ we find,

$$= \frac{1}{2s^3}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]V(u)\int_0^u [\cos(su-2sv)-\cos(su)]V(v)dv\ du\ d\tau + \mathcal{O}(|s|^{-4})$$

$$= -\frac{1}{2s^3}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau \sin[s(\tau-u)]\cos(su)V(u)\int_0^u V(v)dv\ du\ d\tau + \mathcal{O}(|s|^{-4})$$

Now by the trig identity $2\sin[s(\tau-u)]\cos(su) = \sin(s\tau)-\sin(s\tau-2su)$ we find,

$$= -\frac{1}{4s^3}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau [\sin(s\tau)-\sin(s\tau-2su)]V(u)\int_0^u V(v)dv\ du\ d\tau + \mathcal{O}(|s|^{-4})$$

$$= -\frac{1}{4s^3}\int_0^\pi \sin[s(\pi-\tau)]\sin(s\tau)V(\tau)\int_0^\tau V(u)\int_0^u V(v)dv\ du\ d\tau + \mathcal{O}(|s|^{-4})$$

Again, by the trig identity $2\sin[s(\pi-\tau)]\sin(s\tau) = \cos(s\pi-2s\tau)-\cos(s\pi)$ we find,

$$= -\frac{1}{8s^3}\int_0^\pi [\cos(s\pi-2s\tau)-\cos(s\pi)]V(\tau)\int_0^\tau V(u)\int_0^u V(v)dv\ du\ d\tau + \mathcal{O}(|s|^{-4})$$

$$= \frac{\cos(s\pi)}{8s^3}\int_0^\pi V(\tau)\int_0^\tau V(u)\int_0^u V(v)dv\ du\ d\tau + \mathcal{O}(|s|^{-4})$$

$$= \frac{\cos(s\pi)}{8s^3}\int_0^\pi V(\tau)\int_0^\tau V(u)\int_0^u V(v)dv\ du\ d\tau + \mathcal{O}(|s|^{-4})$$

$$= \frac{\cos(s\pi)}{48s^3}\left[\int_0^\pi V(\tau)d\tau\right]^3 + \mathcal{O}(|s|^{-4})$$

We analyze the second to last integral of (B.7) through integration by parts as follows,

$$\frac{1}{2s^2}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau \cos(s\tau-2su)V(u)du\ d\tau$$

$$= -\frac{1}{4s^3}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\int_0^\tau [-2s\cos(s\tau-2su)]V(u)du\ d\tau$$

$$= -\frac{1}{4s^3}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\left[V(u)\sin(s\tau-2su)\Big|_0^\tau - \int_0^\tau \sin(s\tau-2su)V(u)du\right]\ d\tau$$

$$= -\frac{1}{4s^3}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)[V(\tau)\sin(-s\tau)-V(0)\sin(s\tau)]d\tau + \mathcal{O}(|s|^{-4})$$

$$= \frac{1}{4s^3}\int_0^\pi \sin[s(\pi-\tau)]\sin(s\tau)V^2(\tau)d\tau + \frac{V(0)}{4s^3}\int_0^\pi \sin[s(\pi-\tau)]\sin(s\tau)V(\tau)d\tau + \mathcal{O}(|s|^{-4})$$

Now by the trig identity $2\sin[s(\pi-\tau)]\sin(s\tau) = \cos(s\pi - 2s\tau) - \cos(s\pi)$ we find,

$$= -\frac{\cos(s\pi)}{8s^3}\int_0^\pi V^2(\tau)d\tau - \frac{V(0)\cos(s\pi)}{8s^3}\int_0^\pi V(\tau)d\tau + \mathcal{O}(|s|^{-4})$$

Further, we analyze the following integrals from (B.7) to see how they tend to zero,

$$\frac{1}{4s^2}\int_0^\pi \sin(s\pi - 2s\tau)V'(\tau)d\tau = \frac{1}{8s^3}\int_0^\pi [2s\sin(s\pi - 2s\tau)]V'(\tau)d\tau$$
$$= \frac{\cos(s\pi)}{8s^3}(V'(\pi) - V'(0)) - \frac{1}{8s^3}\int_0^\pi \cos(s\pi - 2s\tau)V''(\tau)d\tau$$

and,

$$-\frac{1}{4s^2}\int_0^\pi \sin(s\pi - 2s\tau)V(\tau)\int_0^\tau V(u)du\ d\tau$$
$$= -\left[\frac{\cos(s\pi - 2s\tau)}{8s^3}V(\tau)\int_0^\tau V(u)du\right]_0^\pi + \frac{1}{8s^3}\int_0^\pi \cos(s\pi - 2s\tau)\left[V'(\tau)\int_0^\tau V(u)du + V^2(\tau)\right]d\tau$$
$$= -\frac{\cos(s\pi)}{8s^3}V(\pi)\int_0^\pi V(u)du + \frac{1}{8s^3}\int_0^\pi \cos(s\pi - 2s\tau)V'(\tau)\int_0^\tau V(u)du\ d\tau$$
$$+ \frac{1}{8s^3}\int_0^\pi \cos(s\pi - 2s\tau)V^2(\tau)d\tau$$

With the second integral of the first identity and last two integrals of the second identity equal to $O(|s|^{-4})$ through integration by parts once again. Thus we find from (B.7) that,

$$\begin{aligned}0 = \sin(s\pi) &- \frac{\cos(s\pi)}{2s}\int_0^\pi V(\tau)d\tau + \frac{\sin(s\pi)}{4s^2}(V(\pi)+V(0)) - \frac{\sin(s\pi)}{8s^2}\left[\int_0^\pi V(\tau)d\tau\right]^2 \\ &-\frac{\cos(s\pi)}{8s^3}\int_0^\pi V^2(\tau)d\tau - \frac{V(0)\cos(s\pi)}{8s^3}\int_0^\pi V(\tau)d\tau + \frac{\cos(s\pi)}{48s^3}\left[\int_0^\pi V(\tau)d\tau\right]^3 \\ &+\frac{\cos(s\pi)}{8s^3}(V'(\pi) - V'(0)) - \frac{\cos(s\pi)}{8s^3}V(\pi)\int_0^\pi V(u)du + \mathcal{O}(|s|^{-4})\end{aligned} \tag{B.8}$$

From which we see by substituting in α_1,

$$\begin{aligned}0 = \sin(s\pi) &+ \frac{\sin(s\pi)}{4s^2}(V(\pi)+V(0)) + \frac{\cos(s\pi)}{8s^3}(V'(\pi) - V'(0)) - \frac{\cos(s\pi)}{8s^3}\int_0^\pi V^2(\tau)d\tau \\ &-\frac{\pi V(0)\cos(s\pi)}{4s^3}\alpha_1 - \frac{\pi\cos(s\pi)}{s}\alpha_1 - \frac{\pi\cos(s\pi)}{4s^3}V(\pi)\alpha_1 - \frac{\pi^2\sin(s\pi)}{2s^2}\alpha_1^2 \\ &+\frac{\pi^3\cos(s\pi)}{6s^3}\alpha_1^3 + \mathcal{O}(|s|^{-4})\end{aligned} \tag{B.9}$$

For large s, this vanishes close to the integers, and for sufficiently large n, the corresponding eigenvalues are simple, which can be seen by taking the derivative of (B.8) to see it does not vanish for $s \approx n$, n large. We now divide (B.9) by $\cos(s\pi)$ and solve for $\tan(s\pi)$ to see,

$$\begin{aligned}\tan(s\pi) = -\frac{\tan(s\pi)}{4s^2}(V(\pi)+V(0)) - \frac{1}{8s^3}(V'(\pi)-V'(0)) + \frac{1}{8s^3}\int_0^\pi V^2(\tau)d\tau + \frac{\pi V(0)}{4s^3}\alpha_1 \\ \frac{\pi}{s}\alpha_1 + \frac{\pi V(\pi)}{4s^3}\alpha_1 + \frac{\pi^2\tan(s\pi)}{2s^2}\alpha_1^2 - \frac{\pi^3}{6s^3}\alpha_1^3 + \mathcal{O}(|s|^{-4})\end{aligned} \tag{B.10}$$

We now determine the large-n behavior of $s_n = n + \dfrac{\alpha_1}{n} + \delta_n$ using series expansion for large n. Notice we have $\delta_n = O(n^{-2})$ and the following, which can be seen from (B.10) above,

$$\tan(s_n\pi) = \tan\left(\left(n + \frac{\alpha_1}{n} + \delta_n\right)\pi\right) = \tan\left(\left(\frac{\alpha_1}{n} + \delta_n\right)\pi\right) = \frac{\pi\alpha_1}{n + \frac{\alpha_1}{n} + \delta_n} + \mathcal{O}(n^{-2})$$

We let $s_n = n + \dfrac{\alpha_1}{n} + \delta_n$ and substitute $\dfrac{\pi\alpha_1}{n + \frac{\alpha_1}{n} + \delta_n} + \mathcal{O}(n^{-2})$ for each $\tan(s\pi)$ on the right of (B.10) as follows, noting that the $O(n^{-2})$ portion gives $O(n^{-4})$ after substitution,

$$\begin{aligned}\tan\left(\left(\frac{\alpha_1}{n} + \delta_n\right)\pi\right) &= -\frac{\pi\alpha_1(V(\pi)+V(0))}{4\left(n + \frac{\alpha_1}{n} + \delta_n\right)^3} - \frac{(V'(\pi)-V'(0))}{8\left(n + \frac{\alpha_1}{n} + \delta_n\right)^3} \\ &+ \frac{1}{8\left(n + \frac{\alpha_1}{n} + \delta_n\right)^3}\int_0^\pi V^2(\tau)d\tau + \frac{\pi V(0)\alpha_1}{4\left(n + \frac{\alpha_1}{n} + \delta_n\right)^3} + \frac{\pi\alpha_1}{\left(n + \frac{\alpha_1}{n} + \delta_n\right)} \\ &+ \frac{\pi V(\pi)\alpha_1}{4\left(n + \frac{\alpha_1}{n} + \delta_n\right)^3} + \frac{\pi^3\alpha_1^3}{2\left(n + \frac{\alpha_1}{n} + \delta_n\right)^3} - \frac{\pi^3\alpha_1^3}{6\left(n + \frac{\alpha_1}{n} + \delta_n\right)^3} + \mathcal{O}(n^{-4})\end{aligned} \tag{B.11}$$

Now, for $s_n = n + \dfrac{\alpha_1}{n} + \delta_n$, we note we have the following for large n using geometric series,

$$\frac{1}{s_n} = \frac{1}{n + \frac{\alpha_1}{n} + \delta_n} = \frac{1}{n}\sum_{k=0}^{\infty}\left(-\frac{\alpha_1}{n^2} - \frac{\delta_n}{n}\right)^k = \frac{1}{n} - \frac{\alpha_1}{n^3} + \mathcal{O}(n^{-4})$$

Similarly we find for second and third powers of $\dfrac{1}{s_n}$,

$$\frac{1}{s_n^2} = \frac{1}{\left(n + \frac{\alpha_1}{n} + \delta_n\right)^2} = \frac{1}{n^2} + \mathcal{O}(n^{-4}) \text{ and } \frac{1}{s_n^3} = \frac{1}{\left(n + \frac{\alpha_1}{n} + \delta_n\right)^3} = \frac{1}{n^3} + \mathcal{O}(n^{-4})$$

Substituting this information into (B.11) gives us,

$$\tan\left(\left(\frac{\alpha_1}{n}+\delta_n\right)\pi\right) = -\frac{\pi\alpha_1(V(\pi)+V(0))}{4n^3} - \frac{(V'(\pi)-V'(0))}{8n^3} + \frac{\pi\alpha_1}{n} - \frac{\pi\alpha_1^2}{n^3} + \frac{1}{8n^3}\int_0^\pi V^2(\tau)d\tau + \frac{\pi V(0)\alpha_1}{4n^3} + \frac{\pi V(\pi)\alpha_1}{4n^3} + \frac{\pi^3\alpha_1^3}{2n^3} - \frac{\pi^3\alpha_1^3}{6n^3} + \mathcal{O}(n^{-4}) \tag{B.12}$$

Notice that the right side of (B.12) is small for sufficiently large n, justifying the next steps. Simplifying, and taking the inverse tangent of (B.12) and using its series expansion for large n gives,

$$\delta_n\pi = -\frac{(V'(\pi)-V'(0))}{8n^3} - \frac{\pi\alpha_1^2}{n^3} + \frac{1}{8n^3}\int_0^\pi V^2(\tau)d\tau + \frac{\pi^3\alpha_1^3}{2n^3} - \frac{\pi^3\alpha_1^3}{6n^3} - \frac{\pi^3\alpha_1^3}{3n^3} + \mathcal{O}(n^{-4})$$

$$\begin{aligned}\delta_n &= -\frac{(V'(\pi)-V'(0))}{8\pi n^3} - \frac{\alpha_1^2}{n^3} + \frac{1}{8\pi n^3}\int_0^\pi V^2(\tau)d\tau + \mathcal{O}(n^{-4})\\ &= -\frac{(V'(\pi)-V'(0))}{8\pi n^3} + \frac{1}{8\pi n^3}\int_0^\pi V^2(\tau)d\tau - \frac{\alpha_1^2}{n^3} + \mathcal{O}(n^{-4})\end{aligned}$$

So that we see,

$$s_n = n + \frac{\alpha_1}{n} + \left[-\frac{1}{8\pi}(V'(\pi)-V'(0)) + \frac{1}{8\pi}\int_0^\pi V^2(\tau)d\tau - \alpha_1^2\right]\frac{1}{n^3} + \mathcal{O}(n^{-4})$$

and conclude that,

$$\alpha_3 = -\frac{1}{8\pi}(V'(\pi)-V'(0)) + \frac{1}{8\pi}\int_0^\pi V^2(\tau)d\tau - \alpha_1^2$$

□

We consider an eigenvalue problem with so-called mixed boundary conditions. Let $I=[0,\pi]$ and let $V\in C^\infty([0,\pi])$ be real. Eigenvalues λ_n and eigenfunctions $y_n(x)$ for this problem are defined by

$$-\frac{d^2y_n(x)}{dx^2} + V(x)y_n(x) = \lambda_n y_n(x), \qquad y_n'(0)=0, \qquad y_n(\pi)=0.$$

Exercise 2.8.

Proof. We first define our operator,

$$Py_n(x) := -\frac{d^2y_n(x)}{dx^2} = \lambda_n y_n(x), \qquad y_n'(0)=0, \qquad y_n(\pi)=0$$

First note $\lambda_n = 0, -s_n^2$ result in trivial solutions. Now by letting $\lambda_n = s_n^2$, we look for solutions of the type,

$$y_n(x) = A\cos(s_n x) + B\sin(s_n x) \Rightarrow y_n'(x) = -As_n\sin(s_n x) + Bs_n\cos(s_n x)$$

Which by applying BC gives,

$$0 = y_n'(0) = Bs_n \Rightarrow B = 0 \Rightarrow y_n(x) = A\cos(s_n x) \Rightarrow 0 = y_n(\pi) = A\cos(\pi s_n)$$

$$\Rightarrow s_n = n - \frac{1}{2} \Rightarrow \lambda_n = \left(n - \frac{1}{2}\right)^2 \text{ for } n \in \mathbb{N}$$

Hence we find the zeta function for our operator P,

$$\zeta_P(s) = \sum_{n=1}^{\infty} \lambda_n^{-s} = \sum_{n=1}^{\infty} \left(n - \frac{1}{2}\right)^{-2s} \text{ for } \mathrm{Re}(s) > \frac{1}{2}$$

Now we relate our zeta function to the zeta function of Riemann by noting the following,

$$\zeta_R(2s) = 2^{-2s}\zeta_P(s) + 2^{-2s}\zeta_R(2s) \Rightarrow \zeta_P(s) = (4^s - 1)\zeta_R(2s)$$

$$\zeta_P'(s) = \ln 4(4^s)\zeta_R(2s) + 2(4^s - 1)\zeta_R'(2s)$$

$$\zeta_P'(0) = -\frac{1}{2}\ln 4 = -\ln 2$$

□

Exercise 2.9 i).

Proof. Consider the BVP,

$$Py_n(x) := -\frac{d^2 y_n(x)}{dx^2} + V(x)y_n(x) = \lambda_n y_n(x), \qquad y_n'(0) = 0, \qquad y_n(\pi) = 0$$

By theorem 2.3 we can relate this to the following IVP,

$$P\psi(x,\lambda) = -\frac{d^2\psi(x,\lambda)}{dx^2} + V(x)\psi(x,\lambda) = \lambda\psi(x,\lambda), \quad \lambda \in \mathbb{C}$$

with initial conditions,

$$\psi(0,\lambda) = 1, \qquad \frac{d}{dx}\psi(x,\lambda)\Big|_{x=0} = 0$$

□

Exercise 2.9 ii).

Proof. Let $\psi(x,\lambda)$ denote the solution of the IVP,

$$-\frac{d^2\psi(x,\lambda)}{dx^2}+V(x)\psi(x,\lambda)=\lambda\psi(x,\lambda), \qquad \psi(0,\lambda)=1, \qquad \frac{d}{dx}\psi(x,\lambda)\Big|_{x=0}=0$$

We let $\lambda=s^2$ and multiply the differential equation above by $\sin[s(x-\tau)]$ and integrate to find,

$$-\int_0^x \sin[s(x-\tau)]\psi''(\tau,\lambda)d\tau+\int_0^x \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda)d\tau=s^2\int_0^x \sin[s(x-\tau)]\psi(\tau,\lambda)d\tau$$

Now we consider the first integral on the left and perform integration by parts,

$$\int_0^x \sin[s(x-\tau)]\psi''(\tau,\lambda)d\tau=\sin[s(x-\tau)]\psi'(\tau,\lambda)\Big|_0^x+s\int_0^x \cos[s(x-\tau)]\psi'(\tau,\lambda)d\tau$$

$$=-\sin(sx)\psi'(0,\lambda)+s\cos[s(x-\tau)]\psi(\tau,\lambda)\Big|_0^x-s^2\int_0^x \sin[s(x-\tau)]\psi(\tau,\lambda)d\tau$$

$$=s\psi(x,\lambda)-s\cos(sx)\psi(0,\lambda)-s^2\int_0^x \sin[s(x-\tau)]\psi(\tau,\lambda)d\tau$$

$$=s\psi(x,\lambda)-s\cos(sx)-s^2\int_0^x \sin[s(x-\tau)]\psi(\tau,\lambda)d\tau$$

Substituting back in and solving for $\psi(x,\lambda)$ gives,

$$\psi(x,\lambda)=\cos(sx)+\frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda)d\tau$$

□

Exercise 2.9 iii).

Proof. We first put $\psi(x,\lambda)=e^{|t|x}F(x)$, and note from ii) we have,

$$F(x)=e^{-|t|x}\cos(sx)+\frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)e^{|t|(\tau-x)}F(\tau)d\tau$$

It follows for $\mu=\max_{0\le x\le\pi}|F(x)|$, that

$$\mu\le 1+\frac{\mu}{|s|}\int_0^\pi |V(\tau)|d\tau \Rightarrow \mu\le\frac{1}{1-\frac{1}{|s|}\int_0^\pi |V(\tau)|d\tau}$$

as long as $|s| > \int_0^\pi |V(\tau)|d\tau$. Hence we have $\psi(x,\lambda) = O(e^{|t|x})$ for $|s| > s_0$ large enough. Substituting this back into ii.) we continue,

$$\psi(x,\lambda) = \cos(sx) + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)O(e^{|t|x})d\tau$$

$$= \cos(sx) + \mathcal{O}(|s|^{-1}e^{|t|x})$$

□

Exercise 2.9 iv).

Proof. Note we have from ii.),

$$\psi(x,\lambda) = \cos(sx) + \frac{1}{s}\int_0^x \sin[s(x-\tau)]V(\tau)\psi(\tau,\lambda)d\tau$$

which solves the IVP,

$$-\frac{d^2\psi(x,\lambda)}{dx^2} + V(x)\psi(x,\lambda) = \lambda\psi(x,\lambda), \qquad \psi(0,\lambda) = 1, \qquad \frac{d}{dx}\psi(x,\lambda)\Big|_{x=0} = 0$$

We now look for λ such that $\psi(\pi,\lambda) = 0$ in order to satisfy the second BC of our original BVP. Thus we obtain the following equation for eigenvalues $\lambda_n = s_n^2$,

$$0 = \psi(\pi,\lambda) = \cos(\pi s) + \frac{1}{s}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\psi(\tau,\lambda)d\tau$$

Further, for $s_n \in \mathbb{R}$ we have from iii.) that,

$$\psi(\tau,\lambda) = \cos(s\tau) + \mathcal{O}(|s|^{-1})$$

Using this, we continue by substituting,

$$0 = \cos(\pi s) + \frac{1}{s}\int_0^\pi \sin[s(\pi-\tau)]V(\tau)\left[\cos(s\tau) + \frac{1}{s}\int_0^\tau \sin[s(\tau-\mu)]V(\mu)\psi(\mu,\lambda)d\mu\right]\, d\tau$$

$$= \cos(\pi s) + \frac{1}{s}\int_0^\pi \sin[s(\pi-\tau)]\cos(s\tau)V(\tau)d\tau + \mathcal{O}(|s|^{-2})$$

$$= \cos(\pi s) + \frac{1}{2s}\int_0^\pi [\sin(\pi s) + \sin(s\pi - 2s\tau)]V(\tau)d\tau + \mathcal{O}(|s|^{-2})$$

$$= \cos(\pi s) + \frac{\sin(\pi s)}{2s}\int_0^\pi V(\tau)d\tau + \frac{1}{2s}\int_0^\pi \sin(s\pi - 2s\tau)V(\tau)d\tau + \mathcal{O}(|s|^{-2})$$

Note by previous calculations we have the second integral is $O(|s|^{-2})$, so we conclude

$$0 = \cos(\pi s) + \frac{\sin(\pi s)}{2s}\int_0^\pi V(\tau)d\tau + \mathcal{O}(|s|^{-2}) = \cos(\pi s) + \frac{\pi\sin(\pi s)}{s}\alpha + \mathcal{O}(|s|^{-2})$$

For large s, this vanishes close to $n + \frac{1}{2}$ for $n \in \mathbb{Z}$. Further, taking the derivative wrt s allows us to conclude these eigenvalues are simple. We now divide the previous equation by $\sin(\pi s)$ and solve for $\cot(\pi s)$ to see,

$$\cot(\pi s) = -\frac{\pi}{s}\alpha + \mathcal{O}(|s|^{-2})$$

We determine the large-n behavior by substituting $s_n = n + \frac{1}{2} + \delta_n$,

$$\cot\left(\pi\left[n + \frac{1}{2} + \delta_n\right]\right) = \cot\left(\pi\left[\frac{1}{2} + \delta_n\right]\right) = -\frac{\pi}{n + \frac{1}{2} + \delta_n}\alpha + \mathcal{O}(n^{-2})$$

We note we have the following for large n using geometric series,

$$\frac{1}{s_n} = \frac{1}{n + \frac{1}{2} + \delta_n} = \frac{1}{n + \frac{1}{2}}\sum_{k=0}^{\infty}\left(-\frac{\delta_n}{n + \frac{1}{2}}\right)^k = \frac{1}{n + \frac{1}{2}} + \mathcal{O}(n^{-2})$$

Thus, taking the inverse cotangent and using its series expansion gives,

$$\pi\left[\frac{1}{2} + \delta_n\right] = \frac{\pi}{2} + \frac{\pi\alpha}{n + \frac{1}{2}} + \mathcal{O}(n^{-2})$$

$$\frac{1}{2} + \delta_n = \frac{1}{2} + \frac{\alpha}{n + \frac{1}{2}} + \mathcal{O}(n^{-2})$$

$$\delta_n = \frac{\alpha}{n + \frac{1}{2}} + \mathcal{O}(n^{-2})$$

So that we see,

$$s_n = n + \frac{1}{2} + \frac{\alpha}{n + \frac{1}{2}} + \mathcal{O}(n^{-2})$$

□

Exercise 2.9 v).

Proof. First we verify the results from exercise 2.7 using contour integral formalism. Consider the eigenvalue problem $P_0 : -\dfrac{d^2y}{dx^2} = \lambda y(x), \quad y'(0) = 0, \quad y(\pi) = 0$. By exercise 2.7, the eigenvalues for P_0 are: $\left(n - \dfrac{1}{2}\right)^2$, $n \in \mathbb{N}$. Further, by the Cauchy Residue theorem, we have that

$\zeta_{P_0}(s) = \dfrac{1}{2\pi i}\oint_\gamma z^{-2s}\dfrac{d}{dz}\left\{\ln\left[\cos(\pi z)\right]\right\}\, dz$, with γ enclosing all poles and not containing the origin (for branch cut $(-\infty, 0]$). Further, we can deform the contour to the imaginary axis and substitute appropriately to see,

$$\zeta_{P_0}(s) = \frac{1}{2\pi i}\int_{\infty}^{-\infty}(iy)^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy$$

$$= \frac{1}{2\pi i}\int_{0}^{-\infty}(iy)^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy + \frac{1}{2\pi i}\int_{\infty}^{0}(iy)^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy$$

$$= \frac{1}{2\pi i}\int_{0}^{\infty}(-iy)^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(-\pi iy)\right]\right\}\, dy - \frac{1}{2\pi i}\int_{0}^{\infty}(iy)^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy$$

$$= \frac{1}{2\pi i}\int_{0}^{\infty}e^{-2s\ln(-i)}y^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy - \frac{1}{2\pi i}\int_{0}^{\infty}e^{-2s\ln(i)}y^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy$$

$$= \frac{\sin(\pi s)}{\pi}\int_{0}^{\infty}y^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy$$

$$= \frac{\sin(\pi s)}{\pi}\int_{0}^{1}y^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy + \frac{\sin(\pi s)}{\pi}\int_{1}^{\infty}y^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy$$

The first integral converges for $\mathrm{Re}(s) < 1$ while the second integral converges for $\mathrm{Re}(s) > \frac{1}{2}$. Now we consider the second integral as follows,

$$\frac{\sin(\pi s)}{\pi}\int_{1}^{\infty}y^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy = \frac{\sin(\pi s)}{\pi}\int_{1}^{\infty}y^{-2s}\frac{d}{dy}\left[\ln\left(\frac{e^{i\pi iy}+e^{-i\pi iy}}{2}\right)\right]\, dy$$

$$= \frac{\sin(\pi s)}{\pi}\int_{1}^{\infty}y^{-2s}\frac{d}{dy}\left\{\ln\left[e^{\pi y}\left(1+e^{-2\pi y}\right)\right]\right\}\, dy$$

$$= \sin(\pi s)\int_{1}^{\infty}y^{-2s}\, dy + \frac{\sin(\pi s)}{\pi}\int_{1}^{\infty}y^{-2s}\frac{d}{dy}\left[\ln\left(1+e^{-2\pi y}\right)\right]\, dy$$

$$= \frac{\sin(\pi s)}{2s-1} + \frac{\sin(\pi s)}{\pi}\int_{1}^{\infty}y^{-2s}\frac{d}{dy}\left[\ln\left(1+e^{-2\pi y}\right)\right]\, dy$$

Summarizing we have,

$$\zeta_{P_0}(s) = \frac{\sin(\pi s)}{\pi}\int_{0}^{1}y^{-2s}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy + \frac{\sin(\pi s)}{2s-1} + \frac{\sin(\pi s)}{\pi}\int_{1}^{\infty}y^{-2s}\frac{d}{dy}\left[\ln\left(1+e^{-2\pi y}\right)\right]\, dy$$

$$\zeta'_{P_0}(0) = \int_{0}^{1}\frac{d}{dy}\left\{\ln\left[\cos(\pi iy)\right]\right\}\, dy - \pi + \int_{1}^{\infty}\frac{d}{dy}\left[\ln\left(1+e^{-2\pi y}\right)\right]\, dy$$

$$= \ln\left[\cos(\pi i)\right] - \pi - \ln\left(1+e^{-2\pi}\right) = \ln\left(e^{\pi}+e^{-\pi}\right) - \ln(2) - \pi - \ln\left(1+e^{-2\pi}\right) = -\ln(2)$$

Now we consider the following IVP,

$$P\psi(x,\lambda) = -\frac{d^2\psi(x,\lambda)}{dx^2} + V(x)\psi(x,\lambda) = \lambda\psi(x,\lambda), \qquad \psi(0,\lambda) = 1, \qquad \frac{d}{dx}\psi(x,\lambda)\Big|_{x=0} = 0$$

Note eigenvalues are determined by $\psi(\pi,\lambda) = 0$, which implies eigenvalues are poles of the following equation,

$$\frac{d}{d\lambda}\ln[\psi(\pi,\lambda)] = \frac{\psi'(\pi,\lambda)}{\psi(\pi,\lambda)}$$

Hence we have the integral representation of the zeta function of our operator with $\lambda = \beta^2$,

$$\zeta_P(s) = \frac{1}{2\pi i}\oint_\gamma \beta^{-2s}\frac{d}{d\beta}\ln[\psi(\pi,\beta^2)]d\beta$$

for $\frac{1}{2} < \text{Re}(s) < 1$ with γ enclosing all poles and not containing the origin (for branch cut $(-\infty, 0]$).

Proceeding as in the evaluation of $\zeta'(0)$, noting the leading terms do not depend on the potential, we find,

$$\zeta_P(s) = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} \beta^{-2s}\frac{d}{d\beta}\ln[\psi(\pi,\beta^2)]d\beta \text{ for } \frac{1}{2} < \text{Re}(s) < 1$$

We now substitute $\beta = \pm ik$ in above with the upper and lower portion of the contour respectively to find,

$$\begin{aligned}\zeta_P(s) &= \frac{1}{2\pi i}\left\{\int_{i\infty}^{0} \beta^{-2s}\frac{d}{d\beta}\ln[\psi(\pi,\beta^2)]d\beta + \int_{0}^{-i\infty} \beta^{-2s}\frac{d}{d\beta}\ln[\psi(\pi,\beta^2)]d\beta\right\}\\
&= \frac{1}{2\pi i}\left\{\int_{\infty}^{0} (ik)^{-2s}\frac{d}{dk}\ln[\psi(\pi,-k^2)]dk + \int_{0}^{\infty} (-ik)^{-2s}\frac{d}{dk}\ln[\psi(\pi,-k^2)]dk\right\}\\
&= \frac{1}{2\pi i}\left\{\int_{\infty}^{0} (e^{i\pi/2}k)^{-2s}\frac{d}{dk}\ln[\psi(\pi,-k^2)]dk + \int_{0}^{\infty} (e^{-i\pi/2}k)^{-2s}\frac{d}{dk}\ln[\psi(\pi,-k^2)]dk\right\}\\
&= \frac{1}{2\pi i}\left(e^{i\pi s} - e^{-i\pi s}\right)\int_0^\infty k^{-2s}\frac{d}{dk}\ln[\psi(\pi,-k^2)]dk\\
&= \frac{\sin(\pi s)}{\pi}\int_0^\infty k^{-2s}\frac{d}{dk}\ln[\psi(\pi,-k^2)]dk \text{ for } \frac{1}{2} < \text{Re}(s) < 1\end{aligned}$$

Now we notice that by lemmas 2.12 and 2.13, which do not rely on a particular IVP, we have,

$$\zeta_P'(0) = -\ln\left[\frac{\psi(\pi,0)}{\psi_0(\pi,0)}\right] + \zeta_{P_0}'(0) \text{ where } P_0 = -\frac{d^2}{dx^2}$$

Notice that we know $\zeta_{P_0}'(0) = -\ln 2$. Further, we note $\psi_0(x,0) = 1$ is the solution to,

$$P_0\psi_0(x,0) = -\frac{d^2\psi_0(x,0)}{dx^2} = 0, \qquad \psi_0(0,0) = 1, \qquad \frac{d}{dx}\psi_0(x,0)\Big|_{x=0} = 0$$

So that we can conclude,

$$\zeta_P'(0) = -\ln\left[\frac{\psi(\pi,0)}{1}\right] - \ln 2 = -\ln(2\psi(\pi,0))$$

□

Appendix C

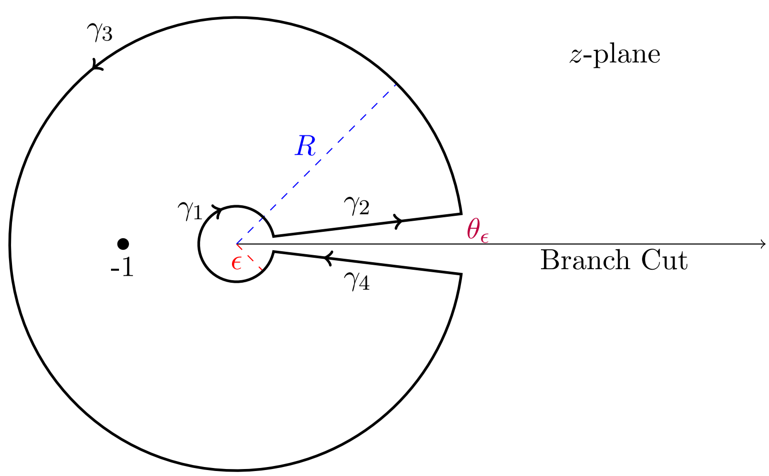
Additional Exercises with Solutions

In this appendix, we will offer additional exercises of interest with hints, solutions, and remarks.

Exercise C.1. Show that

$$\int_0^\infty \frac{s^{-\nu}}{s+1}\, ds = \frac{\pi}{\sin(\pi\nu)} \text{ for } 0 < \mathrm{Re}(\nu) < 1$$

Proof. Consider $\oint_\gamma \frac{z^{-\nu}}{z+1} dz$ with the branch cut $[0, \infty)$ and the path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ defined as below,



with $R :=$ radius of the outer circle, $R > 1$; $\epsilon :=$ radius of the inner circle, $0 < \epsilon < 1$; and $\theta_\epsilon :=$ the angle between the positive real line and γ_2 (similarly for γ_4), where $\theta_\epsilon \to 0$ as $\epsilon \to 0$.

First, we notice the only pole in γ is the simple pole at $z=-1$, so, noting that $\log(z)=\ln(r)+i\theta$, we have with $f(z) := \dfrac{z^{-\nu}}{z+1}$,

$$\text{Res}[f(-1)] = \lim_{z\to -1}(z+1)f(z) = (-1)^{-\nu} = e^{-\nu\log(-1)} = e^{-\nu(\ln(1)+i\pi)} = e^{-\nu\pi i}$$

Thus we have by the Residue Theorem that,

$$\oint_\gamma f(z)dz = 2\pi i\ \text{Res}[f(-1)] = 2\pi i e^{-\nu\pi i}$$

Next we calculate each of the integrals for γ_1, γ_2, γ_3, and γ_4.
γ_1: We use $z=re^{i\theta}$ with $r=\epsilon$ and $\theta_\epsilon \le \theta \le 2\pi-\theta_\epsilon$, to see,

$$\oint_{\gamma_1} f(z)dz = \int_{2\pi-\theta_\epsilon}^{\theta_\epsilon} \frac{(\epsilon e^{i\theta})^{-\nu}}{\epsilon e^{i\theta}+1}\epsilon i e^{i\theta}d\theta$$

Notice,

$$\left|\frac{(\epsilon e^{i\theta})^{-\nu}}{\epsilon e^{i\theta}+1}\epsilon i e^{i\theta}\right| = \frac{\epsilon^{-\text{Re}(\nu)+1}e^{\text{Im}(\nu)\theta}}{|\epsilon e^{i\theta}+1|} \le \frac{\epsilon^{-\text{Re}(\nu)+1}e^{2\pi|\text{Im}(\nu)|}}{1-\epsilon}$$

Which we see goes to 0 as $\epsilon \to 0$, since $0<\text{Re}(\nu)<1$. Thus we have,

$$\oint_{\gamma_1} f(z)dz \to 0 \text{ as } \epsilon \to 0$$

γ_3: We use $z=re^{i\theta}$ with $r=R$ and $\theta_\epsilon \le \theta \le 2\pi-\theta_\epsilon$, to see,

$$\oint_{\gamma_3} f(z)dz = \int_{\theta_\epsilon}^{2\pi-\theta_\epsilon} \frac{(Re^{i\theta})^{-\nu}}{Re^{i\theta}+1}Rie^{i\theta}d\theta$$

Notice,

$$\left|\frac{(Re^{i\theta})^{-\nu}}{Re^{i\theta}+1}Rie^{i\theta}\right| = \frac{R^{-\text{Re}(\nu)+1}e^{\text{Im}(\nu)\theta}}{|Re^{i\theta}+1|} \le \frac{R^{-\text{Re}(\nu)+1}e^{2\pi|\text{Im}(\nu)|}}{R-1}$$

Which we see goes to 0 as $R\to\infty$, since $0<\text{Re}(\nu)<1$. Thus we have,

$$\oint_{\gamma_3} f(z)dz \to 0 \text{ as } R \to \infty$$

γ_2: We use $z=te^{i\theta}$ with $\epsilon<t<R$ and $\theta=\theta_\epsilon$, to see,

$$\oint_{\gamma_2} f(z)dz = \int_\epsilon^R \frac{(te^{i\theta_\epsilon})^{-\nu}}{te^{i\theta_\epsilon}+1}e^{i\theta_\epsilon}dt$$

We calculate,

$$(te^{i\theta_\epsilon})^{-\nu} = e^{-\nu(\ln(t)+i\theta_\epsilon)} = t^{-\nu}e^{-\nu i\theta_\epsilon}$$

So that,

$$\oint_{\gamma_2} f(z)dz = \int_\epsilon^R e^{(-\nu+1)i\theta_\epsilon}\frac{t^{-\nu}}{te^{i\theta_\epsilon}+1}dt$$

Now we see,

$$\oint_{\gamma_2} f(z)dz \to \int_0^\infty \frac{t^{-\nu}}{t+1}dt \text{ as } \epsilon \to 0,\ R \to \infty$$

γ_4: We use $z = te^{i\theta}$ with $\epsilon < t < R$ and $\theta = 2\pi - \theta_\epsilon$, to see,

$$\oint_{\gamma_4} f(z)dz = \int_R^\epsilon \frac{(te^{i(2\pi-\theta_\epsilon)})^{-\nu}}{te^{i(2\pi-\theta_\epsilon)}+1}e^{i(2\pi-\theta_\epsilon)}dt$$

$$= -\int_\epsilon^R e^{(-\nu+1)i(2\pi-\theta_\epsilon)}\frac{t^{-\nu}}{te^{i(2\pi-\theta_\epsilon)}+1}dt$$

Now we see,

$$\oint_{\gamma_4} f(z)dz \to -e^{(-\nu+1)2\pi i}\int_0^\infty \frac{t^{-\nu}}{t+1}dt = -e^{-\nu 2\pi i}\int_0^\infty \frac{t^{-\nu}}{t+1}dt \text{ as } \epsilon \to 0,\ R \to \infty$$

Thus we obtain the following from comparing the results from the residue theorem to the sum of the integrals above after sending $\epsilon \to 0$ and $R \to \infty$,

$$\begin{aligned}2\pi i e^{-\nu\pi i} &= \oint_\gamma f(z)dz = \oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz + \oint_{\gamma_3} f(z)dz + \oint_{\gamma_4} f(z)dz \\ &= (1 - e^{-\nu 2\pi i})\int_0^\infty \frac{t^{-\nu}}{t+1}dt\end{aligned}$$

So that we see,

$$\int_0^\infty \frac{t^{-\nu}}{t+1}dt = \frac{2\pi i e^{-\nu\pi i}}{1-e^{-\nu 2\pi i}} = \frac{2\pi i}{e^{\nu\pi i}-e^{-\nu\pi i}} = \pi\left(\frac{2i}{e^{\nu\pi i}-e^{-\nu\pi i}}\right) = \frac{\pi}{\sin(\pi\nu)}, \quad 0 < \mathrm{Re}(\nu) < 1$$

□

Remark C.1. Making the change of variables $t = as$, $a > 0$, one obtains for $0 < \mathrm{Re}(\nu) < 1$,

$$\int_0^\infty \frac{t^{-\nu}dt}{t+a} = a^{-\nu}\int_0^\infty \frac{s^{-\nu}ds}{s+1} = a^{-\nu}\frac{\pi}{\sin(\pi\nu)}$$

or

$$a^{-\nu} = \frac{\sin(\pi\nu)}{\pi}\int_0^\infty \frac{t^{-\nu}dt}{t+a}, \quad a > 0,$$

and hence

$$(a-z)^{-\nu} = \frac{\sin(\pi\nu)}{\pi} \int_0^\infty \frac{t^{-\nu} dt}{t+a-z}, \quad a > 0,\ z < 0.$$

By analytic continuation w.r.t. z one finally gets

$$(a-z)^{-\nu} = \frac{\sin(\pi\nu)}{\pi} \int_0^\infty \frac{t^{-\nu} dt}{t+a-z}, \quad 0 < \mathrm{Re}(\nu) < 1,\ a > 0,\ z \in \mathbb{C}\backslash[a,\infty).$$

This makes it **plausible** that for a self-adjoint, nonnegative operator $A \geq 0$ in a complex, separable Hilbert space $\mathcal{H}$, one obtains

$$(A - zI_{\mathcal{H}})^{-\nu} = \frac{\sin(\pi\nu)}{\pi} \int_0^\infty dt\, t^{-\nu} (A + (t-z)I_{\mathcal{H}})^{-1}, \quad 0 < \mathrm{Re}(\nu) < 1,\ z \in \mathbb{C}\backslash[0,\infty).$$

That is, fractional powers of the resolvent of A are reduced to a parameter integral over just the resolvent of A.

As mentioned in class, this can indeed be verified (and extended to larger classes of not necessarily self-adjoint operators A)! $\diamond$

Exercise C.2. Show that

$$\int_0^\infty t^{\mu-1}(1+\beta t^p)^{-\nu}dt = \frac{1}{p}\beta^{-\frac{\mu}{p}}\mathrm{B}\left(\frac{\mu}{p}, \nu - \frac{\mu}{p}\right),$$

$$|\arg\beta| < \pi,\ p > 0,\ 0 < \mathrm{Re}(\mu) < p\ \mathrm{Re}(\nu).$$

Proof. First we note the following definition for Beta functions with $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$,

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \left(= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\right)$$

Next we show the following relation by substituting $t = \dfrac{u}{1+u}$ so that $dt = \dfrac{1}{(1+u)^2}du$,

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

$$= \int_0^\infty \left(\frac{u}{1+u}\right)^{x-1}\left(1 - \frac{u}{1+u}\right)^{y-1}\frac{1}{(1+u)^2}du$$

$$= \int_0^\infty u^{x-1}(1+u)^{-x+1}(1+u)^{-y+1}(1+u)^{-2}du$$

$$= \int_0^\infty u^{x-1}(1+u)^{-(x+y)}du$$

Now we calculate as follows,

$$\int_0^\infty x^{\mu-1}(1+\beta x^p)^{-\nu}dx = \int_0^\infty \frac{p}{p}\beta^{\frac{\mu}{p}-\frac{\mu}{p}+1-1}x^{\mu-p+p-1}(1+\beta x^p)^{-\nu}dx$$

$$= \frac{1}{p}\beta^{-\frac{\mu}{p}}\int_0^\infty \beta^{\frac{\mu}{p}-1}x^{\mu-p}(1+\beta x^p)^{-\nu}p\beta x^{p-1}dx$$

$$= \frac{1}{p}\beta^{-\frac{\mu}{p}}\int_0^\infty (\beta x^p)^{\frac{\mu}{p}-1}(1+\beta x^p)^{-\nu}p\beta x^{p-1}dx$$

Now we let $t = \beta x^p$, so that $dt = p\beta x^{p-1}dx$, to see,

$$= \frac{1}{p}\beta^{-\frac{\mu}{p}}\int_0^\infty t^{\frac{\mu}{p}-1}(1+t)^{-\nu}dt$$

$$= \frac{1}{p}\beta^{-\frac{\mu}{p}}\int_0^\infty t^{\frac{\mu}{p}-1}(1+t)^{-(\frac{\mu}{p}+\nu-\frac{\mu}{p})}dt$$

as we have $0 < \dfrac{\mathrm{Re}(\mu)}{p} < \mathrm{Re}(\nu)$, implying $\mathrm{Re}(\frac{\mu}{p}) > 0$ and $\mathrm{Re}(\nu - \frac{\mu}{p}) > 0$.

Now applying the Beta function relation from above gives the desired result,

$$\int_0^\infty t^{\mu-1}(1+\beta t^p)^{-\nu}dt = \frac{1}{p}\beta^{-\frac{\mu}{p}}\mathrm{B}\left(\frac{\mu}{p}, \nu - \frac{\mu}{p}\right)$$

□

Remark C.2. One might naively hope that this formula now yields an alternative approach to fractional powers of operators, but unfortunately, the elementary substitution $s = t^p$, and taking $\nu = 1$ reduces exercise C.2 to exercise C.1. While values $\nu \neq 1$ yield expressions different from exercise C.1, these new expressions don't seem to simplify matters (in the manner exercise C.1 does by reducing fractional powers of resolvents to straightforward resolvents). ◇

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