RESEARCH STATEMENT

JONATHAN STANFILL

Introduction

My main research interests lie in the spectral theory of differential operators and their associated spectral functions. The analysis of spectral functions typically involves a combination of techniques from spectral theory, (partial) differential equations, complex analysis, special functions, and asymptotic analysis. These topics are of widespread interest since in many areas of mathematics and physics one is often confronted with the problem of extracting relevant information from the spectrum of differential operators, which, in one dimension, are often Sturm–Liouville operators. My research focuses on the spectral ζ-function, heat kernel, Weyl–Titchmarsh–Kodaira m-function, and Donoghue m-function. A brief description of each is given next.

The spectral ζ-function represents a generalization of the more familiar Riemann ζ-function in which the integers are replaced by the non-vanishing positive eigenvalues of a differential operator. It is of fundamental importance for the analysis of complex powers of elliptic operators and for the study of ζ-regularized functional determinants [18, 24]. Furthermore, the analytic continuation of the ζ-function is used in studying the small time asymptotic behavior of the trace of the heat kernel, which is the fundamental solution to the heat equation on a manifold endowed with appropriate boundary conditions. This small time asymptotic behavior is used in order to extract geometric information about the underlying manifold [21, 22]. The widespread use of both of these functions in physics can be found especially in the area of quantum field theory [2, 3].

The Weyl and Donoghue m-functions are indispensable tools in the spectral analysis of self-adjoint extensions, T, of Sturm–Liouville differential operators [1, Ch. 6], [11, 26]. The Weyl m-function was first studied by Weyl and its relation to spectral theory was later investigated by Titchmarsh, who found a simple formula to determine the associated spectral measure (discovered by Kodaira around the same time); hence, the general terminology Weyl–Titchmarsh–Kodaira m-function (for more details and references, see [1, Introd.]). Interest in Weyl and Donoghue m-functions arises from the fact that they are generalized Nevanlinna–Herglotz functions and an analog of the Stieltjes inversion formula applied to them yields that the spectrum (and its subdivisions) of T is related to the singularity structure of the m-functions on the real line.

My current research investigations can be broadly classified in the following areas:

(i) Weyl and Donoghue m-functions for Sturm–Liouville operators [9, 10, 13, 16].
(ii) The study of spectral ζ-functions and functional determinants for Sturm–Liouville operators [5, 6].
(iii) Domain properties of Sturm–Liouville operators [4, 14].
(iv) Hardy–Rellich-type integral inequalities, with special emphasis on those related to Sturm–Liouville operators [12, 13, 15].

These topics belong to extremely active fields of research which deal, for instance, with important problems in mathematical physics, applied mathematics, and analysis. The importance of these studies lies in their potential for providing a deeper understanding of Sturm–Liouville operators, the relation between analysis and geometry, and in the development of novel methods, or improvement of old ones. In addition, spectral functions are powerful tools for the description of physical phenomena at both the microscopic and macroscopic scale.

The methods and techniques used in the analysis of spectral functions and their application to physical problems are accessible to advanced undergraduate and graduate students with a background preparation in complex analysis, (partial) differential equations, and asymptotic methods. Furthermore, research projects regarding the application of spectral functions to problems related to physics are very likely to be funded by national agencies, and I am committed to exploring other related avenues of research that would lead to opportunities for external funding.

Additional Details and Ongoing Projects

In this section, I briefly give more details on previous and ongoing projects. The next section is devoted to the discussion of future research projects

Weyl and Donoghue m-functions.

Assume the standard local integrability hypotheses on the coefficients p, q, r, and consider all self-adjoint realizations corresponding to the Sturm–Liouville differential expression

$$\tau = 1/r(x)[-(d/dx)p(x)(d/dx) + q(x)] \text{ for a.e. } x \in (a, b) \subseteq \mathbb{R}, \text{ in } L^2((a, b); rdx).$$

We are currently investigating the construction of singular Weyl m-functions corresponding to self-adjoint realizations with coupled boundary conditions employing the notions of generalized boundary values as in [8]. This investigation stems from the current work [10] on explicitly constructing the Weyl m-function for the Jacobi differential operator associated with the expression

$$\tau_{\alpha,\beta} = -(1-x)^{-\alpha}(1+x)^{-\beta}(d/dx)\big((1-x)^{\alpha+1}(1+x)^{\beta+1}\big)(d/dx),$$
$$x \in (-1, 1),\ \alpha, \beta \in \mathbb{R}.$$

To the best of our knowledge, the general case of Weyl m-functions associated to coupled boundary conditions has not been fully investigated.

Regarding Donoghue m-functions, in [9] we systematically constructed the Donoghue m-functions in all cases where τ is in the limit circle case at least at one interval endpoint employing generalized boundary values and Krein resolvent identities (see also [16] for the study of Jacobi operators).

Spectral ζ-functions.

In the recent work [5], we employed a newly developed unified approach to the computation of traces of resolvents and ζ-functions to efficiently compute values of spectral ζ-functions at positive integers associated to regular self-adjoint Sturm–Liouville differential expressions τ. Depending on the underlying boundary conditions, we expressed the ζ-function values in terms of a fundamental system of solutions of $\tau y = zy$ and their expansions about the point $z = 0$. In particular, given a self-adjoint operator, $T_{A,B}$, associated to τ (A, B denoting separated or coupled boundary conditions), we have

$$\zeta(s; T_{A,B}) = \frac{1}{2\pi i}\oint_{\gamma} dz\ z^{-s}\left(\frac{d}{dz}\ln(F_{A,B}(z)) - z^{-1}m_0\right),$$

where $F_{A,B}(\,\cdot\,)$ is a certain characteristic function encoding eigenvalues of $T_{A,B}$ as its zeros, m_0 is the multiplicity of zero as an eigenvalue of $T_{A,B}$, γ is a simple contour enclosing $\sigma(T_{A,B})\backslash\{0\}$ in a counterclockwise manner so as to dip under (and hence avoid) the point 0 (cf. Figure 1), and we take the branch cut of z^{-s} to be

$$R_\psi = \{z = te^{i\psi}\,|\,t \in [0, \infty)\}, \quad \psi \in (\pi/2, \pi).$$

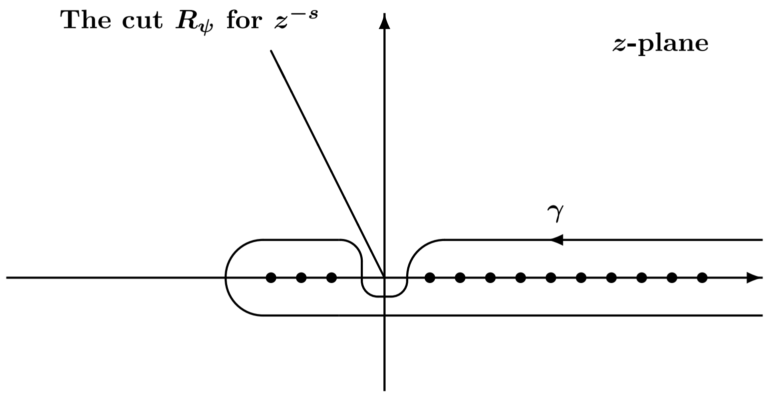

Figure 1. Contour γ in the complex z-plane.

By letting $s = n$, $n \in \mathbb{N}$, one no longer needs a branch cut for the fractional powers of z^{-s} given in Figure 1, reducing the integral along the curve γ to a clockwise oriented integral along the circle C_ε, centered at zero with radius $\varepsilon > 0$,

$$\zeta(n; T_{A,B}) = -\frac{1}{2\pi i} \oint_{C_\varepsilon} dz\, z^{-n} \frac{d}{dz} \ln(F_{A,B}(z)) = -\operatorname{Res}\left[z^{-n} \frac{d}{dz} \ln(F_{A,B}(z));\ z = 0\right], \quad n \in \mathbb{N}.$$

Furthermore, under additional assumptions we gave the full analytic continuation of the ζ-function through a Liouville transformation and provided an explicit expression for the ζ-regularized functional determinant. In particular, one obtains the remarkably simple formula

$$\zeta'(0; T_{A,B}) = i\pi n - \ln(2c|F_{m_0}/\Gamma_{k_0}|),$$

where n is the number of strictly negative eigenvalues of $T_{A,B}$, F_{m_0} is the first coefficient of the small-z expansion of $F_{A,B}(z)$, Γ_{k_0} is a certain non-vanishing term coming from the large-z asymptotic expansion of $F_{A,B}(z)$, and

$$c = \int_a^b dt\ [r(t)/p(t)]^{1/2} > 0.$$

In [6] we extend this method (when appropriate) to singular self-adjoint Sturm–Liouville differential operators. This allows one to treat many more examples such as the *generalized Bessel operator* associated with the expression

$$\tau_{\delta,\nu,\gamma} = x^{-\delta}\left[-\frac{d}{dx} x^\nu \frac{d}{dx} + \frac{(2+\delta-\nu)^2\gamma^2 - (1-\nu)^2}{4} x^{\nu-2}\right],$$
$$\delta > -1,\ \nu < 1,\ \gamma \geqslant 0,\ x \in (0, b).$$

Extending the results for the ζ-regularized functional determinant is more nuanced, and is discussed below as a future research project.

Domain properties of Sturm–Liouville operators.

In order to better study spectral functions associated to all coupled boundary conditions in the singular setting, we revisited the Krein–von Neumann extension in [4] when the underlying symmetric operator is strictly positive. In particular, the boundary conditions for the Krein–von Neumann extension of the strictly positive minimal Sturm–Liouville operator were explicitly expressed in terms of generalized boundary values.

Motivated by the study of Bessel operators in connection with a refinement of Hardy's inequality in [13] (described next), we took a closer look at the underlying Bessel-type operators with more general inverse square singularities at the interval endpoints in [14]. More precisely, we considered quadratic forms and operator realizations in $L^2((a, b); dx)$ associated with differential expressions of the form

$$\tau_{s_a,s_b} = -\frac{d^2}{dx^2} + \frac{s_a^2 - (1/4)}{(x-a)^2} + \frac{s_b^2 - (1/4)}{(x-b)^2} + q(x), \quad x \in (a, b),$$
$$s_a, s_b \in [0, \infty),\ q \in L^\infty((a, b); dx),\ q \text{ real-valued a.e. on } (a, b),$$

where $(a, b) \subset \mathbb{R}$ is a bounded interval. As an explicit illustration and application of [4], we described the corresponding Krein–von Neumann extensions via hypergeometric functions.

Integral inequalities.

In [13], we employed Bessel-type operators and their Weyl m-functions in proving the inequality

$$\int_0^\pi dx\, |f'(x)|^2 \geqslant \frac{1}{4}\int_0^\pi dx\, \frac{|f(x)|^2}{\sin^2(x)} + \frac{1}{4}\int_0^\pi dx\, |f(x)|^2, \quad f \in H_0^1((0, \pi)),$$

where both constants $1/4$ appearing in the above inequality are optimal. In addition, this inequality is strict in the sense that equality holds if and only if $f \equiv 0$. This inequality is derived using the exactly solvable, singular, Dirichlet-type Schrödinger operator associated with the differential expression

$$\tau_s = -\frac{d^2}{dx^2} + \frac{s^2 - (1/4)}{\sin^2(x)}, \quad s \in [0, \infty),\ x \in (0, \pi).$$

The new inequality represents a refinement of one of Hardy's classical inequalities and it improves upon its well-known extension involving the distance to the boundary of $(0, \pi)$.

Extending [7], we illustrate in [15] how factorizations of singular, even-order partial differential operators yield an elementary approach to classical inequalities of power weighted Hardy–Rellich-type. More precisely, for $\alpha, \beta, \gamma \in \mathbb{R}$, we introduce the three-parameter n-dimensional homogeneous scalar differential expression

$$T_{\alpha,\beta,\gamma} = |x|^{\gamma/2}\big[-\Delta + \alpha|x|^{-2}x\cdot\nabla + \beta|x|^{-2}\big], \quad x \in \mathbb{R}\backslash\{0\},\ n\in\mathbb{N},\ n\geqslant 2,$$

and its formal adjoint, $T^+_{\alpha,\beta,\gamma}$. We show nonnegativity of $T^+_{\alpha,\beta,\gamma}T_{\alpha,\beta,\gamma}$ on $C_0^\infty(\mathbb{R}^n\backslash\{0\})$ implies, for $\alpha,\beta,\gamma\in\mathbb{R}$ and $f\in C_0^\infty\big(\mathbb{R}^n\backslash\{0\}\big),\ n\in\mathbb{N},\ n\geqslant 2$, the inequality

$$\begin{aligned}\int_{\mathbb{R}^n} |x|^{\gamma}|(\Delta f)(x)|^2\ d^n x \geqslant [(n-4+\gamma)\alpha - 2\beta]\int_{\mathbb{R}^n}|x|^{\gamma-2}|(\nabla f)(x)|^2\ d^n x\\ -\alpha(\alpha-4+2\gamma)\int_{\mathbb{R}^n}|x|^{\gamma-4}|x\cdot(\nabla f)(x)|^2\ d^n x\\ +\beta[(n-4)(\alpha-2)-\beta+(n+\alpha+\gamma-6)\gamma]\int_{\mathbb{R}^n}|x|^{\gamma-4}|f(x)|^2 d^n x.\end{aligned}$$

A particular choice of parameters in this fundamental inequality yields known Hardy–Rellich-type inequalities, including a power weighted Rellich inequality and an improvement of a power weighted extension of an inequality due to Schmincke (cf. [25]).

Future Directions of Research

Lastly, I describe a few of the projects I am interested in pursuing in the immediate future, ending with a potential project involving undergraduate students related to integer partitions, generating functions, and special functions.

Weyl m-functions for operators associated to exceptional orthogonal polynomials.

In recent years, there has been much research and interest on exceptional orthogonal polynomials since the systematic study started in [17]. The study of the spectral theory of the associated differential operators and their Weyl m-functions would be a natural investigation given [10]. *In particular, through an appropriate Darboux transformation the Weyl m-function for the Jacobi operator will yield the Weyl m-function for the exceptional Jacobi operator.* Furthermore, some of the known results, such as completeness results regarding the polynomial eigenfunctions, would then follow directly from properties of the Weyl m-function, such as its Nevanlinna–Herglotz property.

ζ-regularized functional determinants for singular Sturm–Liouville operators.

Given the general results for ζ-regularized functional determinants for regular Sturm–Liouville operators in [5] and the consideration of spectral ζ-functions for singular Sturm–Liouville operators in [6], one might wonder what can be shown regarding functional determinants for singular Sturm–Liouville operators. *The main difficulty is that the large spectral parameter asymptotics of the underlying solutions of the differential equation are much more intricate than in the regular case.* It should be mentioned that even in the regular case given in [5], extra assumptions are needed on the coefficients of the different expression in order to apply a Liouville transformation and compute the needed asymptotic expansion. Consequently, it would be of interest to classify the large spectral parameter asymptotics of solutions based on assumptions on the coefficients of their associated differential expression. This would provide a unified method in the computation of functional determinants.

Donoghue m-functions in the limit point case.

A natural extension of the results in [9] would be to complete the analysis of Donoghue m-functions associated to singular Sturm–Liouville operators by considering the case in which both endpoints of the interval are in the limit point case. The techniques utilized in [9] would need to be modified in order to investigate this problem. One approach would consist in inserting a reference point in the interval and constructing the associated Donoghue m-function for the full interval problem from the two interval problems using the results in [9].

Spectral theory of Bessel-type Heun operators.

In our recent study [14], an example of interest was the Schrödinger operator with an inverse square potential located at each end of the interval considered. In particular, the associated differential equation was shown to be a specific case of the confluent Heun differential equation

$$w''(\xi) + \left(\frac{\gamma}{\xi} + \frac{\delta}{\xi - 1} + \varepsilon\right) w'(\xi) + \frac{\nu\xi - \mu}{\xi(\xi - 1)} w(\xi) = 0, \quad \gamma, \delta, \varepsilon, \mu, \nu, \in \mathbb{C},\ \xi \in (0, 1),$$

in normal form. The explicit form of principal and nonprincipal solutions (in terms of ${}_2F_1$ hypergeometric functions) to the associated differential equation as well as the explicit characterization of the Krein–von Neumann extension in terms of these solutions were given in [14, Appendix A]. It would be of interest to investigate the spectral theory and spectral functions associated to these Bessel-type Heun operators as they appear in many applications in current physics literature, from quantum gravity to general relativity.

Trace of the heat kernel of the Navier–Lamé operator.

In [23], Liu studied the first two coefficients of the trace of the heat kernel of the *Navier–Lamé operator* on a compact n-dimensional Riemannian manifold, Ω, with smooth boundary $\partial\Omega$. These two coefficients are of interest as they provide the precise information for the volume of the elastic body, Ω, and the surface area of the boundary, $\partial\Omega$, in terms of the spectrum of the Navier-Lamé operator with Dirichlet or Neumann boundary conditions. Approaching this problem by means of the spectral ζ-function would simplify the computations and provide a different method to verify the first two coefficients. Furthermore, it would also allow one to *compute additional coefficients of the small-t asymptotic expansion of the trace of the heat kernel* by utilizing the well-known relation between the aforementioned coefficients and the residue of the associated spectral ζ function at particular points.

Generalized boundary values for even-order Sturm–Liouville problems.

An interest of mine would be to extend the generalized boundary values for singular, second-order Sturm–Liouville problems discussed in [8] to singular, higher even-order Sturm–Liouville problems. These results would allow one to extend our previous work on spectral functions to singular, higher order differential operators using the notion of generalized boundary values. The outcome of this research will certainly allow us to analyze the spectral functions for a wide array of Sturm-Liouville problems which are otherwise intractable with current techniques.

Undergraduate Research: Integer Partitions and Generating Functions.

Motivated in part by [19] and [20], potential undergraduate research projects could include further investigations of the relationship between specific integer partitions and their generating functions. In particular, it would be of interest to revisit the open problems and conjectures given in [19] as well as to employ the general techniques using theta functions outlined in [19] and [20] to investigate congruences of other integer partitions. These investigations are particularly suitable for an undergraduate student since they only require a basic knowledge of combinatorics, power series, and specific special functions (the latter of which can easily be provided to the student by me). At the beginning of the project, the students would rely on an algebraic computer program (such as Mathematica) to gain relevant insights on possible congruences and relations. These projects would also be prime candidates for collaboration with other mathematics faculty members, especially those interested in combinatorics and integer partitions as well as leading undergraduate student research.

References

[1] J. Behrndt, S. Hassi, and H. De Snoo, *Boundary Value Problems, Weyl Functions, and Differential Operators*, Monographs in Math., Vol. 108, Birkhäuser, Springer, 2020.

[2] B. S. DeWitt, *Quantum field theory in curved spacetime*, Phys. Rep. C **19**, 295–357 (1975).

[3] B. S. DeWitt, *The Global Approach to Quantum Field Theory*, International Series of Monographs on Physics, Vol. 114, Oxford University Press, Oxford, 2003.

[4] G. Fucci, F. Gesztesy, K. Kirsten, L. L. Littlejohn, R. Nichols, and J. Stanfill, *The Krein–von Neumann extension revisited*, Applicable Anal., 25p. (2021). DOI:10.1080/00036811.2021.1938005

[5] G. Fucci, F. Gesztesy, K. Kirsten, and J. Stanfill, *Spectral ζ-functions and ζ-regularized functional determinants for regular Sturm–Liouville operators*, to appear in Res. Math. Sci., 44 pp., arXiv:2101.12295.

[6] G. Fucci, F. Gesztesy, K. Kirsten, and J. Stanfill, *Traces, determinants, and spectral ζ-functions for singular Sturm–Liouville operators*, in preparation.
[7] F. Gesztesy and L. L. Littlejohn, *Factorizations and Hardy–Rellich-type inequalities*, in *Non-Linear Partial Differential Equations, Mathematical Physics, and Stochastic Analysis, The Helge Holden Anniversary Volume*, F. Gesztesy, H. Hanche–Olsen, E. R. Jakobsen, Yu. Lyubarskii, N. H. Risebro, and K. Seip (eds.), EMS Congress Reports, EMS, ETH-Zürich, 2018, pp. 207–226.
[8] F. Gesztesy, L. L. Littlejohn, and R. Nichols, *On self-adjoint boundary conditions for singular Sturm–Liouville operators bounded from below*, J. Diff. Eq. **269**, 6448–6491 (2020).
[9] F. Gesztesy, L. L. Littlejohn, R. Nichols, M. Piorkowski, and J. Stanfill, *Donoghue m-Functions for Singular Sturm–Liouville Operators*, submitted, 2021, arXiv:2107.09832.
[10] F. Gesztesy, L. L. Littlejohn, M. Piorkowski, and J. Stanfill, *The Jacobi operator and its Weyl–Titchmarsh–Kodaira m-functions*, in preparation.
[11] F. Gesztesy, S. Naboko, R. Weikard, and M. Zinchenko, *Donoghue-type m-functions for Schrödinger operators with operator-valued potentials*, J. d'Analyse Math. **137**, 373–427 (2019).
[12] F. Gesztesy, R. Nichols, and J. Stanfill, *A survey of some norm inequalities*, Complex Anal. Operator Th. **15**, No. 23 (2021).
[13] F. Gesztesy, M. M. H. Pang, and J. Stanfill, *Bessel-type operators and a refinement of Hardy's inequality*, in *From Operator Theory to Orthogonal Polynomials, Combinatorics, and Number Theory. A Festschrift in honor of Lance L. Littlejohn's 70th birthday*, F. Gesztesy and A. Martinez-Finkelshtein (eds.), Operator Theory: Advances and Applications, Birkhäuser, Springer, to appear, arXiv:2102.00106.
[14] F. Gesztesy, M. M. H. Pang, and J. Stanfill, *On domain properties of Bessel-type operators*, preprint, 2021, arXiv:2107.09271.
[15] F. Gesztesy, M. M. H. Pang, and J. Stanfill, *Factorizations and power weighted Rellich-type inequalities*, in preparation.
[16] F. Gesztesy, M. Piorkowski, and J. Stanfill, *The Jacobi operator and its Donoghue m-functions*, in preparation.
[17] D. Gómez-Ullate, N. Kamran, and R. Milson, *An extended class of orthogonal polynomials defined by a Sturm–Liouville problem*, J. of Math. Anal. and App. **359**, 1, 352–367 (2009).
[18] S.W. Hawking, *Zeta Function Regularization of Path Integrals in Curved Space-Time*, Comm. Math. Phys. **55**, 133–148 (1977).
[19] D. Herden, M. Sepanski, J. Stanfill, C. Hammon, J. Henningsen, H. Ickes, J. Menendez, T. Poe, I. Ruiz, and E. Smith, *Counting the number of parts divisible by k in all the partitions of n whose parts have multiplicity less than k*, submitted, 13 pp., arXiv:2010.02788.
[20] D. Herden, M. Sepanski, J. Stanfill, C. Hammon, J. Henningsen, H. Ickes, and I. Ruiz, *Partitions with designated summands not divisible by 2^ℓ, 2, and 3^ℓ modulo 2, 4, and 3*, submitted, 22 pp., arXiv:2101.04058.
[21] M. Kac, *Can one hear the shape of a drum?*, Am. Math. Mon. **73**, 1–23 (1966).
[22] K. Kirsten, *Spectral Functions in Mathematics and Physics*, CRC Press, Boca Raton, 2002.
[23] G. Liu, *Geometric invariants of spectrum of the Navier–Lamé operator*, J. of Geometric Anal., (2021). DOI:10.1007/s12220-021-00639-8
[24] D.B. Ray and I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. Math. **7**, 145–210 (1971).
[25] U-W. Schmincke, *Essential self-adjointness of a Schrödinger operator with strongly singular potential*, Math. Z. **124**, 47–50 (1972).
[26] A. Zettl, *Sturm–Liouville Theory*, Mathematical Surveys and Monographs, Vol. 121, Amer. Math. Soc., Providence, RI, 2005.