Calculus II Guided Notes, Baylor

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Text: Single Variable Calculus: Early Transcendentals, 4th Edition, Jon Rogawski and Colin Adams

The course covers techniques of integration, applications of integration, and infinite series:

Techniques of Integration

- Review: Substitution [Section 5.7]
- Integration by parts [Section 7.1]
- Trigonometric integrals and trigonometric substitution [Sections 7.2 and 7.3]
- Partial fractions [Section 7.5]
- L'Hôpital's rule [Section 4.5]
- Improper integrals [Section 7.7]

Applications of Integration

- Area between curves [Section 6.1]
- Integrals in geometry: volume, density, average value [Section 6.2]
- Integrals in geometry: volumes of revolution [Sections 6.3 and 6.4]
- Integrals in geometry: arc length and surface area [Section 8.2]
- Integrals in physics: work and energy [Section 6.5]
- Solving differential equations by separation of variables [Sections 9.1 and 9.4]

Infinite Series

- Sequences and series [Sections 10.1 and 10.2]
- Convergence tests for series [Section 10.3, 10.4, and 10.5]
- Power series [Section 10.6]
- Taylor polynomials [Section 10.7]
- Taylor series, complex numbers, and Euler's formula [Section 10.8]

0 Lecture

Outline:

1. Welcome, syllabus
2. Calculus II in a Nutshell

0.1 Calculus II in a Nutshell

Students are often left with the impression that Calculus II is a hodgepodge of many unrelated topics and ideas. However, Calculus II, or *integral calculus of a single variable*, is really only about two topics: integrals and series, and the need for the latter can be motivated by the former. The purpose of this first lecture is to explore this a little bit so that you should have a pretty good idea what Calculus II is really all about.

Let's start in familiar territory: Integrals you have already encountered in Calculus I since Calculus II picks up right were you left off in Calculus I. During the first weeks of Calculus II you are going to learn several new techniques to evaluate integrals. For example, consider the integral

$$\int e^{-x}\cos x\,dx.$$

This integral can't be evaluated easily by using the methods you learned in Calculus I (try it!), but soon after learning about a method called *integration by parts* (which is related to the product rule for derivatives) you will be able to obtain that

$$\int e^{-x}\cos x\,dx = \frac{1}{2}\,e^{-x}\left(\sin x - \cos x\right) + C.$$

With all these new techniques, will we be able to evaluate (at least in principle) just about any integral, as long as the integrand is a "nice" function? The answer may surprise you. Let's look at the integral

$$\int \sin\left(x^2\right)dx.$$

Using Calculus I ideas, we could define a function $S(x)$ as a definite integral as follows:

$$S(x) = \int_0^x \sin\left(t^2\right)dt.$$

By the *Fundamental Theorem of Calculus* (FTC, Part II), the function $S(x)$ is an antiderivative of the function $\sin\left(x^2\right)$ and hence

$$\int \sin\left(x^2\right)dx = S(x) + C.$$

Expressing an indefinite integral in terms of a definite integral feels like cheating! What we want is a formula for $S(x)$ that does not involve a definite integral. It turns out that there is no such formula in terms of any of the functions that you have encountered in your career so far! But, as we will see later, there is a different kind of formula for $S(x)$ in terms of what is called a *power series*, the most important topic in Calculus II.

Before talking about power series, let's return to familiar territory. Some of the simplest functions that you are familiar with are polynomials. For example, $f(x) = x - x^3/6$ is a polynomial function. Amazingly, most of the "higher" functions such as the trigonometric and exponential functions are essentially almost polynomials, just with infinitely many terms. For example, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\ ,$$

where the three dots ($\cdots$) indicate that the terms go on forever in the obvious fashion. Here the denominators (the numbers with the exclamation marks next to them) are simple products called *factorials*: $3! = 3\cdot2\cdot1 = 6$, $5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$, *etc.*

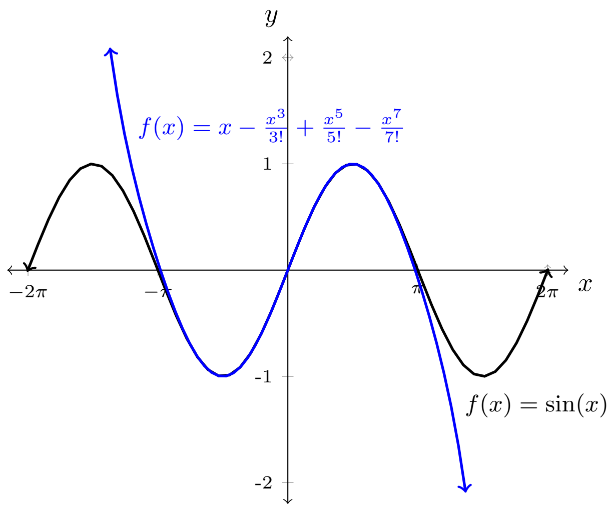
The right-hand side of this mysterious formula for $\sin x$ is what we call a power series in x. Power series are often presented in Σ-notation. (Σ is the capital Greek letter sigma, which should remind you of the initial letter of sum.) In Σ-notation, the formula for $\sin x$ is as follows:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \ .$$

You may still wonder how to interpret this formula. Suppose that we only consider the first few terms of the power series, say only up to the x^7-term. Then, as long as x is relatively close to 0, we get a good approximation of $\sin x$ by a polynomial function of degree 7:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ .$$

Just how good is this polynomial approximation? Let's compare the two graphs:



Over the interval $(-\pi/2, \pi/2)$, the two graphs are so close to each other that they are indistinguishable to the naked eye. In fact, for $-\pi/2 < x < \pi/2$, it follows from *Taylor's theorem*[1] that

$$\left| \sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \right) \right| < \frac{(\pi/2)^9}{9!} \approx 0.000160441 \ .$$

As we include more and more terms of the power series, we get better and better approximations, even for x that are not close to 0. Finally, we get the exact formula for $\sin x$ for every real number x if we use the whole power series with infinitely many terms.

Let's return to the integral of $\sin(x^2)$. By replacing x by x^2 in our power series formula for $\sin x$ we obtain

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \ .$$

[1] You will encounter Taylor's theorem in Calculus II as an application of integration by parts, the technique of integration that was mentioned at the beginning.

Now integrating $\sin x^2$ is as easy as integrating a polynomial:

$$\int \sin x^2\,dx = \int \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right) dx$$

$$= \left(\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{10}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots\right) + C$$

This means that we found a formula for the function $S(x) = \int_0^x \sin(t^2)\,dt$ after all:

$$S(x) = \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{10}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots ,$$

or, in fancy pants Σ-notation,

$$S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!(4n+3)} .$$

By using the same power series trick, we can evaluate many other important integrals and special functions. For example, we can obtain a power series formula for the *Gauss' error function*

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt ,$$

which plays a fundamental role in statistics. The power series formula for $\mathrm{Erf}(x)$ is

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right) .$$

Given this formula (and a calculator), you may try to find the approximate value of $\mathrm{Erf}(1/\sqrt{2})$. If you do it correctly, you will obtain $\mathrm{Erf}(1/\sqrt{2}) \approx 0.683$, which has the following interpretation: In a data sample that is normally distributed, about 68.3% of the data values are within one standard deviation of the mean.

The examples above may have already convinced you that power series will be useful in several applications of mathematics. We will now look at the ultimate reason why you may want to learn about power series. It is related to the fundamental reason why science students have to study calculus to begin with. In all of the exact sciences, many (if not most) of the laws of nature can be expressed as differential equations. Therefore, solving differential equations (and understanding their solutions!) is of paramount importance.

A differential equation is an equation involving an unknown function and its derivatives. An *ordinary differential equation* (ODE) is a differential equation involving an unknown function, say $y = F(x)$, of only one independent variable. The simplest such equation is

$$\frac{dy}{dx} = f(x),$$

where $f(x)$ is some given function. A solution in this case is simply a function $y = F(x)$ such that $F'(x) = f(x)$. In other words, solving this differential equation is equivalent to finding the indefinite integral of $f(x)$. In Calculus II, you will learn a few tricks to solve slightly more general equations by integration, but as you might have guessed, not every differential equation can be solved by integration and we will often have to look for power series solutions.

Of particular importance are second order[2] equations that show up in the study of waves of various kinds. For example, when studying the vibrations of a drum, one is led to the following second order equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0,$$

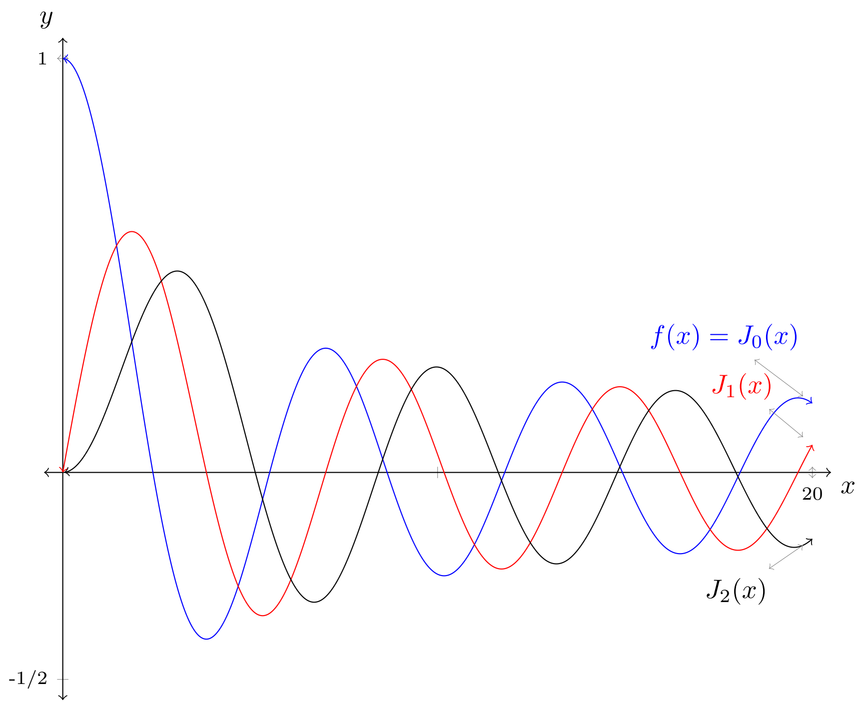
where α is a nonnegative number. This equation is called the *Bessel equation* and its canonical solutions are the so-called *Bessel functions*. The easiest way to describe a Bessel function is as a power series:

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha} .$$

Here the the expression $\Gamma(n+\alpha+1)$ is a value of the Gamma function, which you may encounter in Calculus II (depending on the instructor) as part of the discussion of *improper integrals*. In the special case when $\alpha = 0$, the value $\Gamma(n+\alpha+1)$ is equal to $\Gamma(n+1) = n!$ and it follows[3] that

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots .$$

Here is what the graphs of $J_0(x)$, $J_1(x)$, and $J_2(x)$ look like:



So, what should you take away from all this? After taking Calculus II, you will have gained a whole new perspective on what we mean by a "function". The functions from your youth (such as polynomials, trigonometric functions, exponential functions, and logarithmic functions) are but a small collection of functions that arise in the sciences. If you want to understand the applications of calculus and differential equations (even if it is just to understand a Wikipedia entry!), then you need to learn about series. Applications range from Physical Chemistry (energies of reactions, entropy, heat, phase transitions, chemical potential), to phenomena in Engineering and Physics (laws of nature), to Medicine (imagery, half-life), to general Mathematics (including fractals!), and even arise in discussing some Greek paradoxes.

In other words, you need to take Calculus II.

[2] A second order equation involves the second derivative of the unknown function.

[3] In case you want to verify this, you need to know that $\Gamma(1) = 0! = 1$.

1 Lecture

Outline:

1. Review of concepts: derivatives
2. Review basic rules: power, product, chain
3. Review of concepts: integrals
4. 5.7 The Substitution Method

1.1 Derivatives

Let $y = f(x)$ be a function of real numbers with real values. Its **derivative** at x is provided this limit exists.

For instance, if $f(x) = 2x$ then $f'(x) = 2$ for every x because

1.1.1 Power rule

You have probably memorized that

For example, the derivative of x^2 is $2x$.

1.1.2 Product/quotient rule

More generally,

You can actually use this over and over again to get the power rule.

Another example: if $f(x) = xe^x$, then

The quotient rule is similar:

For example,

$$\frac{d}{dx}\frac{\ln x}{x^{1/2}} =$$

1.1.3 Chain rule

Recall that the **composition** of two functions f and g is given by

$$(f \circ g)(x) = f(g(x)).$$

The chain rule is

$$(f \circ g)'(x) = \qquad\qquad\qquad , \textbf{ or } \frac{dy}{dx} =$$

whichever way helps you remember what's going on. (Often taught as "inner" and "outer" functions.) For example,

$$\frac{d}{dx}e^{x^2} =$$

1.2 Integrals

Let f be a continuous function on $[a, b]$. The **definite integral** of f over $[a, b]$ is defined as the limit of **Riemann sums** as the width of blocks goes to zero. By the **Fundamental Theorem of Calculus,**

where F is any **antiderivative** of f. Remember that we can interpret the definite integral as the **area under the curve, $f(x)$, on $[a, b]$** or the **net change of $F(x)$ on $[a, b]$**. We can think of the **indefinite integral** of f as the *set* of all antiderivatives, meaning there is an unknown constant C. For example,

$$\int x^2 dx =$$

To compute a **definite integral,** it doesn't matter which value of C we choose. For instance,

$$\int_0^1 x^2 dx =$$

from which we see that C is irrelevant for definite integrals.

1.3 Substitution

Thanks to the Fundamental Theorem of Calculus, integration is basically the opposite of differentiation, so if you're good with derivatives, you should be good at integrals. Still, it can be tricky to do, for example, the chain rule backwards.

The chain rule backwards is called **substitution.**

Change of Variables Formula:

$$\int \underbrace{f(u(x))}_{f(u)} \underbrace{u'(x)\ dx}_{du} = \int f(u)\ du$$

We saw that the derivative of e^{x^2} is $2xe^{x^2}$. Now suppose we want to compute an indefinite integral:

$$\int xe^{x^2} dx.$$

The answer should have something to do with e^{x^2}. How can we be systematic?

Do not forget that when using substitution for a definite integral we must consider the limits of integration!

Change of Variables Formula:

$$\int_a^b \underbrace{f(u(x))}_{f(u)} \underbrace{u'(x)\ dx}_{du} = \int_{u(a)}^{u(b)} f(u)\ du$$

Now suppose we want to compute a definite integral:

$$\int_1^2 xe^{x^2} dx.$$

Then all our previous work holds, but we also need to consider the change to the limits of integration.

Example. Evaluate

$$\int x^2 \sin\left(x^3\right) dx.$$

Example. Evaluate

$$\int \tan\theta\ d\theta.$$

The trick with all substitutions is to find the u which will make $f(x)dx$ turn into $g(u)du$ where $g(u)$ is nice and easy. Easier said than done! Indeed, substitution only works when you can find such a nice u. Otherwise, you need to try other tricks.

Suggestions for Substitution

i) If a function is raised to a power or in a radical, i.e., $[f(x)]^2,\ [f(x)]^{-1},\ \sqrt{f(x)}$, let $u = f(x)$.

ii) If a function appears in the argument of another function, i.e., $\sin(f(x)),\ \ln(f(x)),\ e^{f(x)}$, let $u = f(x)$.

Note that i) is a particular case of ii) with the outer function being the power function, and these suggestions do not always work!

2 Lecture

Outline:

1. 7.1 Integration by Parts

2.1 Integration by parts

By now you should be able to do the following techniques forwards (derivatives) and backwards (integrals):

$$\text{power rule: } \frac{d}{dx}x^n = nx^{n-1} \rightarrow \int x^n dx = \frac{1}{n+1}x^{n+1} + C \; (n \neq -1)$$
$$\text{chain rule: } \frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \rightarrow \int g(u(x))\frac{du}{dx}dx = \int g(u)du \text{ (substitution)}$$

Now we extend this to the product rule:

This is called **integration by parts** (or partial integration).

Another way to write the product rule for easier recall is to simply suppress dx to write

Strategy: pick out a u and dv in your integral, such that it is advantageous to pass the derivative from v to u in order to make integration easier.

Example. Compute $\int x \cos x dx$.

Example. Find $\int \ln x dx$. This is a surprising example, because it doesn't seem like there are any "parts"!

Example. Find $\int e^x \cos x dx$. Here's an example were something funny happens.

Example. Compute $\int x^n e^x dx$ for any positive integer n.

Example. Show that $\int_0^\pi \cos x \cos 2x \; dx = 0$.

Suggestions for Substitution If one of the functions in the integrand can be eliminated or made simpler by taking a few derivatives (or integrals), choose this to be u (or dv). Example:

$$u: \; x^n, \; \ln x, \; (\ln x)^n, \text{inverse trigonometric functions}$$

If neither function will be made simpler by derivatives or integrals (multiplication of trigonometric and exponential), you will have to use integration by parts until the original integral reappears and then solve for it.

3 Lecture

Outline:

1. 7.2 Trigonometric Integrals, Part I

3.1 Trigonometric integrals

Many integrals involve trigonometric functions. To compute them, you will need to remember two or three fundamental identities:

$$\cos^2 x + \sin^2 x = 1, \quad \sin 2x = 2\sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x.$$

Note

$$\cos 2x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1, \quad \cos 2x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x.$$

Together with the fact that $(d/dx)\sin x = \cos x$ and $(d/dx)\cos x = -\sin x$, we can compute many integrals of the form

$$\int \sin^m x \cos^n x \; dx.$$

Example. Compute $\int \cos^{100} x \sin x \; dx$.

3.2 Odd powers of cosine or sine

If there's an odd power of either cosine or sine, then we can always reduce down to a single cosine or sine using $\cos^2 x + \sin^2 x = 1$, then use substitution.

Example. Compute $\int \sin^3 x \; dx$.

Example. Compute $\int \sin^2 x \cos^3 x \; dx$.

3.3 Even powers of cosine and sine

In general, even powers are more difficult to compute. How do we compute $\int \sin^2 x \; dx$? We could use integration by parts, actually, but let's use a different trick instead.

You can do the same to get $\int \cos^2 x \; dx = \frac{x}{2} + \frac{1}{2} \sin x \cos x + C$.

With higher powers it gets tougher. We use integration by parts to get a **recursive formula.** (Your book calls it a **reduction formula**.) We want to find the integral of, say, $\cos^n x$. Let $u = \cos^{n-1} x, dv = \cos x \; dx$.

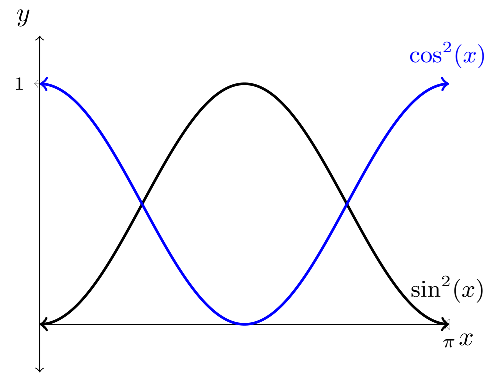
We can use this to compute daunting integrals.

Example. Compute $\int \sin^2 x \cos^4 x dx$.

Another way would be to use the double angle formula directly instead of relying on the reduction formula. There are usually multiple options as you learn more techniques as seen here and in the next example.

Interesting Example. Compute $\int_0^\pi \sin^2 x \; dx$. From previous work we know

But let us look at it a little differently which could help you in the future! Consider the following graphs



4 Lecture

Outline:

1. 7.2 Trigonometric Integrals, Part II

4.1 Trigonometric integrals: tangents and secants

Starting with $\cos^2 x + \sin^2 x = 1$, divide by $\cos^2 x$ to get

$$1 + \tan^2 x = \sec^2 x.$$

We also have

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x = 1 + \tan^2 x, \quad \frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{-\sin x}{-\cos^2 x} = \tan x \sec x.$$

Remembering these two facts, as well as the basic facts we already know, we can compute lots of integrals involving tangents and secants.

Example. Find $\int \tan^2 x \; dx$.

Other examples involve reducing down in clever ways. Normally you substitute $u = \tan x$ or $u = \sec x$, so either $du = \sec^2 x \; dx$ or $du = \tan x \sec x \; dx$. If what's left over is reducible down to powers of u, then the strategy works. We look for even powers of $\tan x$ and $\sec x$, since $\tan^2 x + 1 = \sec^2 x$.

Example. Compute $\int \tan^3 x \sec^5 x \; dx$.

Example. Compute $\int \tan^2 x \sec^4 x dx$.

Example. Find $\int \tan^3 x \; dx$.

Example. Find $\int \sec x \; dx$. One way to do this systematically requires a technique called **partial fractions**, which we haven't done yet. Still, I will only use it to get through one part; everything else is just trigonometric identities.

Table of Trigonometric Integrals (will be given)

$$\int \sin^n x \; dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \; dx$$

$$\int \cos^n x \; dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \; dx$$

$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^n x \; dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x \; dx$$

$$\int \csc x \; dx = -\ln|\csc x + \cot x| + C$$

$$\int \csc^n x \; dx = -\frac{1}{n-1}\csc^{n-2} x \cot x + \frac{n-2}{n-1}\int \csc^{n-2} x \; dx$$

$$\int \tan x \; dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$\int \tan^n x \; dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x \; dx$$

$$\int \cot x \; dx = -\ln|\csc x| + C = \ln|\sin x| + C$$

$$\int \cot^n x \; dx = -\frac{1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x \; dx$$

5 Lecture

Outline:

1. 7.3 Trigonometric Substitution

5.1 Trigonometric substitution

Finding the area under part of a circle. Suppose I want the area under the top half of the unit circle, but only on the interval $0 \le x \le 1/2$. The equation for the top half of the circle is $y = \sqrt{1-x^2}$. So I want to evaluate

$$\text{Area} = \int_0^{1/2} \sqrt{1-x^2}dx.$$

How? What if we could write $1 - x^2 = w^2$ so that $\sqrt{1-x^2} = w$?

With this idea in mind, we try a **trigonometric substitution** recalling that $\sin^2\theta + \cos^2\theta = 1$.

It is possible to verify this geometrically, by splitting the region into a triangle and a wedge.

If a circle has radius a instead of 1, then the top half is given by $y = \sqrt{a^2 - x^2}$. If we want to integrate this, we use $x = a\sin\theta$ instead of $x = \sin\theta$:

Example. Some tough integrals can be computed using the same substitution. Compute $\int \frac{x^2}{(4-x^2)^{3/2}} dx$.

Tangents, secants Recall that

$$\tan^2\theta + 1 = \sec^2\theta \;\Rightarrow\; \sec^2\theta - 1 = \tan^2\theta.$$

So if we get something like $\sqrt{x^2+a^2}$ we will try $x = a\tan\theta$ where $\sqrt{x^2+a^2} = a\sec\theta$. If we get $\sqrt{x^2-a^2}$ we will try $x = a\sec\theta$ where $\sqrt{x^2-a^2} = a\tan\theta$.

Example. Evaluate $\int \frac{1}{\sqrt{x^2+9}} dx$.

Example. Evaluate $\int \frac{dx}{x^2\sqrt{4x^2-36}}$.

Completing the square. *Any* quadratic polynomial can be converted into a square plus some number. Then we can use trigonometric substitution (or partial fractions, as we will see).

For example, suppose we want to compute $\int \frac{dx}{(x^2-6x+11)^2}$.

If there is time: Now let's use trig subs.

6 Lecture

Outline:

1. 7.5 The Method of Partial Fractions; Part I

6.1 Partial fractions

Consider the integral

$$\int \frac{du}{u^2 - 1}.$$

We could use trig subs to do this, but that's actually more trouble than it's worth. Instead we notice that

We have used the method of **partial fractions**. There is a systematic technique we can use for any rational function. (Recall that a *rational function* is just a fraction where top and bottom are polynomials.)

Any polynomial can be factored into *linear* (or first-order) and *quadratic* (or second-order) terms. For example,

$$x^3 - 2x^2 + x = x(x - 1)^2.$$

If the polynomial appears in the denominator of a fraction, then the idea is to break up the fraction into a sum where each term has *one* of the factors in the denominator. Think about this as the reverse (or inverse operation) of finding common denominators.

6.2 Distinct linear factors

Let's try $\int \frac{4x+1}{x^2-4} dx$.

6.3 Repeated linear factors

In the last example all three factors were distinct. When there is a repeated factor, it actually contributes more than one partial fraction. For example, let's try $\int \frac{3x-9}{(x-1)(x+2)^2}\,dx$.

Remark. For repeated linear factors of degree higher than 2, write a fraction for each power, that is, you should have the same number of fractions for the repeated factor as its power.

Example. In this example we will just focus on the partial fraction decomposition rather than the integral as well so we can get a feel for how to solve something with more factors. Consider

$$\frac{1}{x(x-2)^3}.$$

7 Lecture

Outline:

1. 7.5 The Method of Partial Fractions; Part II

7.1 Irreducible quadratic factors

Any factor of the form $x^2 + a^2$ is **irreducible** in the sense that it cannot be factored into linear terms. We can deal with all irreducible quadratic factors by completing the square and either using substitution or inverse tangents.

Example. Let's try $\int \frac{18}{(x+3)(x^2+9)} dx$.

Remark. If there is a repeated irreducible factor, it again contributes more than one partial fraction.

7.2 Long division

Let's try $\int \frac{x^3+1}{x^2-4} dx$.

8 Lecture

Outline:

1. 4.5 L'Hôpital's Rule; Part I

8.1 L'Hôpital's Rule

One interesting way to compare two functions $f(x)$ and $g(x)$ around a point $x = a$ is to compute the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

This includes $a = \pm\infty$, that is the limit when x gets really large (or very negative).

Before getting to L'Hôpital's Rule, let's remember the basic techniques! *It is important not to lose these techniques, because L'Hôpital's Rule doesn't always apply!* If $\lim_{x\to a} f(x)$ exists, and if $\lim_{x\to a} g(x)$ exists and is not zero, then just plug in $x = a$:

$$\lim_{x \to 3} \frac{x^3 - 8}{x - 2} =$$

If both limits exist but the bottom is zero, then the limit might not exist, or it may be infinity. The case "0/0" is ambiguous, but sometimes cancellation is enough to find out what the limit is.

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} =$$

When you can't easily find such cancellation, there's L'Hôpital's Rule.

Theorem 1. *Suppose a is a number, that f and g are differentiable around a, and either $f(a) = g(a) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. Then if $g'(x)$ is nonzero near a (but not necessarily at $x = a$) we get*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \tag{1}$$

provided this limit exists or is $\pm\infty$.

Proof. □

You can use (1) for easy limits:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} =$$

But it's more interesting for harder ones:

$$\lim_{x \to 2} \frac{x^2 - 4}{\sin \pi x} =$$

In this case, we get a 0/0 limit, but cancellation looks pretty hopeless, so it's a good thing we have L'Hôpital's Rule.

Remark. Don't forget to check for $0/0$ or $\pm\infty/\pm\infty$! Let us review the first example,

$$\lim_{x\to 3} \frac{x^3-8}{x-2} =$$

What if we had mistakenly used L'Hôpital's Rule? We would have said

$$\lim_{x\to 3} \frac{x^3-8}{x-2} \text{“} = \text{”}$$

WRONG! If the hypothesis of the theorem does not hold then we cannot apply the theorem!

8.2 Comparing two functions asymptotically

Quick Sort and *Bubble Sort* are two algorithms for sorting a list. If the list has size n, Quick Sort requires a maximum of $n \ln n$ steps, whereas Bubble Sort requires a maximum of n^2 steps. Which is more efficient?

In other words, we want to compare two functions *asymptotically*. Both $f(x) = x \ln x$ and $g(x) = x^2$ go to infinity as x goes to infinity. But one might go faster than the other. To make this precise, let's compute

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x \ln x}{x^2}.$$

We will need to use L'Hôpital's Rule at infinity:

Theorem 2. *Suppose $f(x)$ and $g(x)$ are both differentiable for x large and that $g'(x)$ is nonzero for x large. If $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are either both zero or both infinite, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists or is infinite.

Let's apply it. Note that we can also perform some cancellation *before* applying the rule:

$$\lim_{x\to\infty} \frac{x \ln x}{x^2} =$$

Now let's interpret. As x gets very large, the ratio between $x \ln x$ and x^2 gets very small. In other words, $x \ln x$ is much smaller than x^2 whenever x is big. We sometimes write this with the notation $x \ln x \ll x^2$.

Example. Which is faster, an algorithm that takes $(\ln n)^2$ steps or one that takes $\sqrt{n}$ steps?

To answer this, we find

Once again, the logarithmic function wins.

General fact. Exponentials grow faster than any polynomial. More formally,

$$x^n \ll e^x \text{ for any number } n.$$

To see this, we can use L'Hôpital's Rule.

It shouldn't be surprising to know that $\ln x$ is much smaller than x^n for x large, whenever n is a positive number. You can also show this using L'Hôpital's Rule.

9 Lecture

Outline:

1. 4.5 L'Hôpital's Rule; Part II

9.1 More examples of L'Hôpital's Rule

Motivated by the idea of comparing two functions, we can use L'Hôpital's Rule to calculate lots of limits, not just as x gets large. Remember, we always need to check that the fraction is of the form $0/0$ or $\pm\infty/\pm\infty$ before we can use it.

Example. Find

$$\lim_{x\to\pi/2} \frac{\cos^2 x}{1-\sin x}.$$

The form $0\cdot\infty$. When you get this form, you can rewrite as a fraction to get $0/0$ or $\pm\infty/\pm\infty$ and apply L'Hôpital's Rule. For example,

$$\lim_{x\to 0+} x\ln x =$$

The form $\infty-\infty$. No, it's not always zero! For example,

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x}\right) =$$

On the other hand,

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) =$$

Using logarithms for the forms 0^0 and 1^∞. Let's find the following limits:

$$\lim_{x\to 0+} x^x, \quad \lim_{x\to 0}(1+4x)^{1/2x}.$$

The form 0^∞ in not indeterminate. If $\lim_{x\to a} f(x)^{g(x)}$ is of the form 0^∞, then $\lim_{x\to a} e^{g(x)\ln(f(x))}$ has an exponent of the form $-\infty$. Hence the limit is **always** 0 in this case!

10 Lecture

Outline:

1. 7.7 Improper Integrals; Part I

10.1 Improper integrals

We often think of integrals as areas of regions between curves. But some areas can be unbounded: they stretch out to infinity. Still, the area could be finite or infinite. We call this an **improper integral**.

If the integral is finite, we say it **converges**. Otherwise, we say it **diverges**. To find out whether it converges, we need to take the **limit of proper integrals**.

$$\int_a^\infty f(x)dx = \qquad\qquad\qquad , \int_{-\infty}^a f(x)dx = \qquad\qquad\qquad ,$$

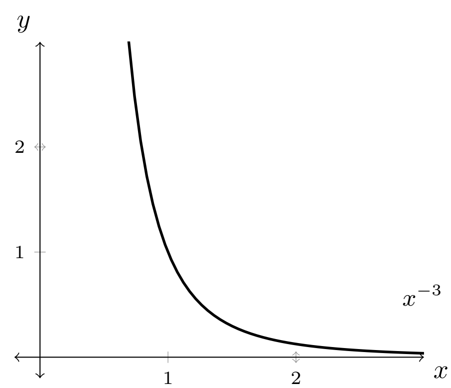
$$\int_{-\infty}^\infty f(x)dx =$$

A very important function in many areas of science that is defined through improper integrals is the **gamma function** which is define for $s > 0$ (actually for complex numbers whose real part is greater than zero) as

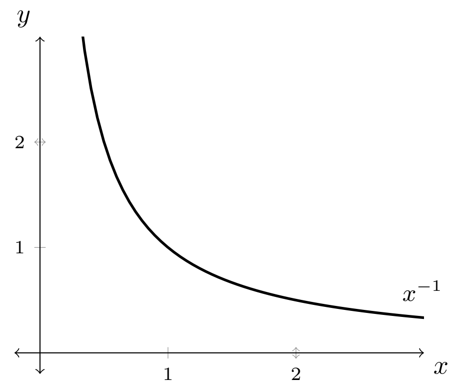
$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\ dt.$$

See the extra credit problem on Homework 4 for more about it and to see how it generalizes the factorial.

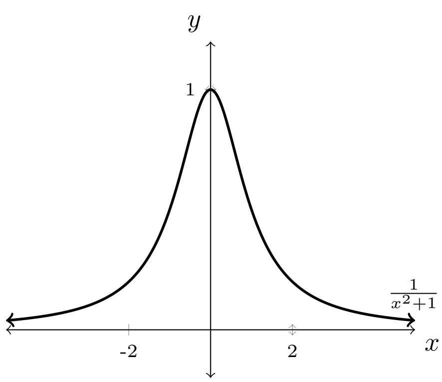
Example. Let's find $\int_2^\infty \frac{dx}{x^3}$. In other words, we're finding the area under the curve $y = x^{-3}$ over the interval $2 \leq x < \infty$. So the region is unbounded and the graph is given below.



Example. Does $\int_1^\infty \frac{dx}{x}$ converge?



Example. What about $\int_{-\infty}^\infty \frac{dx}{x^2+1}$? We need to split into two sides, positive and negative.



***p*-integrals.** Suppose p is any number and $a > 0$. Consider

$$\int_a^\infty \frac{dx}{x^p}.$$

If p is large then we expect this to converge. If p is small (or negative) we expect this to diverge. In fact, for $R > a,\ p \neq 1$ we get

If $p > 1$, then $1/R^{p-1} \to 0$ as $R \to \infty$. In this case the integral converges and we get (verify with first example)

If $p < 1$ then $1/R^{p-1} = R^{1-p} \to \infty$ as $R \to \infty$. So the integral diverges, and we write $\int_a^\infty \frac{dx}{x^p} = \infty$. If $p = 1$, we get $\int_a^R \frac{dx}{x} = \ln(R) - \ln(a) \to \infty$ as $R \to \infty$, so the integral again diverges (see second example).

11 Lecture

Outline:

1. 7.7 Improper Integrals; Part II

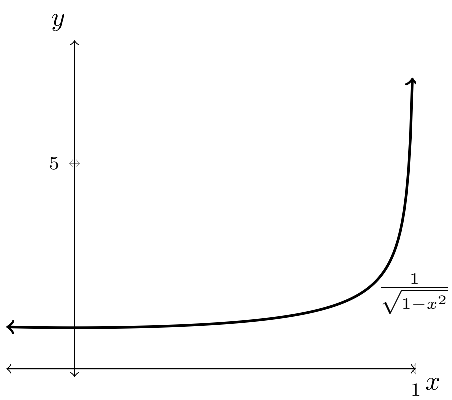
11.1 Unbounded functions

An area can be unbounded in the horizontal or vertical direction. So another kind of improper integral is one for which the function itself blows up, i.e. goes to infinity. If $f(x)$ goes to $\pm\infty$ as $x \to a$, then we define

$$\int_a^b f(x)dx = \qquad\qquad , \int_a^b f(x)dx = \qquad\qquad ,$$

and if the function blows up at both ends, we add both limits together.

Example. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.



p-**integrals.** Consider

$$\int_0^a \frac{dx}{x^p}.$$

The function $\frac{1}{x^p}$ blows up as $x \to 0$. We expect the integral to converge for *small or negative* p, the opposite of what we expected before. If $0 < R < a$ then

If $p > 1$ then $\frac{1}{R^{p-1}} \to \infty$ as $R \to 0^+$, so the integral diverges. If $p < 1$ then it goes to zero and we get

If $p = 1$ then the integral also diverges because

11.2 Comparison Test

Sometimes all we want to know is whether an integral converges or diverges; we don't care about the value. So we compare the integral to another one we know about already.

If $f(x) \geq g(x) \geq 0$, then if $\int_a^\infty f(x)dx$ converges, so does $\int_a^\infty g(x)dx$. Logically, if $\int_a^\infty g(x)dx$ diverges, so does $\int_a^\infty f(x)dx$.

Example. Does $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converge? A good comparison technique is to **make the denominator smaller to get a larger function**.

Example. Does $\int_1^\infty \frac{dx}{\sqrt{x}+e^{3x}}$ converge? Well, if we compare to $\frac{1}{\sqrt{x}}$, then we might think not, but that would be a mistake. **Exponentials go to zero super-fast**, so their integrals converge.

Example. Does $\int_0^{1/2} \frac{dx}{x^8+x^2}$ converge?

Remark. Comparisons can be difficult at first. You have to know which part of a function is bigger than the others, and where that's true! In general, we tend to care most about what happens for x small and for x large. So you need to know the difference in behavior for functions as $x \to 0$ and as $x \to \infty$. And when you're integrating, it's not enough just to know what the limits are; you have to know how much area accumulates under the curve!

Exam 1 Review

The first exam will cover techniques of integration.

Substitution [Section 5.7]

This method is used when a function and its derivative appear in the integrand. Remember to return to the original variable for the final answer!

Change of Variables Formula:

$$\int_a^b \underbrace{f(u(x))}_{f(u)} \underbrace{u'(x)\ dx}_{du} = \int_{u(a)}^{u(b)} f(u)\ du$$

Suggestions for Substitution

i) If a function is raised to a power or in a radical, i.e., $[f(x)]^2,\ [f(x)]^{-1},\ \sqrt{f(x)}$, let $u = f(x)$.

ii) If a function appears in the argument of another function, i.e., $\sin(f(x)),\ \ln(f(x)),\ e^{f(x)}$, let $u = f(x)$.

Note that i) is a particular case of ii) with the outer function being the power function, and these suggestions do not always work!

Examples: $\int 2xe^{x^2}\ dx, \quad \int \frac{x\ dx}{\sqrt{x^2+1}}, \quad \int \frac{\ln x}{x} dx, \quad \int x^2 \sin\left(x^3\right)\ dx$

Integration by parts [Section 7.1]

Extending integrals to the product rule:

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx} \to uv = \int \frac{du}{dx} v dx + u\frac{dv}{dx}dx \quad \text{or} \quad \int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx.$$

This is called **integration by parts.**

Another way to write the product rule is simply

$$d(uv) = u\ dv + v\ du \to uv = \int u\ dv + \int v\ du.$$

Strategy: pick out a u and dv in your integral, such that it is advantageous to pass the derivative from v to u in order to make integration easier.

Suggestions for Substitution If one of the functions in the integrand can be eliminated or made simpler by taking a few derivatives (or integrals), choose this to be u (or dv). Example:

$$u:\ x^n,\ \ln x,\ (\ln x)^n, \text{inverse trigonometric functions}$$

If neither function will be made simpler by derivatives or integrals (multiplication of trigonometric and exponential), you will have to use integration by parts until the original integral reappears and then solve for it.

Examples: $\int x \cos x\ dx, \quad \int x^2 e^x\ dx, \quad \int x \ln x dx, \quad \int e^x \cos x dx$

Trigonometric integrals [Section 7.2]

Many integrals involve trigonometric functions. To compute them, you will need to remember one fundamental identity (the Pythagorean identity):

$$\cos^2 x + \sin^2 x = 1.$$

Divide by $\cos^2 x$ to get

$$1 + \tan^2 x = \sec^2 x.$$

We can then use the provided Table of Trigonometric Integrals to evaluate most of these integrals. Keep in mind that if we have high powers of one or more trigonometric functions, it is most likely advantageous to use the substitution method rather than applying the trigonometric integral identities three or more times!

Examples: $\int \sin^2 x \cos^3 x \ dx, \quad \int \tan^{20} x \sec^4 x \ dx$

Trigonometric substitution [Section 7.3]

We can use the Pythagorean identity to evaluate many integrals that have terms that involve the square of our variable and a constant by introducing trigonometric functions cleverly, then returning to our original variable.

Examples: $\int \frac{dx}{\sqrt{x^2-1}}, \quad \int \frac{dx}{\sqrt{9-x^2}}, \quad \int \frac{x \ dx}{x^2+1}$

Partial fractions [Section 7.5]

This method is used to turn a rational function into simpler rational functions that we can easily integrate. In order to do this, we need to remember how to setup the different forms we might encounter.

Examples: $\frac{4x+1}{(x+2)(x-2)}, \quad \frac{3x+9}{(x-1)(x+2)^2}, \quad \frac{18}{(x+3)(x^2+9)}$

L'Hôpital's rule [Section 4.5]

Loosely speaking, L'Hôpital's rule allows us to compare two functions that approach zero or infinity at the same point.

Remark. Don't forget to check for $0/0$ or $\pm\infty/\pm\infty$!

Examples: $\lim_{x\to 0^+} x \ln x, \quad \lim_{x\to\infty} \frac{x \ln x}{x^2}, \quad \lim_{x\to 0^+} x^x$

Improper integrals [Section 7.7]

We often think of integrals as areas of regions between curves, but some areas can be unbounded with the area being finite or infinite. We call this an **improper integral**. If the integral is finite, we say it **converges**. Otherwise, we say it **diverges**. To find out whether it converges, we need to take the **limit of proper integrals**.

$$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx, \ \int_{-\infty}^a f(x)dx = \lim_{R\to-\infty} \int_R^a f(x)dx, \ \int_{-\infty}^\infty f(x)dx = \int_a^\infty f(x)dx + \int_{-\infty}^a f(x)dx$$

If $f(x)$ goes to $\pm\infty$ as $x \to a$, then we define

$$\int_a^b f(x)dx = \lim_{R\to a^+} \int_R^b f(x)dx, \ \int_a^b f(x)dx = \lim_{R\to b^-} \int_a^R f(x)dx,$$

and if the function blows up at both ends, we add both limits together.

12 Lecture

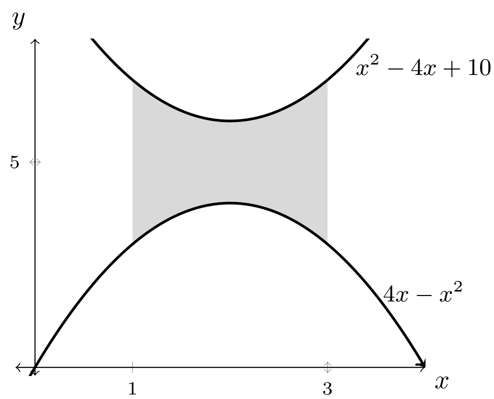
Outline:

1. 6.1 Area Between Two Curves

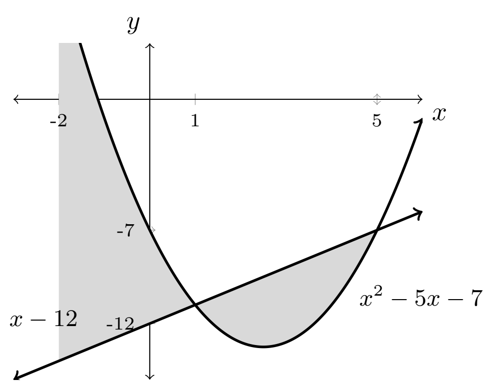
12.1 Area between two curves

If $f > g$ then the curve $y = f(x)$ is above the curve $y = g(x)$. The area between the two curves can be computed by integration.

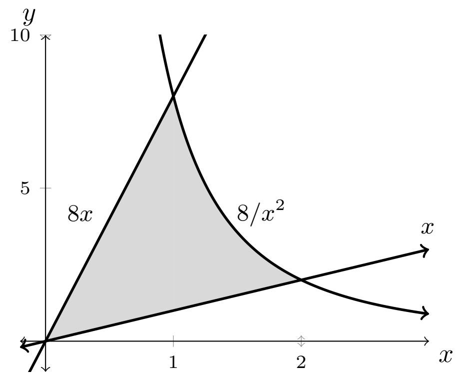
Example. Find the area of the region between the graphs of $f(x) = x^2 - 4x + 10$ and $g(x) = 4x - x^2$ over $1 \leq x \leq 3$.



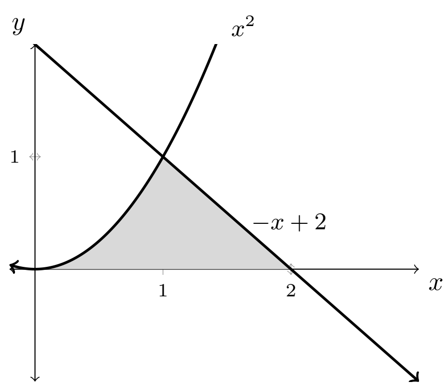
Changing sides. Let's find the area between the graphs of $f(x) = x^2 - 5x - 7$ and $g(x) = x - 12$ over $[-2, 5]$.



A region enclosed by several curves. Let's find the area enclosed by the graphs $y = 8/x^2, y = 8x$, and $y = x$.



Integrating along the y-axis. Sometimes it's advantageous to switch roles of x and y and write x as a function of y. For example, let's find the area of the region enclosed by $y = x^2, y = -x + 2$, and $y = 0$.



Remark: We could also have solved this as before by evaluating

Summary: If you are given two or more functions without a specified interval, find their intersections to determine the enclosed region and find its area.

13 Lecture

Outline:

1. 6.2 Setting Up Integrals: Volume, Density, Average Value; Part I
2. Volume

13.1 Volumes using cross sections

To compute the volume of a solid body, divide the body into N horizontal slices of thickness $\Delta y = (b-a)/N$. Then the ith slice extends from y_{i-1} to y_i, and let V_i denote the volume of this slice.

Notice if N is very large, then Δy is very small and the slices are very thin. Furthermore, the ith slice has area $A(y_{i-1})$ and height Δy, and hence, $V_i \approx A(y_{i-1})\Delta y$. Then the entire volume is obtained by summing all the volumes of the slices,

$$V = \sum_{i=1}^{N} V_i \approx \sum_{i=1}^{N} A(y_{i-1})\Delta y.$$

This sum should look familiar as it is the left-endpoint approximation to the integral $\int_a^b A(y)\ dy$, where $A(y)$ denotes the area function. Assuming A is continuous, the approximation will converge to the integral as $N \to \infty$. This allows us to conclude that **the volume of the solid is equal to the integral of its cross-sectional area**, that is,

$$\textbf{Volume of the solid body } = \int_a^b A(y)\ dy.$$

Because it might be visually easier, your book proposes using the y-axis on the graph with objects that stand up straight. But this is just a matter of orientation.

Example. Find the volume of a pyramid of height 12 whose base a square of side 4.

Example. Compute the volume of the solid whose base is the region between the parabola $y = 4 - x^2$ and the x-axis, and whose vertical (in the "z direction") cross sections are semicircles perpendicular to the y-axis.

Example. Compute the volume of a sphere of radius R.

People knew how to do this long before calculus, but thankfully we came after them!

Cavalieri's principle: If two solids have equal cross-sectional areas, then they have equal volume. (For example, just take two stacks with the same number of quarters. Even if the quarters aren't stacked perfectly one on top of the other, they have the same volume.)

14 Lecture (not covered on homework or exams)

Outline:

1. 6.2 Setting Up Integrals: Volume, Density, Average Value; Part II
2. Linear mass density
3. Flow rate
4. Average value of a function

14.1 Linear mass density and total mass

Density can vary as a continuously changing function. Some objects can be thought of as one-dimensional, for instance a rod. If $\rho(x)$ is the mass density at length x from one end of the rod, then the total mass of the rod is

$$\int_0^L \rho(x)dx$$

where L is the length of the rod.

Example. If a 2 meter rod has linear mass density $\rho(x) = 1 + x(2 - x)$ kg/m, then its total mass is

There are other one-dimensional quantities. A **population density** could depend only on distance from the city center, for example. To find the total population, we need to integrate circles:

$$\text{Population with radius } R = \int_0^R 2\pi r\rho(r)dr.$$

Example. Suppose the population density depends on the distance from the city center like $\rho(r) = 15(1 + r^2)^{-1/2}$ thousands per square kilometer. This makes sense, since the farther out you go, the fewer people there are. How many people are there between 10 and 30 km from the city center?

14.2 Flow rate for a laminar flow

The flow of a fluid through a pipe being **laminar** means the velocity of each particle depends only on its distance from the center of the tube. So just as for population density, we get

$$\text{Flow rate} = \int_0^R 2\pi r v(r)dr$$

where R is the radius of the pipe and $v(r)$ is the velocity of particles.

Example. According to Poiseuille's Law, the velocity of blood flowing in a blood vessel of radius R is $v(r) = k(R^2 - r^2)$ (where k is a constant). What is the total flow rate?

This actually shows the danger of reduced blood flow, say from plaque buildup. The flow is proportional to R^4, so if the radius, R, is reduced by say 1/2, the flow rate is reduced by a factor of 16.

14.3 Average value

The average value of N numbers is found by adding them and dividing by N:

$$\frac{a_1 + \cdots + a_N}{N}.$$

If we take N values of a function f from equally spaced intervals in $[a, b]$, and then we let $N \to \infty$, we get the average value of f on $[a, b]$:

$$\text{Average value } = \frac{1}{b-a} \int_a^b f(x)dx$$

Example. Let's find the average value of $f(x) = \sin(x)$ on the interval $[0, \pi]$. Recall that this is the interval on which sin is non-negative. We get

Remark: Note with an average of a list of numbers, the average may never be obtained in the list but it always is for a continuous function by the Mean Value Theorem.

Extra Example. A ball is thrown in the air vertically from ground level with initial velocity 18 m/s. Find the average height and average speed over the time interval extending from the ball's release to its return to ground level. (The acceleration due to gravity is 9.8 m/s^2)

Step 1: find a function for the ball's height and velocity at time t.

Step 2: find out when the ball returns to ground level.

Step 3: find the average of $h(t)$ and $v(t)$ over $[0, t_{max}]$.

15 Lecture

Outline:

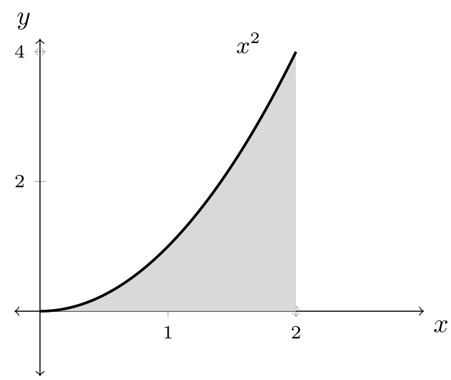
1. 6.3 Volumes of Revolution: Disks and Washers

15.1 Volumes of revolution using disks and washers

If $f(x) \geq 0$ and we take the area under the graph above the x-axis, we can revolve it around the x-axis to get interesting symmetrical solids. The volume can be found by integrating cross sections (which are **disks**) with area $A(x) = \pi r^2 = \pi f^2(x)$, since the radius of each cross section is $f(x)$. Hence,

$$V = \pi \int_a^b f^2(x)dx.$$

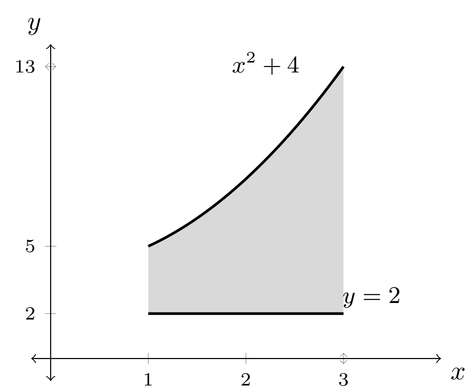
Example. Revolve the area under the curve $y = x^2$ over $0 \leq x \leq 2$ about the x-axis to get a shape similar to the opening of a musical horn.



What is the volume?

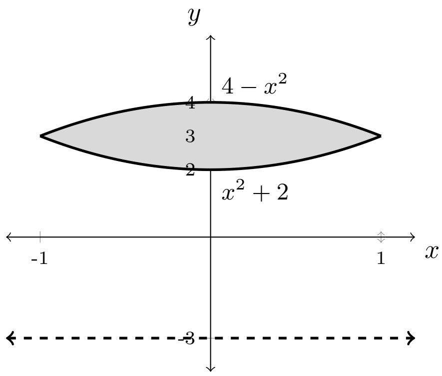
We can do the same thing with areas between two curves $y = f(x)$ and $y = g(x)$ which are both above the x-axis, but then the cross sections are **washers**–there's a hole in the middle. We need to delete the hole from the area: $A(x) = \pi f^2(x) - \pi g^2(x)$.

Example. We make a lamp shade by revolving the area between $y = x^2 + 4$ and $y = 2$ over $1 \leq x \leq 3$ around the x-axis.

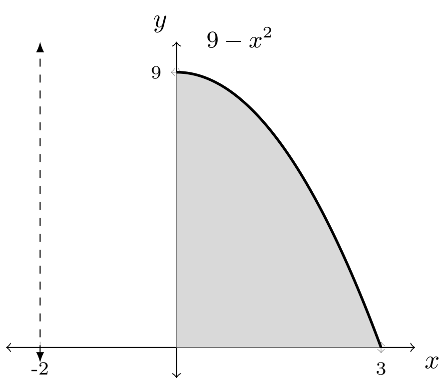


The volume we get is

Example. We make a wedding band by revolving the area enclosed by $y = 4 - x^2$ and $y = x^2 + 2$ around the horizontal line $y = -3$. To find the limits of integration, and which curve goes on top, we compare:



Example. If we want to revolve around a vertical axis instead, we can still use washers or disks, but we have to find x as a function of y. For instance, take the region between the graph $y = 9 - x^2$ and the x-axis over $0 \leq x \leq 3$ and revolve it around the axis $x = -2$.



16 Lecture

Outline:

1. 6.4 Volumes of Revolution: Cylindrical Shells

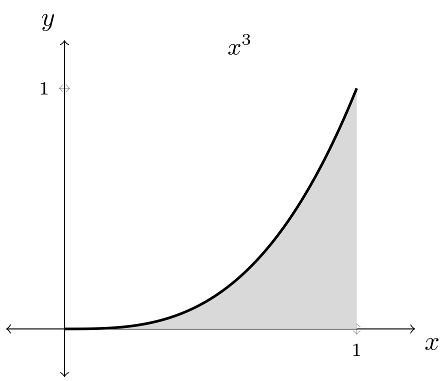
16.1 Volumes of revolution using cylindrical shells

The method of cylindrical shells is different from earlier methods, because we don't look at cross sections perpendicular to a central axis. **Instead, we look at shells whose height is parallel to the central axis.** This is often useful if we take the region under a graph $y = f(x)$ and revolve it around the y-axis. The result is that we consider the surface areas of the different shells and add them up.

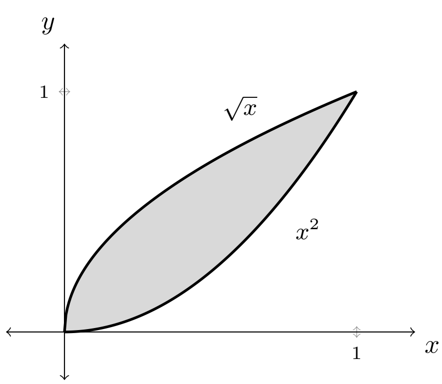
Notice that each of the surface areas can be found by "cutting" and flattening them to see they are simply rectangles with width given by the circumference of the cylinder considered, $2\pi x$, and height given by the function, $f(x)$. This gives

$$V = 2\pi \int_a^b (\text{radius})(\text{height of shell})dx = 2\pi \int_a^b x f(x)\ dx.$$

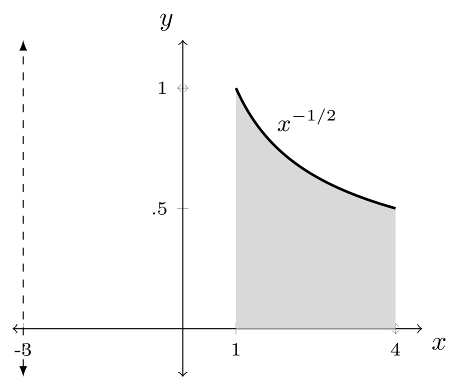
Example. Let's take the region under the graph $y = f(x) = x^3$ (and over the x-axis) between $0 \leq x \leq 1$ and revolve it around the y-axis. We'll get a sort of bowl with square corners.



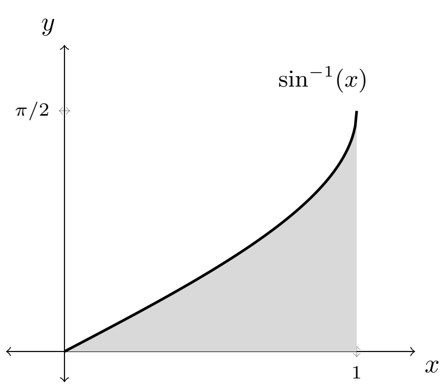
Example. Take the region enclosed by $y = \sqrt{x}$ and $y = x^2$ and revolve it around the y-axis. First, we need to know what this region looks like.



Example. This time we'll revolve around a *different* vertical axis, say $x = -3$. Take the region under the graph $y = x^{-1/2}$ over the interval $1 \leq x \leq 4$.



Example. For our last example, we'll go back to revolving **around the x-axis**. But we'll see that using the normal disk method will be very tricky, and the shell method will help. We take the region between the graph $y = \sin^{-1}(x)$ and the x-axis over $0 \leq x \leq 1$ and revolve it around the x-axis.



17 Lecture

Outline:

1. 8.2 Arc Length and Surface Area

17.1 A Short Warm Up

Let's motivate the topics of arc length and surface area using geometric shapes we know first. Consider a circle with radius r.

If the derivative of the area of a circle gave circumference, what about the derivative of the volume of a sphere?

Remark: We need to be careful with other shapes as this is a result of changing the same in all directions, that is, we need a notion similar to radius. For instance, to consider a square and a cube, we need to label each side as $2x$ so that we change the same when fixing the center. Then we find $A_s = (2x)^2 = 4x^2$, $A_s' = 8x$ = perimeter = arc length of the square, and $V_c = (2x)^3 = 8x^3$, $V_c' = 24x^2 = 6 \cdot 4x^2 = 6 \cdot A_s$ = surface area of the cube.

17.2 Arc length

Suppose we want to find the length of a curve given by $y = f(x)$, with starting point $x = a$ and end point $x = b$.

Example. Let's say I throw a ball into the air at an initial speed of 9.8 m/s vertically at 1 m/s horizontally. Using some physics, I can determine that the path of the ball is given by $y = 9.8x - 4.9x^2$ on the interval $0 \leq x \leq 2$, where x is the horizontal distance traveled by the ball. What is the total distance the ball travels through the air?

Conceptual understanding. Consider the two arcs $y = \sqrt{x}$ and $y = x^2$ over the same interval $0 \leq x \leq 1$. Note that on this interval we can write $y = \sqrt{x}$ as $x = y^2$ with $0 \leq y \leq 1$. In other words, the two arcs are just mirror images of each other. They should have the same arc length.

What does the formula say?

17.3 Area of a surface of revolution

Let's say we take a curve $y = f(x)$ over an interval $[a, b]$ like before, only this time we revolve it around the x-axis to get a *surface*. What is its area? Again, we break up the surface into pieces. Each piece is a ring whose width can be approximated by $\sqrt{1 + (f'(x))^2}\Delta x$ and radius $f(x)$. Therefore the area is the circumference times the width, which is about $2\pi f(x)\sqrt{1 + (f'(x))^2}\Delta x$. We integrate this to get the total surface area:

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx.$$

Example. What is the surface area of the paraboloid obtained by revolving the graph of $y = \sqrt{x}$ around the x-axis over $0 \leq x \leq 1$?

Example. What is the surface area of a sphere of radius R?

By simply changing the limits of integration, we get a simple formula for the area of any spherical cap of height h. Indeed, if we integrate not from $-R$ to R but instead from $R - h$ to R, the area we get is $A = 2\pi Rh$. Which is kind of odd, when you think about it!

What it is really saying is the surface area is increasing with respect to the height by the circumference.

18 Lecture

Outline:

1. 6.5 Work and Energy

Great quote from the textbook:

For those who want some proof that physicists are human, the proof is in the idiocy of all the different units which they use for measuring energy. –Richard Feynman, *The Character of Physical Law*

18.1 Work and Energy

A *constant* force F that pushes an object a straight distance d is said to perform a certain amount of **work** given by

$$W = F \cdot d.$$

Work is equivalent to **energy**; we can think of it as the energy expended by the trip.

In the International System (SI), the units for work are Newton meters ($\mathrm{N} \cdot \mathrm{m}$), also known as Joules (J). Recall that one *Newton* is the weight of one kilogram: $1\mathrm{N} = 1\mathrm{kg} \times \mathrm{m/s}^2$. In the British system, the units are simply foot pounds, because pounds are already a measure of weight, hence of force.

But if the force you apply changes at each point along the way, how do we compute the total work done? Let's say the path lies on the x-axis and the force is given by $F(x)$. Loosely speaking, we divide up the interval into little pieces of length dx and add up to get

$$W = \int_a^b F(x)dx$$

where a is the starting point and b is the end.

Hooke's Law. If you stretch a spring out a distance x from its natural resting position, it will exert a *restoring force* equal to $-kx$, where $k > 0$ is a constant depending on the spring. Let's say, for example, $k = 200N/m$. What is the amount of work required to stretch the spring a distance of 5 cm from rest?

Emptying out a tank. A spherical tank of radius 5 m is filled with water. Calculate the work performed in pumping out the water through a spout of height 1 m at the top.

Note: the density of water is 1 g/cm^3 = 1000 kg/m^3.

Building a pyramid. The Great Pyramid of Giza in Egypt is 146 m high and has a square base of side 230 m. The density of the stone is estimated at 2000 kg/m^3. Find the work required to build the pyramid.

19 Lecture

Outline:

1. 9.1 Differential Equations, Part I

19.1 Differential Equations

A differential equation is any equation involving both a function and its derivatives. (We are only dealing with *ordinary* differential equations. A *partial* differential equation involves a function and its partial derivatives in several variables.) For example:

$$\frac{dy}{dx} = y,\ \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y^3 = 0,\ \frac{d^3y}{dx^3} = 0,\ \text{etc.}$$

Recall that this notation, the Leibniz notation, is used to emphasize the rates of change considered and will be paramount in solving differential equations. A *solution* to a differential equation is any *function* that makes it true. For instance, $y = e^x$ is a solution of $\frac{dy}{dx} = y$.

If *all* of the solutions to a differential equation are known, and we can write them down by a formula, we call this the **general solution**.

A differential equation is **linear** if it can be written like

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b(x),$$

where $a_0(x), a_1(x), \ldots, a_n(x), b(x)$ are functions of the independent variable x and $y, y', y'', \ldots, y^{(n)}$ are the unknown function y and its derivatives. This is analogous to a linear equation in $n+1$ unknowns. Most differential equations are **nonlinear**.

The **order** of a differential equation is the number of higher derivatives appearing in it. So the order of $y'' + y' - 2y = 0$ is 2, while the order of $y' = y$ is 1. Most differential equations that are useful in applications tend to be just first or second order, but not always.

Separable equations

We already know how to solve the simplest kind of differential equation, $y' = f(x)$. A slightly less simple kind looks like

$$y' = f(x)g(y).$$

This kind is called **separable,** because we can **separate variables**. To do this, write y' as dy/dx. Then move everything with y in it to the left side, and everything with x in it to the right side. We get

The next step is to integrate. If we're lucky, we'll be able to find a formula for y that looks something like

In this case we've found the **general solution** to the separable equation.

Example. Let's find the general solution of $y\frac{dy}{dx} = x$.

Remark: One needs to be very careful when solving a differential equation where either the logarithmic or exponential equations appear at any step of the work as is seen in the next two examples.

Example. Let's find the general solution of $y' = e^{x+y}$. In fact this is separable because $e^{x+y} = e^x e^y$.

Initial value problems

For a first-order differential equation, there is usually at most one solution if we also specify an **initial value**, that is, if we specify that the solution must have some particular value at some particular value of x (or time, t). In terms of physical applications, this usually means that we specify a starting point and then let the dynamics tell us the **trajectory** from that initial value.

Example. Solve the **initial value problem** $y' = -ty, y(0) = 3$. Note that the independent variable is t instead of x.

20 Lecture

Outline:

1. 9.1 Differential Equations, Part II (A variety of applications)

20.1 Exponential growth and decay

Differential equations are often used to **model** real situations. This means a physical quantity is supposed to solve a differential equation. A very typical model is

$$\frac{dy}{dt} = k(y - b).$$

We can solve this by separating variables.

$$\frac{dy}{y-b} = k\ dt \ \Rightarrow\ \ln(|y-b|) = kt + C\ \Rightarrow y = Ae^{kt} + b.$$

Note that A can also be zero. In this case we get the **equilibrium solution** $y = b$.

This model describes exponential growth or decay.

- $k > 0$ implies exponential growth; y goes to ∞ or $-\infty$ in the long run, depending on the sign of A.
- $k < 0$ implies exponential decay; y goes to b in the long run.

20.2 Short Term Population growth

We can use exponential growth to model, at least for an initial growth period, population growth as in the following example.

Example. In the laboratory, the *Escherichia coli* bacteria grows such that the rate of change of the population is proportional to the population present. Assume that 1000 bacteria are initially present, and 1500 are present after 1 hour. Determine the population, $P(t)$, after t hours.

20.3 Half-Life and Doubling Time

Often it is beneficial to look at how long it will take for half of something to decay, or for it to double, which is what we mean by half-life and doubling time respectively. One can find that from $P(t) = P_0e^{kt}$, if $k < 0$, **half-life** is given by $\ln(0.5)/k$ and if $k > 0$, the **doubling time** is given by $\ln(2)/k$. This is straightforward enough to check:

$$P(\ln(0.5)/k) =$$

and

$$P(\ln(2)/k) =$$

Example. A patient is administered 400 mg of penicillin, and after 50 minutes, 300 mg remain in her bloodstream. Let $A(t)$ represent the amount of penicillin (in mg) in her bloodstream t minutes after the drug was administered. Assume that the drug is leaving her bloodstream at a rate proportional to the amount in her bloodstream. Determine $A(t)$ and the half-life of the resulting exponential decay.

20.4 Newton's law of cooling

Let y be the temperature of an object in an *ambient temperature* T_0. Then there is a *cooling constant* $k > 0$ such that $y' = -k(y - T_0)$, so that the object's temperature will tend toward T_0 in the long run. The units on k are inverse time, e.g. $(\text{min})^{-1}$.

Example. A metal bar has cooling constant $k = 2.1 \text{ min}^{-1}$. Suppose the bar is submerged in water held at 10 degrees Celsius. Then the temperature of the bar $y(t)$ satisfies $y' = -2.1(y - 10)$. So $y(t) = 10 + Ae^{-2.1t}$. Here the *initial temperature* is $y(0) = 10 + A$, so $A = y(0) - 10$.

Suppose we ask the opposite problem. After 30 seconds, we know the bar has cooled to 80 degrees. How warm was it to begin with?

20.5 Free fall with air resistance (terminal velocity)

Suppose you are skydiving. Your mass is m, so the size of the force of gravity downward is mg. But thankfully air also resists proportionally to your speed, that is, if v is your velocity there's a constant $k > 0$ (measured in units of mass/time, e.g. kg/s) such that the force of air resistance against you is $-kv$. If positive v means you're going up and negative v means you're going down, the total force is

$$F = -mg - kv.$$

But $F = ma = mv'$, so $mv' = -mg - kv$ or

$$v' = -\frac{k}{m}v - g = -\frac{k}{m}(v + \frac{mg}{k}).$$

The general solution is $v(t) = -\frac{mg}{k} + Ae^{-tk/m}$. In the long run, your velocity will be $-\frac{mg}{k}$ rather than increase indefinitely; this is called **terminal velocity**. For more on this, see Homework 8, Exercise 66.

Example. Let's say your mass is 80 kg and the air resistance constant is $k = 8$ kg/s.

20.6 Annuities

An annuity is an investment which gains interest while money is also withdrawn. The initial investment P_0 is called the **principal**. After time t the investment has grown (or shrunk) to $P(t)$, satisfying the differential equation

$$P' = rP - N = r\left(P - \frac{N}{r}\right),$$

where r is the interest rate (usually given as a percentage per unit time) and N is the withdrawal rate (given as an amount of money per unit time). The general solution is $P(t) = \frac{N}{r} + Ae^{rt}$. This suggests exponential growth, but everything depends on the sign of A. If $A > 0$, we get endless growth, but if $A < 0$, we get total disaster!

To see whether $A > 0$ or $A < 0$, we compare $\frac{N}{r}$ to the principal. Indeed, $P_0 = P(0) = \frac{N}{r} + A$, so $A = P_0 - \frac{N}{r}$. If $P_0 > \frac{N}{r}$, then the investment grows forever. If not, we will quickly lose all our money! In other words, do not withdraw too quickly. The rule we must follow is $N < P_0 r$, meaning that the principal times the interest rate determines how much we can withdraw.

21 Lecture

Outline:

1. 9.4 The Logistic Equation

21.1 The logistic equation

In the last section we saw differential equations that had solutions being exponential functions. These differential equations were used to model population growth of bacteria, but we noted this was only valid for short time periods as a population cannot grow indefinitely. That brings us to the idea of changing the formulation of the problem considered previously in order to model both short and long term growth.

Let $k > 0$ be a growth constant, and $A > 0$ a **carrying capacity** (or saturation constant). If $y(t)$ represents the population at time t, we let $A - y(t)$ represent the room available for growth. Now we assume that the rate of change of y is proportional to the amount present, $y(t)$, and the amount of room for growth, $A - y(t)$. Translating this into a differential equation, we obtain the **logistic differential equation**,

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right) = \frac{k}{A}y(A - y).$$

For small y, y grows approximately exponentially. The logistics equation has two constant solutions: the **equilibrium** or **steady state** solutions $y = 0$ and $y = A$.

21.2 Solving the logistic equation

The logistic equation is separable (assuming $y \neq 0,\ A$) so that we can divide:

Note: This is different from the solution given in the book:

$$y = \frac{A}{1 - e^{-kt}/B}.$$

However, the two are really not different if $C \neq 0$, since we can set $B = -C$ and get the same formula.

21.3 Stable and unstable equilibrium

We can see that the solution has a limit as $t \to \infty$. In fact,

$$\lim_{t\to\infty} \frac{AC}{C + e^{-kt}} = \frac{AC}{C} = A.$$

So as long as $y_0 \neq 0$, we know that y goes to A in the long run. This is why we call A a **stable equilibrium**. On the other hand, 0 is an **unstable equilibrium** because any solution that begins at any other point, no matter how close to 0, will quickly move away from 0.

This immediately implies that if the solution is **ever equal** to 0 or A, then the solution is the constant solution $y = 0$ or $y = A$ (i.e. it is always 0 or A).

21.4 Example: deer population

A deer population grows logistically with growth constant $k = 0.4$/year and carrying capacity $A = 1000$ deer. Starting at 100 deer, what is the population after t years? How long does it take to grow to 500 deer?

Exam 2 Review

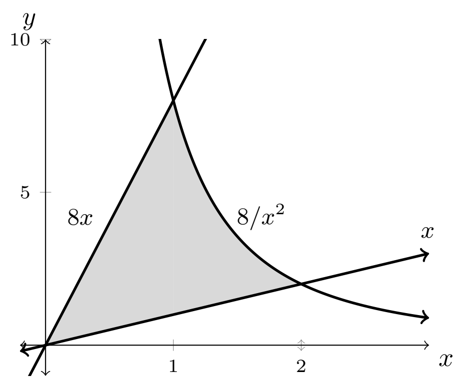
Do not forget that integration techniques from the first test can still show up on this test!

The second exam will cover applications of integration:

Area Between Curves [Section 6.1]

If $f > g$ then the curve $y = f(x)$ is above the curve $y = g(x)$. The area between the two curves can be computed by integration. We can find the area enclosed by more curves by breaking the region into sections where we only consider two curves at a time as seen in the following example.

A region enclosed by several curves. Let's find the area enclosed by the graphs $y = 8/x^2, y = 8x$, and $y = x$.



This is basically going to look like a wedge in the first quadrant. Note that for $x < 0$, the curves don't intersect. The three vertices of this wedge are the points of intersection of the curves: $x = 0$, where $y = 8x$ and $y = x$ meet, and as for the other two:

$$8x = 8/x^2 \Rightarrow x^3 = 1 \Rightarrow x = 1,$$

$$x = 8/x^2 \Rightarrow x^3 = 8 \Rightarrow x = 2.$$

We need to divide up the region between $0 \le x \le 1$ and $1 < x \le 2$. In the first region, the top side is $y = 8x$ and the bottom is $y = x$. In the second, the top is $y = 8/x^2$ and the bottom is $y = x$. So the total area is

$$\int_0^1 7x dx + \int_1^2 \left(\frac{8}{x^2} - x\right) dx = \frac{7}{2} + \left[-\frac{8}{x} - \frac{x^2}{2}\right]_1^2 = \frac{7}{2} + [-4-2] - \left[-8 - \frac{1}{2}\right] = 6.$$

Volume [Section 6.2]

$$\textbf{Volume of the solid body } = \int_a^b A(y)\, dy.$$

This method is used to find the volume of a given shape (pyramid, sphere, cone, etc.) where cross sections can be easily understood and have area $A(y)$. Remember that often **similar triangles** are needed to find a formula for the area.

Disk and Washer Method [Section 6.3]

If $f(x) \ge 0$, we can revolve the area under the curve around the x-axis to get symmetrical solids. The volume can be found by integrating cross sections (which are **disks**) with area $A(x) = \pi r^2 = \pi f^2(x)$, since the radius of each cross section is $f(x)$. Hence,

$$V = \pi \int_a^b f^2(x) dx.$$

Shell Method [Section 6.4]

The method of cylindrical shells is different from earlier methods, because **we look at shells whose height is parallel to the central axis.** The result is that we consider the surface areas of the different shells and add them up. Notice that each of the surface areas can be found by "cutting" and flattening them to see they are simply rectangles with width given by the circumference of the cylinder considered, $2\pi x$, and height given by the function, $f(x)$. This gives

$$V = 2\pi \int_a^b (\text{radius})(\text{height of shell})dx = 2\pi \int_a^b xf(x)\ dx.$$

Arc Length and Surface Area [Section 8.2]

The length of a curve–its **arc length**–is given by the integral

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

The surface area of a surface produced by revolving a curve around the x-axis is given by

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx.$$

Work and Energy [Section 6.5]

A *constant* force F that pushes an object a straight distance d is said to perform a certain amount of **work** given by

$$W = F \cdot d.$$

Units for work are Newton meters ($\text{N} \cdot \text{m}$), also known as Joules (J). Recall that one *Newton* is the weight of one kilogram: $1\text{N} = 1\text{kg} \times \text{m/s}^2$.

How do we compute the total work done? Let's say the path lies on the x-axis and the force is given by $F(x)$. Loosely speaking, we divide up the interval into little pieces of length dx and add up to get

$$W = \int_a^b F(x)dx$$

where a is the starting point and b is the end.

You may be asked to set up, but not solve, problems involving springs and emptying tanks.

Emptying out a tank. A good way to remember how to approach this problem is to think of the integrand as the force required to move a volume of the water, which is how much water there is (volume) times the density of the water (given to you) times the force we are fighting (gravity), multiplied by how far we have to move the water out of the tank and pipe above the tank.

Differential Equations [Section 9.1]

Find general solution of a differential equation using the separation of variables technique.

Examples: $y\dfrac{dy}{dx} = x,\ \dfrac{dy}{dx} = e^{x+y}$

Solve initial value problems by applying initial condition to find the particular solution.

Example. Solve the initial value problem $y' = -ty,\ y(0) = 3$.

First, we find the general solution $y = Ae^{-t^2/2}$, where A can be any constant.

Next, we take into account the initial value to deduce $y = 3e^{-t^2/2}$ is our particular solution.

22 Lecture

Outline:

1. 10.1 Sequences

22.1 Sequences

A sequence, often denoted $\{a_n\}$, is an infinite set of numbers placed in order:

$$a_1, a_2, a_3, \ldots$$

For example, the Fibonacci sequence:

The nth term in the sequence is often written a_n. Sometimes it has a formula.

Or sometimes you can write a_n in terms of other members of the sequence. For the Fibonacci sequence, we can write $a_n =$

As sequences are infinite, the interesting question is whether they approach a certain **limit**.

We say $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$ there exists N large enough such that $|a_n - L| \leq \epsilon$ for every $n \geq N$. In other words, a_n gets closer and closer to L in a way that can be quantified. If no limit exists, we say that $\{a_n\}$ **diverges**.

Example. Consider the sequence

$$\frac{5}{2}, \frac{6}{3}, \frac{7}{4}, \frac{8}{5}, \frac{9}{6}, \frac{10}{7}, \ldots$$

Let's try to understand what this means: If you give me a value for ϵ, say .1, I will choose

The following is one of the most useful tools we have to quickly find the limit of a sequence.

Using what you know about functions. If we can find a formula for a_n, then finding the limit $\lim\limits_{n\to\infty} a_n$ is the same as finding $\lim\limits_{x\to\infty} f(x)$ for some function f. For example, find $\lim\limits_{n\to\infty} \dfrac{n + \ln n}{n^2}$.

Application. You may have learned that $1/9 = .1111111\ldots$. What this means is that $1/9$ is the **limit** of a sequence that converges. Consider the sequence

$$.1,\ .11,\ .111,\ .1111,\ \ldots \text{ or } \frac{1}{10}, \frac{11}{100}, \frac{111}{1000}, \frac{1111}{10000}, \ldots$$

Geometric sequences. If $r \geq 0$, and $c > 0$, we call $a_n = cr^n$ a **geometric sequence**. It is exactly like an exponential function, except it is a sequence.

$$\lim_{n\to\infty} cr^n = \Bigg\{$$

For example, $\dfrac{1}{2^n} \to$, $2(1)^n \to$, and $\left(\dfrac{\pi}{3}\right)^n$.

Squeeze Theorem If $b_n \leq a_n \leq c_n$ and b_n and c_n both have the same limit, then a_n has the same limit.

For example, intuitively, one would think that the sequence defined by $a_n = \left(-\dfrac{1}{d}\right)^n$, $d > 1$ should converge to 0, but how would you show it?

Bounded and monotonic sequences. To say a sequence is **bounded above** or **below** just means it never gets larger (or smaller) than a certain number. A **monotonic** sequence is one that either never decreases or never increases.

For example, $a_n = \frac{1}{n}$ is bounded and monotonic.

Every bounded monotonic sequence converges.

Example. Prove that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded below.

23 Lecture

Outline:

1. 10.2 Summing an Infinite Series; Part I

23.1 Series

A series is simply an infinite sum: $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$. You can think of them as having been obtained by adding up all of the terms in a sequence of numbers. For example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is a series. We could write it as

The question then becomes how do we make sense of adding infinitely many terms? The idea is to examine finite sums of terms at the beginning of the series and see how they behave. A series is said to **converge** to a number L if for any $\epsilon > 0$, there exists an N such that if $n \geq N$ then $\left|\sum_{k=1}^{n} a_k - L\right| \leq \epsilon$.

Note that whether or not the series converges might depend on how we label the terms! For instance, take $1 - 1 + 1 - 1 + \ldots$.

We can think of a series in a couple of different way: either as an ***improper integral*** or as a ***sequence of partial sums***.

23.2 Telescoping series

To illustrate how to use partial sums, consider the series

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

When we compute a few **partial sums**, S_n, we notice a pattern:

We can use partial fractions to investigate. We discover

This means the terms **telescope**, or, all the middle terms cancel leaving only the first and last! We have for the nth partial sum,

23.3 Linearity of Infinite Series

The following theorem shows that infinite series may be added or subtracted like ordinary sums, **provided that the series converge**.

Theorem (Linearity of Infinite Series) If $\sum a_k$ and $\sum b_k$ both converge, then $\sum(a_k \pm b_k)$ and $\sum ca_k$ converge, where c is any constant. Furthermore

$$\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k,$$
$$\sum ca_k = c\sum a_k.$$

This is really helpful as it allows us to possibly rewrite a sum considered into sums that we already understand.

Remark. The fact that the individual series must converge is reminiscent of linearity of limits of functions. Recall that we can only consider limits individually if they each exist, otherwise we have to consider the combination as a whole only!

23.4 Zeno of Elea's paradox: Achilles and the tortoise

One of the first infinite series to be studied is called the geometric series (which we will discuss in detail next lecture). It goes back to Zeno of Elea (c. 490–430 BC) and his paradox involving a race between Achilles and a tortoise.

Achilles is in a footrace with the tortoise. Since Achilles is the fastest runner, the tortoise is allowed a head start of some distance, say 1 kilometer. Suppose that each racer will run at some constant speed, with Achilles being twice as fast. After some finite time, Achilles will have run 1 km, bringing him to the tortoise's starting point. During this time, the tortoise has run a much shorter distance of .5 km. It will then take Achilles some further time to run that distance of .5 km, by which time the tortoise will have advanced .25 km farther; and then more time still to reach this third point, while the tortoise moves ahead. Thus, whenever Achilles arrives somewhere the tortoise has been, he still has some distance to go before he can even reach the tortoise. After n iterations of this, Achilles has run

where as the tortoise has run

Taking the difference of these two finite sequences, one easily concludes that the tortoise will be $\frac{1}{2^n}$ km ahead of Achilles after n iterations. But then how can Achilles ever catch the tortoise (which we know should happen...)? Mathematically speaking, this argument is considering the infinite geometric sum (which is the same for both)

$$1 + \frac{1}{2} + \frac{1}{4} + \dots$$

We will see that this sum adds to the finite number 2, so that both Achilles and the tortoise will be at the 2 km mark at the same time! Furthermore, we know this will happen in a finite amount of time, so there is no conundrum after all. The idea that Zeno was grappling with was the idea of the limits of the partial sums, and understanding that both series are the same, something that would take until the 19th century to fully understand.

24 Lecture

Outline:

1. 10.2 Summing an Infinite Series; Part II

24.1 Geometric series

A geometric *series* has the form $\sum_{k=0}^{\infty} cr^k$ for some $r, c \neq 0$. (It is **not** a geometric *sequence*.) To find the sum, we use a standard (and important) trick. We want to find the partial sum

If we multiply $1 - r$ we get a telescoping sum:

Therefore, as long as $r \neq 1$, we get

Of course if $r = 1$ then we get

Now we let $n \to \infty$. If $|r| < 1$ we get

If $|r| > 1$ we get the same limit, but it *diverges* (goes to infinity) since the numerator would grow without bound. If $r = \pm 1$ the series also goes to infinity since it would grow like n.

If you start further out than at $k = 0$, then we can adjust by factoring:

Example. Suppose I agree to pay you a dollar today, fifty cents tomorrow, and every day for the rest of eternity I pay you half of what I paid you the day before. How much money will you earn?

Application: repeating decimals. Let us once again consider $1/9 = .111111...$ What does this repeating decimal mean as a sum?

Cantor's disappearing table. Imagine a table 2 meters long. Let's remove a quarter of it from the middle. Each of the two remaining pieces is now less than 1 meter long (half the original). Now from each piece let's remove 1/16th, but remember there are two pieces so we really remove 1/8th of the remaining total length! Each of the remaining four pieces is now less than 1/2 meter long (one fourth the original). Keep going in like manner:

Step	Number of Pieces left	Size of each piece $\leq$	How much taken out
1	1	2	1/4
2	2	1	1/8
3	4	1/2	1/16
4	8	1/4	1/32
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^{n-1}	$2(1/2)^{n-1}$	$1/2^n$.

By the end, the size of each piece goes to zero, since $2(1/2)^{n-1} \to 0$. But how much of the table was taken out in total? For that we find the *sum:*

24.2 The nth term divergence test

Suppose $\sum_{k=1}^{\infty} a_k$ converges to L. Look at the nth term a_n. Then

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k \to L - L = 0.$$

So the terms must go to zero. Therefore, if a_n *doesn't* go to zero, it follows that the sum diverges.

In some cases this is really obvious. For example, the geometric series $\sum_{k=0}^{\infty} 2^k$ diverges because all the terms are bigger than 1, so summing them all up clearly gives infinity. In other cases it's somewhat less obvious.

Example. Consider whether the following series converges or diverges:

$$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k}{k+1}$$

The nth term divergence test is not very powerful. Consider the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$.

25 Lecture

Outline:

1. 10.3 Convergence of Series with Positive Terms; Part I

25.1 Series with positive terms

If $a_1, a_2, a_3, \ldots$ are all positive, then there are only two possibilities for their sum:

1. $\sum_{k=1}^{n} a_k$ is bounded above; in this case $\sum_{k=1}^{\infty} a_k$ converges.
2. $\sum_{k=1}^{n} a_k$ is unbounded; in this case the sum is infinity.

Note that if the series converges, then $a_n \to 0$.

We are going to learn some different kinds of comparison tests to see whether a series converges.

25.2 Integral test

We can think of a series as an improper integral, so the comparison principle applies:

The requirement that f be decreasing is only so that we know $a_n \geq f(x) \geq a_{n+1}$ for $n \leq x \leq n+1$.

Harmonic series.

***p*-series.** The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

Example. Show that $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.

25.3 Direct comparison test

Recalling how we compared improper integrals to see if they converged, we see the same applies for series:

If $0 \le a_n \le b_n$ for large n and $\sum_{k=1}^{\infty} b_k$ converges, then so does $\sum_{k=1}^{\infty} a_k$. Therefore the contrapositive is also true: if $\sum_{k=1}^{\infty} a_k$ diverges, then so does $\sum_{k=1}^{\infty} b_k$.

Example. Show that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}3^k}$ converges.

Example. Show that $\sum_{k=1}^{\infty} \frac{1}{(k^2+1)^{1/3}}$ diverges.

Change of variables. The trick I just used is often quite useful: you can start the series from any place you want, and it may help you in your calculations. For example, show that $\sum_{k=1}^{\infty} \frac{1}{(k+1)^{2/3}} = \sum_{j=2}^{\infty} \frac{1}{j^{2/3}}$.

However, the only kind of change of variables that will work is $j = k + m$, where m is some integer. (It's not like integration...)

26 Lecture

Outline:

1. 10.3 Convergence of Series with Positive Terms; Part II

26.1 Limit comparison test

Sometimes we don't have $0 \le a_n \le b_n$, but b_n is "really close" to a_n, and we want to be able to conclude something about $\sum_{k=1}^{\infty} a_k$ because we know $\sum_{k=1}^{\infty} b_k$. For that we have the following:

Theorem 3. *Suppose* $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

1. *If* $0 < L < \infty$, *then* $\sum_{k=1}^{\infty} a_k$ *converges if and only if* $\sum_{k=1}^{\infty} b_k$ *converges.*
2. *If* $L = 0$ *and* $\sum_{k=1}^{\infty} b_k$ *converges, then so does* $\sum_{k=1}^{\infty} a_k$.
3. *If* $L = \infty$ *and* $\sum_{k=1}^{\infty} a_k$ *converges, then so does* $\sum_{k=1}^{\infty} b_k$.

Example. Show that $\sum_{k=2}^{\infty} \frac{k^2}{k^4-k-1}$ converges.

Example. Show that $\sum_{k=3}^{\infty} \frac{1}{\sqrt{k^2+4}}$ diverges.

Making sense of limit comparisons. When you use this test, you need to look for *what's important* in a sequence of terms. For example, when we look at

$$\frac{n^2}{n^4-n-1},$$

we need to see that what's important is

This gives us the terms we want to compare.

The key is to know which terms *dominate* as n gets large, just like we did when comparing functions asymptotically. Here's a hierarchy starting with the fastest growing:

1.
2.
3.
4.
5.

The strategy: look at the dominant term on the top and bottom of your fraction (if the term is a fraction, which it probably is). Then just forget the rest.

Example. Consider

$$\frac{e^{2n} + n + \ln n + e^n}{n^2 - e^n}.$$

27 Lecture

Outline:

1. 10.4 Absolute and Conditional Convergence

27.1 Absolute versus conditional convergence

If $\sum_{k=1}^{\infty} |a_k|$ converges, then the series $\sum_{k=1}^{\infty} a_k$ also converges, and we say it converges **absolutely**.

For example, consider the series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^2}.$$

On the other hand, sometimes $\sum_{k=1}^{\infty} a_k$ converges even though $\sum_{k=1}^{\infty} |a_k|$ does not. This is called **conditional convergence**. The reason is that if you change the order of the terms, you can change the limit of the sum, whereas for an absolutely converging series you cannot.

For example, consider

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}.$$

27.2 Alternating series test

Suppose a_n is a sequence of decreasing positive numbers such that $\lim_{n\to\infty} a_n = 0$. Then the series $\sum_{k=1}^{\infty} (-1)^{k-1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$ converges.

Example. Let $0 < p \leq 1$ and consider the series $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^p}$.

Remark. See Exercise 35 to see the assumption that a_n is a decreasing sequence is necessary.

27.3 Examples

1. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^4}{k^3 + 1}.$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin((k + \frac{1}{2})\pi)}{k^2}.$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{1 + \frac{1}{k}}.$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2 + 1}.$

28 Lecture

Outline:

1. 10.5 Ratio and Root Tests and Strategies for Choosing Tests; Part I
2. Ratio Test
3. Root Test

28.1 Ratio test

Does $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converge? We can't do an integral comparison, because of the factorial. What other kinds of comparison can we make? Well, the series is not geometric, because the ratio between terms would have to be constant. Recall that

$$\sum_{k=0}^{\infty} r^k$$

converges precisely when $|r| < 1$. It turns out this makes a good test even when the series isn't precisely geometric.

Theorem 4 (Ratio test)**.** *Let* $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

1. *If* $r < 1$, *then* $\sum a_k$ *converges absolutely.*
2. *If* $r > 1$, *then* $\sum a_k$ *diverges.*
3. *If* $r = 1$, *then* $\sum a_k$ *may or may not converge.*

Example. $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Example. $\sum_{k=1}^{\infty} \frac{k^3}{2^k}$

Often inconclusive. Look at $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} k^2$.

One more example. Can we learn anything about $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ from the ratio test?

28.2 Root test

Another way to see if a series is close to a convergent geometric series is to take the nth root of a_n.

Theorem 5 (Root test)**.** *Let* $r = \lim_{n\to\infty} |a_n|^{1/n}$.

1. *If* $r < 1$, *then* $\sum a_k$ *converges absolutely.*
2. *If* $r > 1$, *then* $\sum a_k$ *diverges.*
3. *If* $r = 1$, *then* $\sum a_k$ *may or may not converge.*

Example. $\sum_{k=1}^{\infty} \left(\frac{k}{2k+3}\right)^k$

Example. Here's a pretty cool example: $\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{-k^2}$

29 Lecture

Outline:

1. 10.5 Ratio and Root Tests and Strategies for Choosing Tests; Part II
2. So many tests, but how to choose?

29.1 Strategies for choosing tests

Let $\sum_{k=1}^{\infty} a_k$ be given. Keep in mind the series for which convergence or divergence is already known include geometric series and p-series, so these will be very useful in comparison.

1. The nth Term Divergence Test.

Always check this test first as it could give an easy conclusion. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges and we are done! But if $\lim_{n\to\infty} a_n = 0$, we cannot conclude anything...move on to the next step.

2. Positive Series. If all terms in the series are positive, we have the following tests:

(a) The Direct Comparison Test.

Consider whether dropping terms in the numerator or denominator gives a series that we already know convergence or divergence for. If a large series converges, or a smaller series diverges, then the original series does the same.

For example, $\sum_{k=1}^{\infty} \frac{1}{k^2+\sqrt{k}}$ converges by comparison with the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

However, if we try the same strategy on $\sum_{k=2}^{\infty} \frac{1}{k^2-\sqrt{k}}$, we cannot conclude anything from this test. Maybe the Limit Comparison Test will be better suited.

(b) The Limit Comparison Test.

Consider the dominant term in the numerator and denominator, and compare the original series to the ratio of these terms.

For example, look again at $\sum_{k=2}^{\infty} \frac{1}{k^2-\sqrt{k}}$. Since k^2 grows faster than $\sqrt{k}$, we compare with $\frac{1}{k^2}$ to conclude the series converges as

$$\lim_{k\to\infty} \frac{\frac{1}{k^2-\sqrt{k}}}{\frac{1}{k^2}} = 1$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is once again a convergent p-series.

(c) The Ratio Test.

This is often useful when there is a factorial or a constant to the power n present, since the factorial or power will disappear in the ratio.

For example, for $\sum_{k=1}^{\infty} \frac{3^k}{k!}$, applying the Ratio Test yields

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \right| = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1,$$

hence the series converges.

(d) The Root Test.

This is often effective when there is a term of the form $f(n)^{g(n)}$, that is, the base and power is some function of n.

For example, consider $\sum_{k=1}^{\infty} \frac{2^k}{k^{2k}}$. The Root Test yields

$$\lim_{n\to\infty} \left(\frac{2^n}{n^{2n}} \right)^{1/n} = \lim_{n\to\infty} \frac{2}{n^2} = 0 < 1,$$

hence the series converges.

(e) The Integral Test.

When all else fails for a positive, decreasing series, consider the Integral Test by looking at what happens to the integral of the corresponding function.

For example, consider $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$. While all the other tests do not apply easily, the function $\frac{1}{x \ln x}$ is decreasing and $\int_2^{\infty} \frac{dx}{x \ln x} = \infty$. Thus the integral, and hence the series, diverges.

3. Series That Are Not Positive Series.

(a) Alternating Series Test.

If we have an alternating series, show the positive sequence is decreasing and tends to 0. Then the Alternating Series Test shows the series converges, albeit, this only allows one to conclude conditionally.

(b) Absolute Convergence.

If $\sum |a_k|$ (which is now a positive series so we have many test for) converges, then $\sum a_k$ converges absolutely, and therefore is convergent.

Helpful Table

The following is a table that describes the different series tests we have discussed, convergence and divergence criteria, usefulness, and are given in a general good order to look at them if you are not sure what to apply.

Test	Series	Converges if	Diverges if	Useful for
Geometric Series	$\sum_{k=0}^{\infty} cr^k$	$\lvert r\rvert < 1$ to $\frac{c}{1-r}$	$\lvert r\rvert \geq 1$	Sum can start at $k = m$. Converges to $\frac{cr^m}{1-r}$ for $\lvert r\rvert < 1$ by factoring.
p-series Test	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Work the same as improper p-integrals did.
nth Term Divergence Test	$\sum_{k=1}^{\infty} a_k$	n/a	$\lim_{n\to\infty} a_n \neq 0$	When terms do not tend to zero.
Positive Series				All positive terms.
Direct Comparison Test	$\sum_{k=1}^{\infty} a_k$	$0 \leq a_n \leq b_n$ and $\sum_{k=1}^{\infty} b_k$ converges	$0 \leq b_n \leq a_n$ and $\sum_{k=1}^{\infty} b_k$ diverges	Comparing to a known series $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{n\to\infty} \frac{a_n}{b_n} \geq 0$ and $\sum_{k=1}^{\infty} b_k$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges	Dominant term gives a known series $\sum_{k=1}^{\infty} b_k$.
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	No info when equal to one. Useful for factorials or constant raised to k.
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{n\to\infty} \lvert a_n\rvert^{1/n} < 1$	$\lim_{n\to\infty} \lvert a_n\rvert^{1/n} > 1$	No info when equal to one. Useful for terms of the form $f(n)^{g(n)}$.
Integral Test (let $f(n) = a_n$ be positive and decreasing)	$\sum_{k=1}^{\infty} a_k$	$\int_1^{\infty} f(x)\, dx < \infty$	$\int_1^{\infty} f(x)\, dx = \infty$	Useful for positive, decreasing series where integral can be easily understood.
Not Positive Series				
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges	n/a	Can use all the above tests on the series with positive terms. No conclusion if positive series diverges.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$	$\lim_{n\to\infty} a_n = 0$, a_n is positive and decreasing	$\lim_{n\to\infty} a_n \neq 0$	Determining convergence of series that has cancellation from positive and negative terms occurring.

30 Lecture

Outline:

1. 10.6 Power Series; Part I

30.1 Power series

A power series is basically an **infinitely long polynomial**. So it looks like

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

where x is a variable.

Now, a regular polynomial is nice and continuous (and infinitely differentiable) everywhere. But a power series might not even **converge**. How do we know it's even defined?

Example. Consider the power series

$$1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k.$$

For which x does this converge?

Radius of convergence. In the last example, the series converged for $|x| < 1$ but diverged for $|x| \geq 1$. The number 1 is called the *radius of convergence.* It turns out *every* power series has a radius of convergence, which could be infinity or even zero (so the power series would only converge at one point).

Example. Find the radius of convergence of the power series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Example. Find the radius of convergence of the power series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{k=0}^{\infty}(k+1)x^k.$$

Different center.

If a power series has the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

then it is **centered at zero**. When we plug in $x = 0$, we just get one term. So the series always converges there (but possibly nowhere else!).

If we want a series centered around another point, we just shift:

Now c is the center. This will not change the radius of convergence; it will only shift *where* the series converges.

Endpoints. If x is within the radius of convergence, then the series converges, and if it's outside then the series diverges, but what about the endpoints? You have to check those by plugging in each value for x and considering both series separately. Don't worry, it's only two points!

Example. Find where the series $\sum_{k=0}^{\infty} 2^k(x-3)^k$ converges.

Remark. Sometimes the endpoints both converge (closed interval of convergence $[a, b]$), both diverge (open interval of convergence (a, b)), or have different behavior at each one (half-open interval of convergence $(a, b]$ or $[a, b)$). It is often difficult to intuitively see what will happen, so just check each one!

31 Lecture

Outline:

1. 10.6 Power Series; Part II

31.1 Power series that converge to functions

We can make substitutions (change of variables) to get power series formulas for other functions using

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \ |x| < 1.$$

Example. Change x to $-x^2$.

Example. Expand $\dfrac{1}{2+3x}$ in a power series centered at $c = 0$.

Term by term integration and differentiation. If a power series centered at c converges in a certain radius R, then you can take its derivative by differentiating each term (just like with polynomials) for $x \in (c-R, c+R)$. The same holds for integration. The radius of convergence remains the same.

Example. How would we find a power series for $\frac{1}{(1-x)^2}$?

What about for $\ln(1-x)$?

Application. One way to try an solve a differential equation is to use a power series, and solve for the coefficients. This application is only for enrichment; take a differential equations course if you want to know more.

Let's try solving $y' = y, \ y(0) = 1$.

32 Lecture

Outline:

1. 10.7 Taylor Polynomials
2. 10.8 Taylor Series; Part I

32.1 Taylor series

Every function with infinitely many derivatives has, in theory, a power series representation. **It just might not converge.** Whether or not it converges, we call it the **Taylor series** associated with that function.

We pick a center $x = c$. Write

Let's assume the series converges. We want to solve for $a_0, a_1, a_2, a_3, \ldots$ in terms of f and its derivatives.

Plug in $x = c$:

Now take the derivative:

Take the second derivative:

Take the third derivative:

We continue the pattern to get

In other words, the **Taylor series** of f centered at c is

If the Taylor series converges with radius $r > 0$, we say f is **analytic** on the interval $(c - r, c + r)$. Note that the Taylor series must converge to $f(x)$ if it converges at all!

If the center is $c = 0$, sometimes we call the Taylor series **Maclaurin series**.

32.2 Taylor Polynomials

The partial sums of a Taylor series are known as **Taylor Polynomials**. They are a very useful tool for approximating functions so that you consider only finitely many terms as was seen in the first lecture. We define the nth Taylor polynomial, T_n, of $f(x)$ centered at $x = c$ as the nth partial sum

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k.$$

There are lots of theorems on how good these approximations are including error bounds, remainders, and Taylor's theorem, however, we will not go into details here. Similar to series, a **Maclaurin polynomial** is simply a Taylor polynomial with center $c = 0$.

32.3 Examples.

To find a Taylor series (or Taylor polynomial) you need to compute lots of derivatives. Sometimes that is not a problem.

Take e^x.

Take $\sin x$.

Take $\ln x$.

Taylor Polynomial Example. As an example for Taylor polynomials, let's find $T_4(x)$ centered at $c = 2$ for $f(x) = \ln x$.

33 Lecture

Outline:

1. 10.8 Taylor Series; Part II

33.1 Variations on these examples

Example. Find the Taylor series for $x^2 e^{x^3}$ centered at zero.

Example. Here's a warning against forgetting things you know. Let's try to find the Taylor series for $\ln(x^2)$ centered at $x = 1$.

33.2 Application

Limits. I mentioned before that using series can sometimes simplify limits or asymptotics. Here's an example limit you might be tempted to calculate using L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{x - \sin x}{x^3 \cos x}.$$

But you would need to differentiate twice, and that would get complicated. Instead, let's use Taylor series to see that

Integrals. Let's return to the integral of $\sin x^2$ which was introduced in the first lecture of the course.

Now integrating $\sin x^2$ is easy:

Certain Values of Derivatives. We can also use the power series of a function to quickly find the values of derivatives at the center of the series. For example, let $f(x) = \sin x^2$ and find $f^{(10)}(0)$.

Remark: The series here actually tells us that the only nonzero derivatives of $\sin x^2$ are:

Error Bound. We can also use alternating series to easily estimate definite integrals. For example, find $\int_0^1 \sin x^2 \, dx$ within an error of $1/10000$.

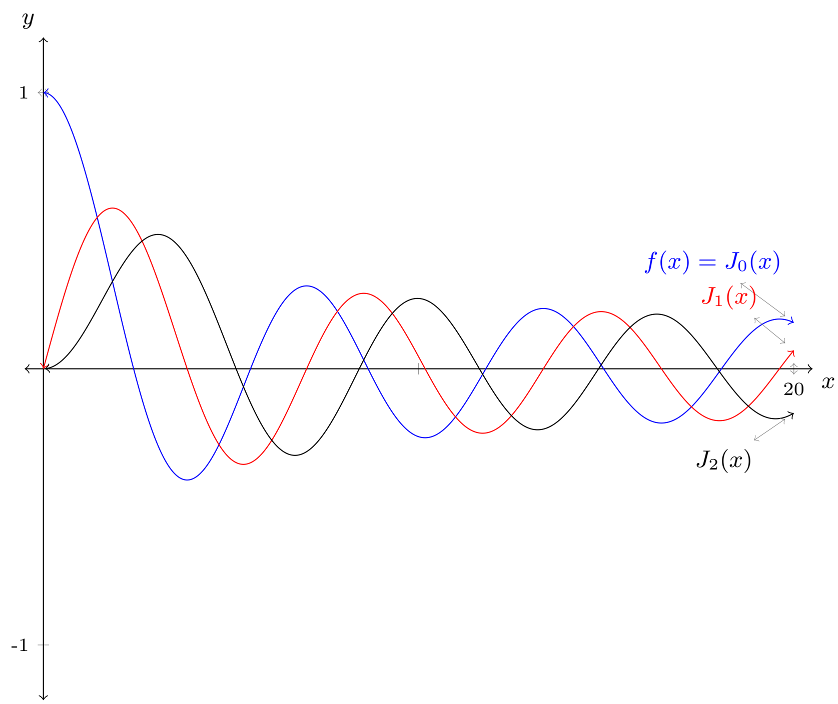
Special functions. Let us end our discussion on Taylor series by reminding ourselves about something else mentioned in the first lecture, the so-called **special functions**. These are functions that appear in a wide range of physics and engineering applications (such as Bessel, hypergeometric, and elliptic functions which are particularly interesting and used to study pendulums), which will be discussed in most Differential Equations courses.

For example, when studying the vibrations of a drum, one is led to the following second order equation:

where α is a nonnegative number. This equation is called the *Bessel equation* and its canonical solutions are the so-called *Bessel functions*. The easiest way to describe a Bessel function is as a power series:

Here the the expression $\Gamma(n+\alpha+1)$ is a value of the Gamma function defined as an improper integrals. In the special case when $\alpha = 0$, the value $\Gamma(n+\alpha+1)$ is equal to $\Gamma(n+1) = n!$ and it follows that

This looks very similar to the expansion for $\cos x$, which is no coincidence. Bessel functions are oscillatory like sine and cosine, however they decay as x grow large. To see this better, here is what the graph of $J_0(x)$ (as well as $J_1(x)$ and $J_2(x)$) looks like:



34 Lecture

Outline:

1. Complex Numbers

34.1 Complex Numbers

Let us begin by discussing the complex numbers, denoted $\mathbb{C}$. The complex numbers are numbers of the form $a + bi$, where a and b are any real number and i is defined to be the square root of -1, that is, $i^2 = -1$. We can realize the complex numbers in the plane by letting what is ordinarily labeled the x-axis correspond to the real part, a, and the y-axis correspond to the imaginary part, b. We can add and multiply complex numbers:

As a result, we can compute polynomial functions of a complex variable, usually denoted z,

We can also measure the distance between two complex numbers as

which is exactly how we would measure the distance between two points in the regular plane (Cartesian coordinates) using Pythagorean's Theorem. This allows one to consider a series of complex numbers and its convergence. In this way, the term radius of convergence now makes sense as it truly is looking at some circle in the complex plane in which your series converges.

In particular, the series for the exponential function, sine function, and cosine function can immediately be extended to the entire complex plane by simply replacing the real variable x with the complex variable z.

34.2 Euler's Formula

This allows one to prove the following theorem, which is surprisingly easy to prove, but extremely useful in application.

Euler's Formula. For all complex numbers z,

$$e^{iz} = \cos z + i \sin z.$$

Proof.

□

Euler's Formula is particularly useful in electrical engineering studying periodic signals.

If we substitute $z = \pi$ into Euler's Formula, we arrive at what is known as **Euler's Identity**, which is probably already familiar to many of you,

Rearranging yields

While this is a useful identity for many reasons, if we take a step back it is quite an amazing equation as it relates 5 of the most important numbers in mathematics and its applications. Namely, and in no particular order, it relates the mathematical constants $0,\ 1,\ e,\ \pi$, and i. That is truly amazing if you really think about it!

34.3 Applications

If Euler's Identity did not impress you, maybe seeing how easy trigonometric identities become using Euler's Formula might! It is fairly easy (and an extra credit problem) to show using Euler's Formula that

which can now be used to find trig identities.

Let's begin easy with Pythagorean's Theorem, that is, $\sin^2 x + \cos^2 x = 1$. How hard is it to prove this using Euler's Formula? Well,

Along these same lines, it is also easy to see

It is often much easier to find these formulas, or come up with new ones, using complex exponential functions as it becomes a purely algebraic manipulation. Even the fact that when you take the derivative of cosine you introduce a negative becomes clear by letting $z = x$, and differentiating since

Exam 3 Review

The third exam will cover sequences, series, and complex numbers:

Sequences [Section 10.1]

A sequence, often denoted $\{a_n\}$, is an infinite set of numbers placed in order:

$$a_1, a_2, a_3, \ldots$$

Be able to find a formula for the nth term of a sequence as well as the limit of a sequence by considering the behavior of an associated function (using L'Hôpital's Rule if needed).

Series [Sections 10.2-10.5]

Be able to use the following techniques and tests in order to determine if a series converges or diverges. **Do not forget the table I provided that lists all of this on one page!**

Let $\sum_{k=1}^{\infty} a_k$ be given. Keep in mind the series for which convergence or divergence is already known include geometric series and p-series, so these will be very useful in comparison. **If the series given is a geometric series, we even have a formula for what the series converges to, if it converges.**

1. The nth Term Divergence Test.

Always check this test first as it could give an easy conclusion. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges and we are done! But if $\lim_{n\to\infty} a_n = 0$, we cannot conclude anything...move on to the next step.

2. Positive Series. If all terms in the series are positive, we have the following tests:

(a) The Direct Comparison Test.

Consider whether dropping terms in the numerator or denominator gives a series that we already know convergence or divergence for. If a large series converges, or a smaller series diverges, then the original series does the same.

For example, $\sum_{k=1}^{\infty} \frac{1}{k^2+\sqrt{k}}$ converges by comparison with the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

However, if we try the same strategy on $\sum_{k=2}^{\infty} \frac{1}{k^2-\sqrt{k}}$, we cannot conclude anything from this test. Maybe the Limit Comparison Test will be better suited.

(b) The Limit Comparison Test.

Consider the dominant term in the numerator and denominator, and compare the original series to the ratio of these terms.

For example, look again at $\sum_{k=2}^{\infty} \frac{1}{k^2-\sqrt{k}}$. Since k^2 grows faster than $\sqrt{k}$, we compare with $\frac{1}{k^2}$ to conclude the series converges as

$$\lim_{k\to\infty} \frac{\frac{1}{k^2-\sqrt{k}}}{\frac{1}{k^2}} = 1$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is once again a convergent p-series.

(c) The Ratio Test.

This is often useful when there is a factorial or a constant to the power n present, since the factorial or power will disappear in the ratio.

For example, for $\sum_{k=1}^{\infty} \frac{3^k}{k!}$, applying the Ratio Test yields

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \right| = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1,$$

hence the series converges.

(d) The Root Test.

Will not be covered on the exam.

(e) The Integral Test.

When all else fails for a positive, decreasing series, consider the Integral Test by looking at what happens to the integral of the corresponding function.

For example, consider $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$. While all the other tests do not apply easily, the function $\frac{1}{x \ln x}$ is decreasing and $\int_2^{\infty} \frac{dx}{x \ln x} = \infty$. Thus the integral, and hence the series, diverges.

3. Series That Are Not Positive Series.

(a) Alternating Series Test.

If we have an alternating series, show the positive sequence is decreasing and tends to 0. Then the Alternating Series Test shows the series converges, albeit, this only allows one to conclude conditionally.

(b) Absolute Convergence.

If $\sum |a_k|$ (which is now a positive series so we have many test for) converges, then $\sum a_k$ converges absolutely, and therefore is convergent.

Power Series [Section 10.6]

Remember that a power series is basically an **infinitely long polynomial**. So it look like

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

where x is a variable. **Be able to find the interval of convergence for a given power series**.

Note that the most useful test for this is often the Ratio Test. Take the limit of the consecutive terms and then refer to the Ratio Test's criteria for convergence to understand where the series converges. If the limit of the ratio does not depend on x and converges by the test, then it converges for all x. If there is x dependence left in the ratio, solve for x in the inequality for convergence to find the radius of convergence. Keep in mind that the endpoints will always have to be considered separately. Simply plug each in and see if the series (which no longer depends on x!) converges or not using any test we know.

Taylor Polynomials [Section 10.7]

We define the nth Taylor polynomial, T_n, of $f(x)$ centered at $x = c$ as the nth partial sum

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k.$$

Taylor Series and Complex Numbers [Section 10.8]

The **Taylor series** of f centered at c is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

If the Taylor series converges with radius $r > 0$, we say f is **analytic** on the interval $(c-r, c+r)$. Note that the Taylor series must converge to $f(x)$ if it converges at all!

If the center is $c = 0$, sometimes we call the Taylor series **Maclaurin series**.

Be able to find a Taylor series for a given function. Note that this means you need to be able to find a "nice" formula for the derivatives if possible. If you are given a function that looks complicated, that is, it is a composition of functions or multiplication of a function by a power of x, remember to use what is known about the "simpler" function instead of trying to find a formula for the derivatives directly.

For example, if you were asked to find the Taylor series for $x^2 e^{x^3}$ centered at zero, there is no reason to take all the derivatives (which would be really complicated!). We just use the series for e^x and plug in x^3 in place of x, then multiply by x^2:

$$x^2 e^{x^3} = x^2 \sum_{k=0}^{\infty} \frac{(x^3)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{3k+2}}{k!} \text{ for all } x.$$

Also be able to use term-by-term differentiation of integration to find a series representation for the derivative or integral of a function. Recall that we used this technique to find

$$\int \sin x^2 \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int x^{4k+2} \, dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+3}}{(2k+1)!(4k+3)} \text{ for all } x.$$

Euler's Formula. For all complex numbers z,

$$e^{iz} = \cos z + i \sin z.$$

Be able to use Euler's Formula to write a complex exponential as the sum of its real and imaginary parts (and vice versa). This is to say, if given the left-hand side of the above formula for a specific z, be able to rewrite in the form of the right-hand side and simplify into $a + bi$, and vice versa.