WEAK TYPE INEQUALITIES FOR ERGODIC STRONG MAXIMAL OPERATORS

PAUL HAGELSTEIN AND ALEXANDER STOKOLOS

Abstract. Fava’s weak type $L \log L$ estimate for strong two-parameter ergodic maximal operators associated to pairs of commuting non-periodic transformations is shown to be sharp. Moreover, given a function $\phi$ on $[0, \infty)$ that is positive, increasing, and $o(\log(x))$ for $x \to \infty$ as well as a pair of commuting invertible non-periodic measure-preserving transformations on a space $\Omega$ of finite measure, a function $f \in L^{\phi}(\Omega)$ is constructed whose associated multiparameter ergodic averages fail to converge almost everywhere in the unrestricted sense.

1. Introduction

A fundamental question in multiparameter harmonic analysis is how an integrable function on, say, $Q = (-\pi, \pi) \times (-\pi, \pi)$ may be recovered from its Fourier coefficients. More precisely, if $f$ is integrable on $Q$ and we define

$$a_{m,n} = \frac{1}{(2\pi)^2} \int_Q f(x,y) e^{-i(mx+ny)} dx \, dy,$$

how may $f$ be expressed in terms of $\{a_{m,n}\}$? The question is a subtle one, especially considering the example due to Charles Fefferman in [4] of a continuous function $f$ on $Q$ such that

$$f(x,y) = \lim_{M,N \to \infty} \sum_{|m| \leq M, |n| \leq N} a_{mn} e^{i(mx+ny)}$$

holds for no $(x,y)$.

One positive result in this vein is the following. If $\int_Q |f| \log^+ |f| < \infty$, then $f$ can be recovered from its Fourier coefficients by considering the Cesàro means of $S_{jk} f$, where $S_{jk} f$ is the partial multiparameter Dirichlet sum of $f$ given by

$$S_{j,k} f(x,y) = \sum_{|m| \leq j, |n| \leq k} a_{mn} e^{i(mx+ny)}.$$
In particular,  

\[
(1) \quad \lim_{m,n \to \infty} \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} S_{jk} f(x, y) = f(x, y)
\]

for a.e. \((x, y)\) in \(Q\). This result is a manifestation of the Jessen-Marcinkiewicz-Zygmund theorem [6], which asserts that the set of rectangles with sides parallel to the coordinate axes differentiates \(L \log L(Q)\). A quantitative analogue of the Jessen-Marcinkiewicz-Zygmund theorem was provided by M. de Guzmán, who showed in [5] that the strong maximal operator \(M\) defined by

\[
Mf(x, y) = \sup_{a_1 < x < a_2, b_1 < y < b_2} \frac{1}{(a_2 - a_1)(b_2 - b_1)} \int_{a_1}^{a_2} \int_{b_1}^{b_2} |f(x, y)| \, dx \, dy
\]
satisfies the weak type estimate

\[
(2) \quad |\{p \in Q : Mf(p) > \lambda\}| \leq C \int_{Q} \frac{|f|}{\lambda} \left(1 + \log^+ \frac{|f|}{\lambda}\right).
\]

Estimate (2) is sharp. To see why this is so, fix \(0 < \lambda < 1\). For \(0 < \epsilon < \lambda\), let \(f_\epsilon = \chi_{[0,\epsilon] \times [0,\epsilon]}\). Note

\[
|\{p \in Q : Mf_\epsilon(p) > \lambda\}| = \left|[0,\epsilon] \times \left[\frac{\epsilon}{\lambda}\right]\right| + \int_{x=\epsilon}^{\frac{\epsilon}{\lambda}} \int_{y=0}^{\frac{\epsilon}{\lambda}} dy \, dx
\]

\[
= \frac{\epsilon^2}{\lambda} + \frac{\epsilon^2}{\lambda} \log\left(\frac{1}{\lambda}\right)
\]

\[
= \int_{Q} \frac{f_\epsilon}{\lambda} \left(1 + \log^+ \frac{|f_\epsilon|}{\lambda}\right).
\]

It is useful to observe that the sharpness of the de Guzmán estimate is reflected in the fact that, given any function \(\phi(x)\) acting on \([0,\infty)\) that is positive, increasing, and \(o(x \log(x))\) as \(x \to \infty\), there exists \(f \in L^1(Q)\) such that \(\phi(f)\) is integrable over \(Q\) and such that

\[
\lim_{m,n \to \infty} \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} S_{jk} f(x, y)
\]
diverges at almost every point \((x, y)\) in \(Q\) (see [10], Vol. II, Ch. XVII for details).

We now turn our attention to ergodic theory analogues of the above results. Let \(U, V\) be one-to-one measure-preserving maps of a measure space
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\( \Omega \) of finite measure onto itself. N. Dunford proved in [2] that

\[
\lim_{m,n \to \infty} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f(U^j V^k x)
\]

converges a.e. in \( \Omega \) provided \( f \in L \log L(\Omega) \). Now, the maximal operator \( M_{UV} \) associated to the above ergodic average is given by

\[
M_{UV}f(x) = \sup_{m,n} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} |f(U^j V^k x)|.
\]

N. Fava proved in [3] that \( M_{UV} \) satisfies an \( L \log L \) weak type inequality similar to the one satisfied by the strong maximal operator \( M \) indicated above, namely,

\[
|\{x : M_{UV} f(x) > \lambda\}| \leq C \int_{\Omega} \frac{|f|}{\lambda} \left(1 + \log^+ \frac{|f|}{\lambda}\right).
\]

Observe that the above inequality readily yields Dunford’s \( L \log L \) convergence result, given the a.e. convergence of (3) for all functions \( f \) in a class of functions dense in \( L \log L(\Omega) \).

The sharpness of the Dunford and Fava results is a subject of recent interest in ergodic theory. A notable result in this regard due to Argiris and Rosenblatt [1] is the following.

**Theorem 1.** Let \( V \) be an invertible ergodic measure-preserving function on a nonatomic separable measure space \( \Omega \) of finite measure.

i) If \( U \) is a power of \( V \), then (3) holds a.e. for all \( f \in L^1(\Omega) \).

ii) If \( f \) is any positive function in \( L^1(\Omega) \) that is not in \( L \log L(\Omega) \), then there exists an ergodic transformation \( U \) such that (3) fails a.e. in \( \Omega \).

In the context of the above theorem, Argiris and Rosenblatt mentioned the problem of, given a function \( f \) in \( L^1(\Omega) \) but not in \( L \log L(\Omega) \), whether or not there were any general conditions an ergodic transformation \( U \) commuting with \( V \) could satisfy such that the convergence of (3) would fail a.e. in \( \Omega \). Motivated by this problem, the authors considered the related question of whether there were a general condition that could be placed on ergodic transformations \( U \) and \( V \) such that the Fava estimate (4) would be sharp for \( M_{UV} \) and moreover such that the convergence of (3) would fail a.e. in \( \Omega \) for some function \( f \) in \( L^1(\Omega) \) but not in \( L \log L(\Omega) \). The primary purpose of this paper is to show that the property of \textit{non-perodicity} of the pair of transformations \( U \) and \( V \) provides such a condition. This property, introduced by Katznelson and Weiss in [7], is the following.
Definition 1. A pair of commuting invertible measure-preserving transformations $U$ and $V$ on a measure space $\Omega$ is said to be non-periodic if for any $(m,n) \neq (0,0), (m,n) \in \mathbb{Z}^2$ we have
\[ |\{x \in \Omega : U^m V^n x = x\}| = 0. \]

We will show that the Fava estimate is sharp if $U$ and $V$ are a commuting pair of non-periodic transformations on $\Omega$. Moreover, we will prove that, given a positive, increasing function $\phi(x)$ on $[0, \infty)$ that is $o(\log x)$ as $x \to \infty$, there exists a function $f \in L^\phi(\mathbb{L})(\Omega)$ such that
\[
\lim_{m,n \to \infty} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f(U^j V^k x) 
\]
does not exist for a.e. $x$ in $\Omega$.

2. Sharpness of the Fava estimate

Theorem 2. Let $U$ and $V$ be a commuting pair of non-periodic transformations of a measure space $\Omega$ of finite measure onto itself. Then for any $\lambda \in (0,1)$ there is a function $f \in L(\Omega)$ such that
\[
|\{x : M_{UV} f(x) > \lambda\}| \geq c \int_\Omega \frac{|f|}{\lambda} \left( 1 + \log^+ \frac{|f|}{\lambda} \right),
\]
where $c$ is independent of $f$ and $\lambda$.

Proof. We will need the following multiparameter analogue of Rohlin’s lemma due to Katznelson and Weiss [7].

Lemma 1. Let $U$ and $V$ be two commuting non-periodic measure-preserving transformations on a measure space $\Omega$ of finite measure. Then for any $\epsilon > 0$ and positive integer $N$ there exist sets $B$ and $E$ in $\Omega$ such that $|E| < \epsilon$ and
\[
\Omega = \left( \bigcup_{j,k=1}^{N} B^{j,k} \right) \cup E,
\]
where the $B^{j,k} = U^j V^k B$ are pairwise disjoint.

It suffices to prove the theorem in the case that $\lambda = \frac{1}{N}$ for a given integer $N \geq 2$. Let $\epsilon = \frac{1}{2} |\Omega|$. Let $B$ and $E$ be the sets obtained by applying the above lemma for the pair of measure-preserving transformations $T = V^{-1}$ and $W = U^{-1}$. Let $f(x) = \chi_B(x)$ and define $X_N$ by
\[
X_N = \bigcup_{j,k < N} T^j W^k B.
\]

Let $x \in X_N$. Then $x \in T^n W^m B$ for some $m, n$ such that $mn < N$. So
\[ M_{UV} f(x) \geq \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} f(U^j V^k x) \]
\[ \geq \frac{1}{mn} f(U^m V^n x) = \frac{1}{mn} \chi_B(U^m V^n x) \]
\[ = \frac{1}{mn} \chi_{T^m W^n B}(x) = \frac{1}{mn} > \frac{1}{N} . \]

We then have
\[ X_N \subset \left\{ x \in \Omega : M_{UV} f(x) > \frac{1}{N} \right\} \]
and
\[ \left| \left\{ x \in \Omega : M_{UV} f(x) > \frac{1}{N} \right\} \right| \geq |X_N| = \left| \bigcup_{jk<N} T^j W^k B \right| \]
\[ = \sum_{jk<N} |T^j W^k B| = |B| \sum_{jk<N} 1 \]
\[ \geq c|B| N \log N . \]

Hence
\[ |\{ x \in \Omega : M_{UV} f(x) > \lambda \}| \geq c \int_{\Omega} \left| \frac{f}{\lambda} \left( 1 + \log^+ \frac{|f|}{\lambda} \right) \right| , \]
as desired. \( \Box \)

3. Sharpness of the Dunford Convergence Result

Before we prove the sharpness of the Dunford \( L \log L(\Omega) \) convergence result in the context of non-periodic commuting transformations, it will be convenient for us to first dispense with some preliminary considerations involving the dyadic strong maximal operator \( M^* \) acting on \( L^1(Q) \). \( M^* \) is the geometric maximal operator associated to dyadic rectangles in \( \mathbb{R}^2 \) whose sides are parallel to the axes, i.e.
\[ M^* f(x, y) = \sup_{j, k} \frac{1}{2^{m+n}} \int_{(j+1)2^m}^{(j+2)2^m} \int_{(k+1)2^n}^{(k+2)2^n} |f(u, v)| \, du \, dv . \]

Lemma 2. Let \( 0 < \lambda < \frac{1}{100}, \epsilon > 0 \). Then there exist a dyadic integer \( N_{\lambda, \epsilon} \) and sets \( A_{\lambda, \epsilon}, S_{\lambda, \epsilon} \) in \( Q \) satisfying the following:

i) \( A_{\lambda, \epsilon} \) and \( S_{\lambda, \epsilon} \) are both unions of dyadic squares of sidelength \( N_{\lambda, \epsilon}^{-1} \).

ii) \( |A_{\lambda, \epsilon}| \leq \frac{100 \lambda}{\log(\frac{1}{\lambda})} \).
iii) $|S_{\lambda, \epsilon}| > 1 - \epsilon$.

iv) $M^* \chi_{A_{\lambda, \epsilon}} > \lambda$ on $S_{\lambda, \epsilon}$.

Moreover, for any $(x, y) \in S_{\lambda, \epsilon}$ there exists a (not necessarily dyadic) rectangle $R(x, y)$ containing $(x, y)$ such that

$$\frac{1}{|R(x, y)|} \int_{R(x, y)} \chi_{A_{\lambda, \epsilon}} > \lambda$$

and such that $R(x, y)$ is a union of dyadic squares of sidelength $N_{\lambda, \epsilon}^{-1}$, all of whom intersect the rectangle $[0, x] \times [0, y]$.

**Proof.** Let $N_\lambda$ be the positive integer satisfying $2^{-N_\lambda-1} \leq \lambda < 2^{-N_\lambda}$. Let $A_\lambda^1 = [0, 2^{-N_\lambda}] \times [0, 2^{-N_\lambda}]$ and let $S_\lambda^1 = \{(x, y) \in Q : M^* \chi_{A_\lambda^1}(x, y) > \lambda\}$. Note that

$$S_\lambda^1 = \bigcup_{j=1}^{N_\lambda+1} ([0, 2^{j-1}2^{-N_\lambda}] \times [0, 2^{1-j}])$$

and $|S_\lambda^1| = 2^{-N_\lambda} \left[1 + \frac{N_\lambda}{2}\right]$. Define $\gamma_\lambda$ by

$$\gamma_\lambda = \frac{1 - |S_\lambda^1|}{2^{-2N_\lambda}} = \frac{1 - 2^{-N_\lambda} \left[1 + \frac{N_\lambda}{2}\right]}{2^{-2N_\lambda}}.$$ 

Note that $Q - S_\lambda^1 = \bigcup_{j=1}^{N_\lambda} Q_{\lambda,j}$, where $\{Q_{\lambda,j}\}_{j=1}^{N_\lambda}$ is an a.e. pairwise disjoint collection of dyadic squares of sidelength $2^{-N_\lambda}$. For $j = 1, \ldots, \gamma_\lambda$, let $h_{\lambda,j} : Q \to Q_{\lambda,j}$ be the homothecy mapping $Q$ onto $Q_{\lambda,j}$.

Let $A_\lambda^2 = A_\lambda^1 \cup \left(\bigcup_{j=1}^{\gamma_\lambda} h_{\lambda,j}(A_\lambda^1)\right)$ and $S_\lambda^2 = S_\lambda^1 \cup \left(\bigcup_{j=1}^{\gamma_\lambda} h_{\lambda,j}(S_\lambda^1)\right)$. Note that $M^* \chi_{A_\lambda^2} > \lambda$ on $S_\lambda^2$, $|A_\lambda^2| = |A_\lambda^1| + \gamma_\lambda \cdot 2^{-2N_\lambda} |A_\lambda^1|$, and $|S_\lambda^2| = |S_\lambda^1| + \gamma_\lambda \cdot 2^{-2N_\lambda} |S_\lambda^1|$. Proceeding by induction, we may define $A_\lambda^n$ and $S_\lambda^n$ in terms of $A_\lambda$, $S_\lambda$ by

$$A_\lambda^{n+1} = A_\lambda^n \cup \left(\bigcup_{j=1}^{\gamma_\lambda} h_{\lambda,j}A_\lambda^n\right)$$

and

$$S_\lambda^{n+1} = S_\lambda^n \cup \left(\bigcup_{j=1}^{\gamma_\lambda} h_{\lambda,j}S_\lambda^n\right).$$

Observe that $M^* \chi_{A_\lambda^{n+1}} > \lambda$ on $S_\lambda^{n+1}$ for each $n$,

$$|A_\lambda^n| = |A_\lambda^1| \left(1 + \gamma_\lambda \cdot 2^{-2N_\lambda} + \ldots + (\gamma_\lambda \cdot 2^{-2N_\lambda})^{n-1}\right)$$

and

$$|S_\lambda^n| = |S_\lambda^1| \left(1 + \gamma_\lambda \cdot 2^{-2N_\lambda} + \ldots + (\gamma_\lambda \cdot 2^{-2N_\lambda})^{n-1}\right).$$

To facilitate understanding we illustrate the above formulas for $|A_\lambda^n|$, $|S_\lambda^n|$ in the special case that $\lambda = \frac{1}{10}$ for $n = 1, 2, 3$. For $\lambda = \frac{1}{10}$ we have
$N_\lambda = 3$. $A_{1/10}^1$ is simply a square of sidelength $\frac{1}{8}$ at the lower left hand corner of $Q$, and $S_{1/10}^1$ is a classic dyadic “Bohr staircase” in $Q$ with steps of heights $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}$, and $1$. $A_{1/10}^1$ and $S_{1/10}^1$ are illustrated in Figure 1, where $A_{1/10}^1$ is shaded in black and $S_{1/10}^1$ is shaded in gray.

Note in the above figure that the complement of gray area representing $S_{1/10}^1$ can be expressed as a union of dyadic squares of sidelength $\frac{1}{8}$. There are $4 + 2(6) + 4(7) = 44$ such squares, and we set $\gamma_{1/10}$ to be $44$ and label the squares as $Q_{1/10,1}, \ldots, Q_{1/10,44}$. For $j = 1, \ldots, 44$ we set $h_{1/10,j}$ to be the homothecy mapping $Q$ onto the square $Q_{1/10,j}$. Now $A_{1/10}^2$ is the union of $A_{1/10}^1$ and the $\gamma_{1/10}$ sets of the form $h_{1/10,j}A_{1/10}^1$. Each $h_{1/10,j}A_{1/10}^1$ has measure $\frac{1}{64} \cdot \frac{1}{64} = \left| A_{1/10}^1 \right| \cdot 2^{-2N_{1/10}}$. Hence the total measure of $A_{1/10}^2$ is $\left| A_{1/10}^1 \right| + 44 \left| A_{1/10}^1 \right| \cdot 2^{-2N_{1/10}} = \left| A_{1/10}^1 \right| \left(1 + \gamma_{1/10}2^{-2N_{1/10}}\right)$. Similarly, $S_{1/10}^2$ is the union of $S_{1/10}^1$ and all the sets of the form $h_{1/10,j}S_{1/10}^1$. Each $h_{1/10,j}S_{1/10}^1$ has measure $\frac{1}{64} \cdot \left| S_{1/10}^1 \right|$, and the total measure of $S_{1/10}^2$ is $\left| S_{1/10}^1 \right| \left(1 + \gamma_{1/10}2^{-2N_{1/10}}\right)$. Figure 2 provides a schematic of the sets $A_{1/10}^2$ and $S_{1/10}^2$, where $A_{1/10}^2$ is shaded in black and $S_{1/10}^2$ is shaded in gray.

**Figure 1.** The sets $A_{1/10}^1$ and $S_{1/10}^1$
The construction of the sets \( A_{1/10}^3 \) and \( S_{1/10}^3 \) from \( A_{1/10}^2 \) and \( S_{1/10}^2 \) proceeds essentially in the same way that \( A_{1/10}^2 \) and \( S_{1/10}^2 \) were constructed from \( A_{1/10}^1 \) and \( S_{1/10}^1 \). Note that the intersection of \( A_{1/10}^2 \) with each of the 44 dyadic squares comprising the complement of the Bohr staircase \( S_{1/10}^1 \) is a homothetic copy of \( A_{1/10}^1 \) itself. \( A_{1/10}^3 \) will contain the square \( A_{1/10}^1 \) but be such that its intersection with each of the 44 dyadic squares will contain a homothetic copy of, not \( A_{1/10}^1 \), but the slightly larger \( A_{1/10}^2 \). The measure of \( A_{1/10}^3 \) is the measure of \( A_{1/10}^2 \) plus the measure of what has been added in the 44 dyadic squares. The additional measure within each dyadic square is the ratio of the measure of the dyadic square to that of \( Q \) (i.e. \( 2^{-2N_{1/10}} \)) times the difference in measure of \( A_{1/10}^2 \) and \( A_{1/10}^1 \). Hence the measure of \( A_{1/10}^3 \) is

\[
|A_{1/10}^3| = |A_{1/10}^1| \left( 1 + \gamma_{1/10} 2^{-2N_{1/10}} \right) + \gamma_{1/10} 2^{-N_{1/10}} \left( |A_{1/10}^2| \left( 1 + \gamma_{1/10} 2^{-2N_{1/10}} \right) - |A_{1/10}^1| \right) 
\]

Using the formula for \( A_{1/10}^2 \) above, we then get

\[
|A_{1/10}^3| = |A_{1/10}^1| \left( 1 + \gamma_{1/10} 2^{-2N_{1/10}} \right) + \gamma_{1/10} 2^{-N_{1/10}} \left( |A_{1/10}^1| \left( 1 + \gamma_{1/10} 2^{-2N_{1/10}} \right) - |A_{1/10}^1| \right) 
= |A_{1/10}^1| \left( 1 + \gamma_{1/10} 2^{-2N_{1/10}} + \left( \gamma_{1/10} 2^{-2N_{1/10}} \right)^2 \right). 
\]
The construction for $B_{1/10}^3$ and the formula for its measure follow along similar lines. Figure 3 provides a schematic of the sets $A_{1/10}^3$ and $S_{1/10}^3$, where $A_{1/10}^3$ is shaded in black and $S_{1/10}^3$ is shaded in gray.

**Figure 3.** The sets $A_{1/10}^3, S_{1/10}^3$

Getting back to the proof, note that

$$\lim_{n \to \infty} |S_{1/10}^n| = |S_{1/10}^1| \frac{1}{1 - \gamma_{1/10} \cdot 2^{-2N_{1/10}}} = 1,$$

so there exists a positive integer $M_{1/10}$ such that $|S_{1/10}^{M_{1/10}, \epsilon}| > 1 - \epsilon$. Observe that $A_{1/10}^{M_{1/10}, \epsilon}$ and $S_{1/10}^{M_{1/10}, \epsilon}$ are both unions of pairwise a.e. disjoint dyadic squares of
sidelength \((2^{-N_\lambda})^{M_\lambda, \epsilon}\), a length we will now denote by \(N_{\lambda, \epsilon}^{-1}\). Moreover,

\[
\left| A^{M_\lambda, \epsilon}_\lambda \right| \leq |A^1_\lambda| \left( 1 + \gamma_\lambda \cdot 2^{-2N_\lambda} + \cdots + \left( \gamma_\lambda \cdot 2^{-2N_\lambda} \right)^{M_\lambda, \epsilon} \right)
\]

\[
\leq |A^1_\lambda| \frac{1}{1 - \gamma_\lambda \cdot 2^{-2N_\lambda}}
\]

\[
= 2^{-2N_\lambda} \frac{1}{1 - \frac{1 - 2^{-N_\lambda} \left( 1 + \frac{N_\lambda}{2} \right)}{2^{-2N_\lambda}} \cdot 2^{-2N_\lambda}}
\]

\[
= \frac{2^{-N_\lambda}}{1 + \frac{N_\lambda}{2}}
\]

\[
\leq \frac{100\lambda}{\log \left( \frac{1}{\lambda} \right)}.
\]

Denoting \(A^{M_\lambda, \epsilon}_\lambda\) by \(A_{\lambda, \epsilon}\) and \(S^{M_\lambda, \epsilon}_\lambda\) by \(S_{\lambda, \epsilon}\), we see that \(i\), \(ii\), and \(iii\) of the lemma hold. It remains to show \(iv\). Note that the fact that \(M^*\chi_{A_{\lambda, \epsilon}} > \lambda\) on \(S_{\lambda, \epsilon}\) has already been established. Now, suppose \((x, y) \in S_{\lambda, \epsilon}\). Let \(\gamma(x, y)\) be the smallest positive integer such that \((x, y) \in S^1_{\lambda, \epsilon}\) itself. If \(\gamma(x, y)\) happens to be 1 (i.e. \((x, y) \in S^1_{\lambda}\) itself), \(R_{(x, y)}\) is simply set to be the union of all dyadic squares in \(Q\) of sidelength \(N_{\lambda, \epsilon}^{-1}\) intersecting \([0, x] \times [0, y]\). Alternatively, and assuming without loss of generality that \((x, y)\) does not lie on the boundary of a dyadic square in \(Q\), if \(\gamma(x, y) > 1\) then \((x, y)\) lies in the interior of a dyadic square \(Q_{(x, y)}\) of sidelength \(2^{-N_\lambda \gamma(x, y)}\). Let \(h_{(x, y)}\) be the homothecy mapping \(Q\) onto \(Q_{(x, y)}\). Note that \(h_{(x, y)}^{-1}(x, y) \in S^1_{\lambda}\). Hence we see in this case the desired rectangle \(R_{(x, y)}\) is given by

\[
R_{(x, y)} = h_{(x, y)} \left( R_{h_{(x, y)}^{-1}(x, y)} \right),
\]

completing the proof of the lemma. \(\square\)

By standard methods of transference we now obtain an ergodic analogue of Lemma 2.

**Lemma 3.** Let \(U\) and \(V\) be two commuting non-periodic measure-preserving transformations of a measure space \(\Omega\) of finite measure onto itself. Let \(0 < \lambda < \frac{1}{100}, 0 < \epsilon < 1\). Then there exists a set \(\tilde{A}_{\lambda, \epsilon} \subset \Omega\) such that

\(i)\) \(M_{UV}\chi_{\tilde{A}_{\lambda, \epsilon}} > \lambda\) on \(\Omega\) except on a set of measure less than \(1 - (1 - \epsilon)^2\)

and

\(ii)\) \(\left| \tilde{A}_{\lambda, \epsilon} \right| \leq \frac{100\lambda}{\log \left( \frac{1}{\epsilon} \right)} |\Omega|.
\)
We assume without loss of generality that $|\Omega| = 1$. Let $N_{\lambda, \epsilon}$ be as in Lemma 2. Applying Lemma 1 in the context of the operators $T = V^{-1}$ and $W = U^{-1}$, we obtain sets $B_{\lambda, \epsilon}$ and $E_{\lambda, \epsilon}$ in $\Omega$ such that $|E_{\lambda, \epsilon}| < \epsilon$ and

$$\Omega = \left( \bigcup_{j, k=1}^{N_{\lambda, \epsilon}} W^j T^k B_{\lambda, \epsilon} \right) \cup E_{\lambda, \epsilon},$$

where the $W^j T^k B_{\lambda, \epsilon}$ are pairwise a.e. disjoint.

We now are going to define a collection of translation operators $\tau_{jk}$ mapping the set of dyadic squares in $\mathbb{R}^2$ of sidelength $N_{\lambda, \epsilon}^{-1}$ to itself. In particular, we define the $\tau_{jk}$ by

$$\tau_{jk} \left( [m N_{\lambda, \epsilon}^{-1}, (m+1) N_{\lambda, \epsilon}^{-1}] \times [n N_{\lambda, \epsilon}^{-1}, (n+1) N_{\lambda, \epsilon}^{-1}] \right) = [(m+j) N_{\lambda, \epsilon}^{-1}, (m+j+1) N_{\lambda, \epsilon}^{-1}] \times [(n+k) N_{\lambda, \epsilon}^{-1}, (n+k+1) N_{\lambda, \epsilon}^{-1}].$$

Now, there exist a positive integer $K_{\lambda, \epsilon}$ and a collection of nonnegative integers $j_1, \ldots, j_{K_{\lambda, \epsilon}}, k_1, \ldots, k_{K_{\lambda, \epsilon}}$ such that $0 \leq j_i \leq N_{\lambda, \epsilon}$, $0 \leq k_i \leq N_{\lambda, \epsilon}$, and

$$A_{\lambda, \epsilon} = K_{\lambda, \epsilon} \bigcup_{i=1}^{K_{\lambda, \epsilon}} \tau_{j_i k_i} \left( [0, N_{\lambda, \epsilon}^{-1}] \times [0, N_{\lambda, \epsilon}^{-1}] \right),$$

where the above union is over pairwise a.e. disjoint sets and $A_{\lambda, \epsilon}$ is as in Lemma 2.

Let now

$$\tilde{A}_{\lambda, \epsilon} = \bigcup_{i=1}^{K_{\lambda, \epsilon}} W^{j_i} T^{k_i} B_{\lambda, \epsilon}.$$

Now $|A_{\lambda, \epsilon}| \leq \frac{100\lambda}{\log \left( \frac{1}{\chi} \right)}$ by Lemma 2. The sets $\tau_{j_i k_i} \left( [0, N_{\lambda, \epsilon}^{-1}] \times [0, N_{\lambda, \epsilon}^{-1}] \right)$ being pairwise a.e. disjoint for $i = 1, \ldots, K_{\lambda, \epsilon}$, we must have

$$K_{\lambda, \epsilon} N_{\lambda, \epsilon}^{-2} \leq \frac{100\lambda}{\log \left( \frac{1}{\chi} \right)}.$$

Hence

$$|\tilde{A}_{\lambda, \epsilon}| \leq \frac{100\lambda}{\log \left( \frac{1}{\chi} \right)} N_{\lambda, \epsilon}^2 \frac{1}{N_{\lambda, \epsilon}^2} \leq \frac{100\lambda}{\log \left( \frac{1}{\chi} \right)}.$$

So $ii$ is satisfied.

Let $S_{\lambda, \epsilon}$ be as in Lemma 2. There exist a positive integer $L_{\lambda, \epsilon}$ and a collection of nonnegative integers $r_1, \ldots, r_{L_{\lambda, \epsilon}}, s_1, \ldots, s_{L_{\lambda, \epsilon}}$ such that $0 \leq r_i \leq N_{\lambda, \epsilon}$, $0 \leq s_i \leq N_{\lambda, \epsilon}$, and

$$S_{\lambda, \epsilon} = \bigcup_{i=1}^{L_{\lambda, \epsilon}} \tau_{r_i s_i} \left( [0, N_{\lambda, \epsilon}^{-1}] \times [0, N_{\lambda, \epsilon}^{-1}] \right).$$
where the above union is over a pairwise a.e. disjoint collection of dyadic squares.

Let now
\[ \tilde{S}_{\lambda, \epsilon} = \bigcup_{i=1}^{L_{\lambda, \epsilon}} W^{r_i} T^{s_i} B_{\lambda, \epsilon} \]

Since \(|S_{\lambda, \epsilon}| > 1 - \epsilon\) by Lemma 2 and the sets \(\tau_{r_i s_i} ([0, N^{-1}_{\lambda, \epsilon}] \times [0, N^{-1}_{\lambda, \epsilon}])\) are pairwise a.e. disjoint we must have \(L_{\lambda, \epsilon} N^{-2}_{\lambda, \epsilon} > 1 - \epsilon\). The sets \(W^{r_j} B_{\lambda, \epsilon}\) being a.e. disjoint and the fact that
\[ \left| \bigcup_{i,j=1}^{N_{\lambda, \epsilon}} W^{r_i} T^{s_j} B_{\lambda, \epsilon} \right| > 1 - \epsilon \]

imply that \(N^{-2}_{\lambda, \epsilon} |B_{\lambda, \epsilon}| > 1 - \epsilon\). Hence
\[
\begin{align*}
|\tilde{S}_{\lambda, \epsilon}| & \geq L_{\lambda, \epsilon} |B_{\lambda, \epsilon}| \\
& \geq (1 - \epsilon) N^{-2}_{\lambda, \epsilon} |B_{\lambda, \epsilon}| \\
& > (1 - \epsilon)^2 .
\end{align*}
\]

It remains to show \(M_{UV} \chi_{\tilde{A}_{\lambda, \epsilon}} > \lambda \) on \(\tilde{S}_{\lambda, \epsilon}\). Well, suppose \(p \in \tilde{S}_{\lambda, \epsilon}\). Then for some \(1 \leq r, s \leq L_{\lambda, \epsilon}\), \(p \in W^{r} T^{s} B_{\lambda, \epsilon}\). Now, since \(W^{r} T^{s} B_{\lambda, \epsilon} \subset \tilde{S}_{\lambda, \epsilon}\), \(\tau_{r s} ([0, N^{-1}_{\lambda, \epsilon}] \times [0, N^{-1}_{\lambda, \epsilon}]) \subset S_{\lambda, \epsilon}\). By Lemma 2, there exists a rectangle \(R_{rs}\) of the form
\[ R_{rs} = \bigcup_{j=r}^{r} \bigcup_{k=s}^{s} \tau_{j k} ([0, N^{-1}_{\lambda, \epsilon}] \times [0, N^{-1}_{\lambda, \epsilon}]) \]
such that \(\frac{1}{|R_{rs}|} \int_{R_{rs}} \chi_{A_{\lambda, \epsilon}} > \lambda\). Hence
\[
M_{UV} \chi_{\tilde{A}_{\lambda, \epsilon}} (p) \geq \frac{1}{(r - j_{rs} + 1) (s - k_{rs} + 1)} \sum_{j=0}^{r-j_{rs}} \sum_{k=0}^{s-k_{rs}} \chi_{\tilde{A}_{\lambda, \epsilon}} (U^j V^k p) \\
= \frac{1}{(r - j_{rs} + 1) (s - k_{rs} + 1)} \sum_{j=0}^{r-j_{rs}} \sum_{k=0}^{s-k_{rs}} |\tilde{A}_{\lambda, \epsilon} \cap U^j V^k W^{r} T^{s} B_{\lambda, \epsilon}| |B_{\lambda, \epsilon}|^{-1} \\
= \frac{1}{(r - j_{rs} + 1) (s - k_{rs} + 1)} \sum_{j=0}^{r-j_{rs}} \sum_{k=0}^{s-k_{rs}} |\tilde{A}_{\lambda, \epsilon} \cap W^{r-j} T^{s-k} B_{\lambda, \epsilon}| |B_{\lambda, \epsilon}|^{-1} \\
= \frac{1}{(r - j_{rs} + 1) (s - k_{rs} + 1)} \sum_{j=r_{rs}}^{r} \sum_{k=k_{rs}}^{s} |\tilde{A}_{\lambda, \epsilon} \cap W^{j} T^{k} B_{\lambda, \epsilon}| |B_{\lambda, \epsilon}|^{-1}
\]
\[
\begin{align*}
&= \frac{1}{(r - j_{rs} + 1)(s - k_{rs} + 1)} \sum_{j=j_{rs}}^{r} \sum_{k=k_{rs}}^{s} \frac{|A_{\lambda, \epsilon} \cap \tau_{jk} ([0, N_{\lambda, \epsilon}^{-1}] \times [0, N_{\lambda, \epsilon}^{-1}])|}{N_{\lambda, \epsilon}^{-2}} \\
&\geq \frac{1}{|R_{rs}|} \int_{R_{rs}} \chi_{A_{\lambda, \epsilon}} \\
&> \lambda,
\end{align*}
\]

as desired. \(\square\)

**Theorem 3.** Let \(U\) and \(V\) be a commuting pair of non-periodic measure-preserving transformations of a measure space \(\Omega\) of finite measure onto itself. Let \(\phi\) be a positive increasing function on \([0, \infty)\) that is \(o(\log x)\) as \(x \to \infty\). Then there exists a function \(f \in L\phi(L)(\Omega)\) such that

\[
\lim_{m,n \to \infty} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f(U^j V^k x)
\]

does not exist for a.e. \(x\) in \(\Omega\).

**Proof.** We may assume without loss of generality that \(|\Omega| = 1\). For each positive integer \(n\), choose \(0 < \lambda_n < \frac{1}{100}\) such that

\[
\frac{\phi \left( \frac{n}{\lambda_n} \right)}{\log \left( \frac{1}{\lambda_n} \right)} < \frac{1}{n \cdot 2^n}.
\]

Note that such a \(\lambda_n\) exists since \(\phi(x) = o(\log x)\) as \(x \to \infty\). By Lemma 3, there exists a set \(E_n \subset \Omega\) such that \(M_{UV} \chi_{E_n} > \lambda_n\) on \(\Omega\) except on a set of measure less than \(\frac{1}{n^2}\), where \(|E_n| \leq \frac{100 \lambda_n}{\log \left( \frac{1}{\lambda_n} \right)}\).

Let now \(f_n = \frac{n}{\lambda_n} \chi_{E_n}\). Note that \(M_{UV} f_n > n\) on \(\Omega\) except on a set of measure less than \(\frac{1}{n^2}\). Moreover,

\[
\int_{\Omega} f_n \phi(f_n) = |E_n| \cdot \frac{n}{\lambda_n} \phi \left( \frac{n}{\lambda_n} \right)
\]

\[
\leq \frac{100 \lambda_n}{\log \left( \frac{1}{\lambda_n} \right)} \cdot \frac{n}{\lambda_n} \phi \left( \frac{n}{\lambda_n} \right)
\]

\[
\leq \frac{n \phi \left( \frac{n}{\lambda_n} \right)}{\log \left( \frac{1}{\lambda_n} \right)}
\]

\[
< \frac{1}{2^n}.
\]

Set now \(f = \sup_n f_n\). Observe that \(M_{UV} f = \infty\) a.e. on \(\Omega\) and hence for a.e. \(x\) in \(\Omega\) there exist sequences of positive integers \(j_{x,1}, j_{x,2}, j_{x,3}, \ldots\),
tending to infinity such that
\[
\lim_{n \to \infty} \frac{1}{j_{x,n} k_{x,n}} \sum_{j=0}^{j_{x,n}-1} \sum_{k=0}^{k_{x,n}-1} f(U^j V^k x) = \infty.
\]
Moreover, \( f \in L\phi(L)(\Omega) \) since
\[
\sum_{n=1}^{\infty} \int_{\Omega} f_n \phi(f_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.
\]
As accordingly \( f \in L^1(\Omega) \) we also have
\[
\int_{\Omega} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f(U^j V^k) \leq \|f\|_{L^1(\Omega)}
\]
for all positive integers \( m, n \), and hence for a.e. \( x \in \Omega \)
\[
\lim_{m,n \to \infty} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f(U^j V^k x) = \infty
\]
does not hold (even though \( \limsup_{m,n \to \infty} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f(U^j V^k x) = \infty \)).

The theorem follows. \( \square \)

4. Concluding Remarks

The conceptual framework for the proofs of the above theorems readily extends to higher dimensions. If \( U_1, \ldots, U_n \) are one-to-one measure preserving maps of a measure space \( \Omega \) of finite measure to itself, the associated ergodic maximal operator \( M_{U_1,\ldots,U_n} \) is given by
\[
M_{U_1,\ldots,U_n} f(x) = \sup_{m_1,\ldots,m_n} \frac{1}{m_1 \cdots m_n} \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} |f(U_{j_1} \cdots U_{j_n} x)|.
\]
Fava showed in [3] that \( M_{U_1,\ldots,U_n} \) satisfies the weak type estimate
\[
|\{x \in \Omega : M_{U_1,\ldots,U_n} f(x) > \lambda\}| \leq C \int_{\Omega} \frac{|f|}{\lambda} \left(1 + \log^+ \frac{|f|}{\lambda}\right)^{n-1}.
\]
Generalizing Dunford’s two-dimensional convergence result, Zygmund proved in [9] that
\[
\lim_{m_1,\ldots,m_n \to \infty} \frac{1}{m_1 \cdots m_n} \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} f(U_{j_1} \cdots U_{j_n} x)
\]
converges a.e. in \( \Omega \) provided \( f \in L(\log L)^{n-1}(\Omega) \).
Continuing to follow the terminology of Katznelson and Weiss in [7], we say that a collection $U_1, \ldots, U_n$ of commuting measure-preserving transformations is non-periodic if for any $(m_1, \ldots, m_n) \neq (0, 0, \ldots, 0), (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we have

$$\left| \{ x \in \Omega : U_1^{m_1} \cdots U_n^{m_n} x = x \} \right| = 0 .$$

This extended notion of non-periodicity enables the following higher-dimensional analogue of Theorem 3.

**Theorem 4.** Let $U_1, \ldots, U_n$ be a collection of commuting non-periodic measure-preserving transformations of a measure space $\Omega$ of finite measure onto itself. Let $\phi$ be a positive increasing function on $[0, \infty)$ that is $o(\log x)^{n-1}$ as $x \to \infty$. Then there exists a function $f \in L\phi(L)(\Omega)$ such that

$$\lim_{m_1, \ldots, m_n \to \infty} \frac{1}{m_1 \cdots m_n} \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} f\left(U_1^{j_1} \cdots U_n^{j_n} x\right)$$

does not exist for a.e. $x$ in $\Omega$.

As the proof follows along the same lines as that of Theorem 3, we leave the details to the interested reader.

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**References**

Department of Mathematics, Baylor University, Waco, Texas 76798
E-mail address: paul.hagelstein@baylor.edu

Georgia Southern University, Department of Mathematical Sciences,
203 Georgia Avenue, Statesboro, Georgia 30460-8093
E-mail address: astokolos@georgiasouthern.edu