ORLICZ BOUNDS FOR OPERATORS OF RESTRICTED WEAK TYPE

PAUL ALTON HAGELSTEIN

Abstract. Let $T$ be a sublinear operator mapping the set of measurable functions supported on the unit circle $\mathbb{T}$ into itself. If $T$ is of restricted weak-type $(1,1)$, it is shown that $T$ is a bounded operator from simple functions in $L \log L (\mathbb{T})$ into weak $L^1 (\mathbb{T})$. Moreover, it is shown that if $T$ is a sublinear translation-invariant operator of restricted weak-type $(1,1)$, then $T$ is a bounded operator from simple functions in $L \log L (\mathbb{T})$ into $L^1 (\mathbb{T})$ itself.

We first recall a few basic definitions.

**Definition 1.** An operator $T$ mapping the set of measurable real-valued functions on a measure space to itself is said to be sublinear if and only if for any measurable functions $f$ and $g$, any real number $\alpha$, and any point $p$ in the measure space,

$$|T(f + g)(p)| \leq |Tf(p)| + |Tg(p)|$$

and

$$|T(\alpha f)(p)| = |\alpha| |Tf(p)|.$$

**Definition 2.** The Orlicz class $L \log L (\mathbb{T})$ is the space of measurable functions $f$ supported on the unit circle $\mathbb{T}$ such that

$$\|f\|_{L \log L (\mathbb{T})} = \inf \left\{ c > 0 : \int_{\mathbb{T}} \frac{|f|}{c} \log \left(2 + \frac{|f|}{c}\right) \leq 1 \right\} < \infty.$$

$L \log L ([0, 1])$ is similarly defined.

**Definition 3.** An operator $T$ mapping the space of measurable functions supported on a measure space to itself is said to be of restricted weak-type $(1,1)$ if and only if there exists a finite constant $C$ such that, for every measurable subset $E$ of the measure space and every $\alpha > 0$,

$$|\{x : |T\chi_E(x)| > \alpha\}| \leq C \frac{|E|}{\alpha}.$$

In this paper, we will show that if a sublinear operator $T$ mapping the space of measurable functions on $\mathbb{T}$ to itself is of restricted weak-type $(1,1)$,
then $T$ maps simple functions in $L \log L (\mathbb{T})$ boundedly into weak $L^1 (\mathbb{T})$. If additionally the operator $T$ is translation-invariant, we will show that $T$ maps simple functions in $L \log L (\mathbb{T})$ boundedly into $L^1 (\mathbb{T})$ itself.

The above two results, although fundamental in nature, surprisingly seem to be new. The first result relies on well-known techniques involving Lorentz spaces and a celebrated theorem of E. M. Stein and Norman Weiss involving sums of functions in weak-$L^1 (\mathbb{T})$ [7]. However, due to the very general conditions placed on our operator $T$ here, an auxiliary device such as Lemma 7 is necessary to complete the proof. The proof of the second result involves a circle of ideas originating from E. M. Stein’s work on limits of sequences of translation-invariant operators [5] and developed further by Colzani, Sawyer, and others (see, for example, [1], [2]). Again, the very general conditions placed on the operators we are considering here do not enable our second theorem to be a direct corollary of results currently in the literature; the use of the sublinearity of $T$ and the Rademacher functions in Equation 2 is the key new ingredient for obtaining the desired proof.

We will estimate the $L \log L$ norms of functions by utilizing the Hardy-Littlewood maximal operator, defined as follows:

**Definition 4.** Let $f$ be a measurable function supported on $[0, 1]$. The Hardy-Littlewood maximal function of $f$ is defined by

$$M_{HL} f (x) = \sup_{x_1 < x < x_2} \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} |f(t)| \, dt.$$  

The following lemma of E. M. Stein helps us relate the $L \log L$ norm of a function to its associated Hardy-Littlewood maximal function.

**Lemma 1 ( [3]).** There exist universal constants $0 < c, C < \infty$ such that if $f$ is a measurable function supported on $[0, 1]$, then

$$c \int_0^1 M_{HL} f \leq \|f\|_{L^1([0,1])} \leq C \int_0^1 M_{HL} f.$$  

The following is a simple consequence of Lemma 1.

**Lemma 2.** Let $f$ be a nonnegative function supported on $[0, 1]$. Let $f^*$ be a function supported on $[0, 1]$ such that $f^*$ and $f$ are equidistributed and such that $f^*(x) \geq f^*(y)$ whenever $0 \leq x \leq y \leq 1$. Then there exist universal constants $0 < c, C < \infty$ such that

$$c \int_0^1 (-\log t) f^*(t) \, dt \leq \|f\|_{L^1([0,1])} \leq C \int_0^1 (-\log t) f^*(t) \, dt.$$
Proof. As \( \|f\|_{L_{\log} L([0,1])} = \|f^*\|_{L_{\log} L([0,1])} \), it is enough to show that

\[
\int_0^1 (-\log t) \, f^*(t) \, dt \sim \|f^*\|_{L_{\log} L([0,1])}.
\]

Well, by Lemma 1 we have that

\[
\|f^*\|_{L_{\log} L([0,1])} \sim \int_0^1 M_{HL} f^*(t) \, dt.
\]

As

\[
\frac{1}{t} \int_0^t f^*(u) \, du \, dt
\]

\[
= \int_0^1 f^*(u) \int_t^1 \frac{1}{t} \, dt \, du
\]

\[
= \int_0^1 f^*(u) \, (-\log u) \, du,
\]

as desired. □

A simple consequence of the above two lemmas is the following:

**Lemma 3.** Suppose \( f = \sum a_j \chi_{E_j} \) is supported on \([0,1]\), where each \( a_j \) is nonnegative and \( E_1 \supset E_2 \supset \ldots \). Then there exist universal constants \( 0 < c, C < \infty \) such that

\[
c \|f\|_{L_{\log} L([0,1])} \leq \sum a_j |E_j| \log \left( 2 + \frac{1}{|E_j|} \right) \leq C \|f\|_{L_{\log} L([0,1])}.
\]

**Proof.** Let \( f^* \) be as in the statement of Lemma 2. Using Lemmas 1 and 2, we see then that

\[
\|f\|_{L_{\log} L([0,1])} = \|f^*\|_{L_{\log} L([0,1])}
\]

\[
\sim \int_0^1 M_{HL} f^* = \int_0^1 M_{HL} \left( \sum a_j \chi_{[0,|E_j|]} \right)
\]

\[
= \sum a_j \int_0^1 M_{HL} \chi_{[0,|E_j|]}
\]

\[
\sim \sum a_j |E_j| \log \left( 2 + \frac{1}{|E_j|} \right).
\]

□

The following result due to E. M. Stein and Norman Weiss will also be useful to us.

**Lemma 4 ([7]).** Suppose that for \( j = 1, 2, \ldots \), \( g_j(x) \) is a nonnegative function on a measure space for which \( |\{x : g_j(x) > \alpha\}| < \frac{1}{\alpha} \) when \( \alpha > 0 \).
Let \( \{c_j\} \) be a sequence of positive numbers with \( \sum c_j = 1 \), and set \( K = \sum c_j \log \frac{1}{c_j} \). Then if \( \alpha > 0 \),

\[
\left\{ x : \sum_j c_j g_j(x) > \alpha \right\} < \frac{2(K + 2)}{\alpha}.
\]

Using this lemma, we now show that if a sublinear operator \( T \) acting on measurable functions on \( T \) is of restricted weak-type \((1,1)\), then

\[
|\{x : T^g(x) > \alpha\}| \lesssim \|f\|_{L^\log L(T)} \quad \text{in the special case that } f = \sum 2^j \chi_{E_j}, \text{ where } E_1 \supset E_2 \supset E_3 \supset \ldots.
\]

Afterwards we will show a similar result for general simple functions \( f \).

**Lemma 5.** Let \( T \) be a sublinear operator mapping the set of measurable functions on \( T \) to itself. Suppose also that for any measurable subset \( E \) of \( T \) and \( \alpha > 0 \) the inequality \( |\{x : |T \chi_E(x)| > \alpha\}| \leq \frac{|E|}{\alpha} \) holds. Then there exists a universal constant \( c \) such that if \( \alpha > 0 \) and \( f = \sum_{j=1}^N 2^j \chi_{E_j} \), where \( E_1 \supset E_2 \ldots \supset E_N \), then

\[
|\{x : |Tf(x)| > \alpha\}| \leq c \frac{\|f\|_{L^\log L(T)}}{\alpha}.
\]

**Proof.** It suffices to prove the result for functions supported on \([0,1]\) rather than \( T \). We assume without loss of generality that \( \|f\|_{L^1([0,1])} = 1 \).

Note that

\[
\|T(2^j \chi_{E_j})\|_{W^{KL,1}} \leq 2^j |E_j|.
\]

Now, by the sublinearity of \( T \), we have that \( |Tf| \leq \sum |T(2^j \chi_{E_j})| \). Also, \( \sum 2^j |E_j| = 1 \) since \( \|f\|_{L^1([0,1])} = 1 \). So by Lemma 4,

\[
\|Tf\|_{W^{KL,1}} \lesssim - \sum 2^j |E_j| \log (2^j |E_j|) + 1
\]

\[
\lesssim \sum j 2^j |E_j| - \sum 2^j |E_j| \log (|E_j|) + 1.
\]

By Chebyshev’s inequality, \( |E_j| \leq 2^{-j} \), so \( -\log (|E_j|) \gtrsim j \). So

\[
\|Tf\|_{W^{KL,1}} \lesssim - \sum 2^j |E_j| \log (|E_j|) + 1
\]

\[
\sim \sum 2^j |E_j| \log \left( 2 + \frac{1}{|E_j|} \right)
\]

\[
\sim \|f\|_{L^\log L([0,1])}
\]

by Lemma 3. \( \square \)
Lemma 6. Let $T$ be a sublinear operator mapping the set of measurable functions on $\mathbb{T}$ to itself. Suppose also that for any measurable subset $E$ of $\mathbb{T}$ and $\alpha > 0$ the inequality $|\{x : |T\chi_E(x)| > \alpha\}| \leq \frac{|E|}{\alpha}$ holds. Then there exists a universal constant $c$ such that if $\alpha > 0$ and $f = \sum_{j=1}^{N} 2^{j}\chi_{E_j}$, where $E_1, E_2, \ldots, E_N$ are disjoint subsets of $\mathbb{T}$, then

$$|\{x : |Tf(x)| > \alpha\}| \leq c \frac{\|f\|_{L\log L(\mathbb{T})}}{\alpha}. $$

Proof. Let $f_1 = 2 \cdot \chi_{E_1} \cup \ldots \cup E_N$ and, for $2 \leq j \leq N$, $f_j = 2^{j-1} \cdot \chi_{E_j \cup \ldots \cup E_N}$. Note that $f = \sum_{j=1}^{N} f_j$. So

$$|\{x : |Tf(x)| > \alpha\}| \leq \left| \left\{ x : |Tf_1(x)| > \frac{\alpha}{2} \right\} \right| + \left| \left\{ x : T \left( \sum_{j=2}^{N} f_j \right) (x) > \frac{\alpha}{2} \right\} \right|$$

$$\leq \frac{2}{\alpha} \|f_1\|_{L\log L(\mathbb{T})} + \left| \left\{ x : T \left( \sum_{j=2}^{N} f_j \right) (x) > \frac{\alpha}{2} \right\} \right|$$

$$\leq \frac{2}{\alpha} \|f_1\|_{L\log L(\mathbb{T})} + \frac{2}{\alpha} \| \sum_{j=2}^{N} f_j \|_{L\log L(\mathbb{T})} \quad \text{(by Lemma 5)}$$

$$\leq \frac{1}{\alpha} \|f\|_{L\log L(\mathbb{T})}. \quad \Box$$

Lemma 7. Let $x \in [0, 1]$. Then there exists a sequence $\{a_k\}$ with $a_k = 2^{-j_k}$ for nonnegative integers $j_k$ and a sequence $\{\epsilon_k\}$ with each $\epsilon_k \in \{-1, 0, 1\}$ such that

$$x = \sum_{j=1}^{\infty} \epsilon_j a_j,$$

where $|a_j| \leq 4 \cdot 2^{-2j}$. 

Proof. We choose $a_1 \in \{\frac{1}{8}, 1\}$ and $\epsilon_1 \in \{0, 1\}$ such that $|x - \epsilon_1 a_1| \leq \frac{1}{4}$. We designate the product $\epsilon_1 a_1$ by $s_1$. We can now choose $a_2 \in \{\frac{1}{8}, \frac{1}{4}\}$ and $\epsilon_2 \in \{0, -1, 1\}$ such that $|x - s_1 - \epsilon_2 a_2| \leq \frac{1}{16}$. We denote the sum $s_1 + \epsilon_2 a_2$ by $s_2$.

Continuing inductively, we now suppose $s_k$ has been chosen such that $|x - s_k| \leq 2^{-2k}$. We can choose $\epsilon_{k+1} \in \{0, -1, 1\}$ and $a_{k+1} \in \{0, 2^{-2k-1}, 2^{-2k}\}$ such that, letting $s_{k+1} = s_k + \epsilon_{k+1} a_{k+1}$, we have $|x - s_{k+1}| \leq 2^{-2(k+1)}$.

So $x = \sum \epsilon_k a_k$, with $|a_k| \leq 4 \cdot 2^{-2k}. \quad \Box$

Theorem 1. Let $T$ be a sublinear operator mapping the set of measurable functions on $\mathbb{T}$ to itself. Suppose that, for any measurable subset $E$ of $\mathbb{T}$
and \( \alpha > 0 \),
\[
|\{ x \in T : |T \chi_E (x)| > \alpha \}| \leq \frac{|E|}{\alpha}.
\]
Then if \( f \) is a simple function supported on \( T \) and \( \alpha > 0 \),
\[
|\{ x \in T : |T f (x)| > \alpha \}| \leq c \frac{\|f\|_{L \log L(T)}}{\alpha},
\]
where \( c \) is a universal constant.

Proof. Again it suffices to prove the result for \([0,1]\) replacing \( T \). Suppose \( f = \sum_{j=1}^{N} b_j \chi_{B_j} \), where the \( B_j \) are disjoint and the \( b_j \) are, without loss of generality, nonnegative. Select \( M \) such that
\[
\sum_{j=M+1}^{M+N} \frac{2^{j+1}}{2^{2M}} \leq 2.
\]
By the previous lemma, for each integer \( j \) there exist sequences \( \{a_{j,k}\}, \{\epsilon_{j,k}\} \) such that each \( \epsilon_{j,k} \in \{0, -1, 1\} \), \( 0 \leq a_{j,k} = 2^{j_k} \leq b_j \cdot (8 \cdot 2^{-2^k}) \) for appropriate integers \( j_k \), and
\[
\sum_k \epsilon_{j,k} a_{j,k} = b_j.
\]

We now define the functions \( f_k \) by
\[
f_k = \sum_{j=1}^{N} \epsilon_{j,k} a_{j,k} \chi_{B_j}.
\]
Let \( g_k = f_k \) if \( k = 1, \ldots, M \). If \( j = 1, \ldots, N \), define \( g_{M+j} \) by
\[
g_{M+j} = (f - f_1 - f_2 - \ldots - f_M) \chi_{B_j}.
\]
Note that
\[
\|g_k\|_{L \log L([0,1])} \leq 8 \cdot \|f\|_{L \log L([0,1])} \cdot 2^{-2^k} \quad k = 1, \ldots, M
\]
and
\[
\|g_k\|_{L \log L([0,1])} \leq 8 \cdot \|f\|_{L \log L([0,1])} \cdot 2^{-2^M} \quad k = M + 1, \ldots, M + N.
\]
So
\[
\sum_{k=1}^{M+N} g_k = f
\]
and, by Lemma 6,
\[
|\{ x : |T f (x)| > \alpha \}| \leq \sum_{j=1}^{M+N} \left| \left\{ x : |T g_j (x)| > \frac{\alpha}{2^{j+1}} \right\} \right|
\]
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\[ \lesssim \sum_{j=1}^{M} 2^{j+1} \|g_j\|_{L\log L ([0,1])} + \sum_{j=M+1}^{N} 2^{j+1} \|g_j\|_{L\log L ([0,1])} \]

\[ \lesssim \sum_{j=1}^{M} 2^{j+1} \cdot 2^{-2j} \|f\|_{L\log L ([0,1])} + \sum_{j=M+1}^{N} 2^{j+1} \cdot 2^{-2M} \|f\|_{L\log L ([0,1])}. \]

\[ \lesssim \frac{\|f\|_{L\log L ([0,1])}}{\alpha} . \]

\[ \square \]

Note that it is not true that if \( T \) satisfies the conditions of Theorem 1 then \( T \) necessarily maps \( L \log L (\mathbb{T}) \) into \( L^1 (\mathbb{T}) \). For example, we could define \( T \) by

\[ Tf (e^{i\theta}) = \frac{\|f\|_{L^1 (\mathbb{T})}}{\theta} \chi_{\{|\theta| \leq \frac{\pi}{2}\}} . \]

Note that \( T \) is a linear operator of restricted weak-type \((1,1)\) and maps \( L \log L (\mathbb{T}) \) boundedly into weak \( L^1 \). However, \( T \) does not map \( L \log L (\mathbb{T}) \) into \( L^1 (\mathbb{T}) \).

Of course, the operator given above is not translation-invariant. We shall now see that if \( T \) is a sublinear restricted weak-type \((1,1)\) operator which is also translation-invariant, the operator \( T \) must map simple functions in \( L \log L (\mathbb{T}) \) boundedly into \( L^1 (\mathbb{T}) \) itself. We remark that the following theorem is sharp in the sense that if \( T \) is either the Hardy-Littlewood maximal operator or the Hilbert transform acting on measurable functions on \( \mathbb{T} \), the comparability \( \|Tf\|_{L^1 (\mathbb{T})} \sim \|f\|_{L\log L (\mathbb{T})} \) holds.

**Theorem 2.** Let \( T \) be a translation-invariant sublinear operator mapping the set of measurable functions on \( \mathbb{T} \) to itself. Also suppose that for any measurable set \( E \) in \( \mathbb{T} \) and \( \alpha > 0 \) we have

\[ \left| \{ x \in \mathbb{T} : |T \chi_E (x)| > \alpha \} \right| \leq \frac{|E|}{\alpha} . \]

Then if \( f \) is a simple function supported on \( \mathbb{T} \),

\[ \|T (f)\|_{L^1 (\mathbb{T})} \leq c \|f\|_{L\log L (\mathbb{T})} , \]

where \( c \) is a universal constant.

**Proof.** Our method of proof is in large part motivated by the proof of E. M. Stein’s theorem on limits of sequences of operators [5]. Accordingly, we gather the following definitions and lemmas for use.

**Lemma 8 ([5]).** Let \( E_1, E_2, \ldots \) be a collection of sets in \( \mathbb{T} \) such that \( \sum |E_j| = \infty \). Then there exist sets \( F_1, F_2, \ldots \) in \( \mathbb{T} \) such that each \( F_j \) is a translate of \( E_j \) in \( \mathbb{T} \) and such that almost every point of \( \mathbb{T} \) belongs to infinitely many of the sets \( F_n \).
Definition 5. Let $r_n(t)$ denote the Rademacher functions on $\mathbb{R}$, defined by

$$r_n(t) = r_0(2^nt),$$

where $r_0(t) = 1$ if $0 \leq t \leq \frac{1}{2}$, $-1$ if $\frac{1}{2} < t < 1$, and $r_0(t + 1) = r_0(t)$.

Lemma 9 ([4, 9]). Let $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, and let $F(t) = \sum_{n=0}^{N} a_n r_n(t)$ be a Rademacher series. Let $0 < p < \infty$. Then there exist finite, positive constants $A(p)$, $B(p)$ such that

$$A(p) \|F\|_{L^p([0,1])} \leq \left( \sum_{n=0}^{N} |a_n|^2 \right)^{1/2} \leq B(p) \|F\|_{L^p([0,1])}.$$

Lemma 10. Let $T$ be a translation-invariant sublinear operator mapping the set of measurable functions on $T$ to itself. Suppose also for any measurable set $E$ in $T$ and $\alpha > 0$ that

$$\left| \{ x \in T : |T\chi_E(x)| > \alpha \} \right| \leq \frac{|E|}{\alpha}.$$

Then if $f$ is a simple function supported on $T$ and $\alpha > 0$,

$$(1) \quad \left| \{ x \in T : |Tf(x)| > \alpha \} \right| \leq C \left( \frac{\|f\|_{L^2(T)}}{\alpha} \right)^2,$$

where $C$ is a universal constant.

Proof. By contradiction. Suppose (1) were false. Then there would exist a sequence of simple functions $\{f_n\}$ and a sequence of sets $\{E_n\}$ such that

$$|Tf_n(x)| > 1 \text{ for } x \in E_n$$

and

$$|E_n| > n \|f_n\|_{L^2(T)}^2.$$

By taking subcollections of the original collections of $\{f_n\}, \{E_n\}$, with possible repetitions, we may obtain another set of collections, again denoted by $\{f_n\}, \{E_n\}$, such that $|Tf_n(x)| > 1$ if $x \in E_n$, $\sum |E_n| = \infty$, and $\sum \|f_n\|_{L^2(T)}^2 < \infty$.

As $\sum \|f_n\|_{L^2(T)}^2$ converges, we may find a sequence $\{R_n\}$ of positive numbers such that $R_n \to \infty$, but such that $\sum \|R_n f_n\|_{L^2(T)}^2 = D < \infty$.

Now, for each $g \in T$, we let $\tau_g$ denote the translation operator defined by

$$\tau_g f (x) = f (-g + x).$$

As $\sum |E_n| = \infty$, by Lemma 8 we see that there exists a sequence $\{F_n\}$ of sets in $T$ such that each $F_j$ is a translate of $E_j$ in $T$ and such that almost
every point of $\mathbb{T}$ belongs to an infinite number of the sets $F_n$. We associate to each $F_j$ an element $g_j \in \mathbb{T}$ such that 

$$\chi_{F_j} = \tau - g_j \chi_{E_j}.$$ 

Let $M$ be a positive integer. There exists a positive integer $N$ and a subset $S \subset \mathbb{T}$ of measure greater than $1/2$ such that for all $x$ in $S$, there exists an integer $j_x$ such that $1 \leq j_x \leq N$ and 

$$M < \left| R_{j_x} T (\tau_{g_{j_x}} f_{j_x}) (x) \right|.$$ 

Now, define the function $h(x, t)$ on $\mathbb{T} \times [0, 1]$ by 

$$h(x, t) = \sum_{j=1}^{N} R_j \tau_{g_{j}} f_{j_x} (x) r_{j} (t).$$ 

If $g(x, t)$ is a measurable function on $\mathbb{T} \times [0, 1]$, we define $Tg(x, t)$ by 

$$Tg(x, t) = Tg_{t}(x),$$ 

where $g_{t}(x) = g(x, t)$.

Now, let $x_0 \in S$. For some $j$ such that $1 \leq j \leq N$ we have that 

$$\left| R_j T (\tau_{g_{j}} f_{j}) (x_0) \right| > M.$$ 

We assume without loss of generality that $j = 1$.

Now, if $0 < t < 1$ and $t$ is not of the form $2^j$ for some integer $j$, the sublinearity of $T$ implies that 

$$M < \left| T (R_1 \tau_{g_{1}} f_{1}) (x_0) \right| \leq \frac{1}{2} \left[ T \left( R_1 \tau_{g_{1}} f_{1} (x) + \sum_{j=2}^{N} R_j \tau_{g_{j}} f_{j_x} (x) r_{j} (t) \right) (x_0) \right]$$

$$+ \left| T \left( R_1 \tau_{g_{1}} f_{1} (x) + \sum_{j=2}^{N} R_j \tau_{g_{j}} f_{j_x} (x) r_{j} (1 - t) \right) (x_0) \right|.$$

So $\{|t \in [0, 1] : |Th(x_0, t)| > M\} \geq 1/4$. As $|S| > \frac{1}{2}$, we then have that

$$\{|(x, t) \in \mathbb{T} \times [0, 1] : |Th(x, t)| > M\} \geq \frac{1}{8}.$$

Note that Lemma 9 implies that 

$$\|h\|_{L^2(\mathbb{T} \times [0, 1])}^2 = \int_{x \in \mathbb{T}} \int_{t=0}^{1} \left( \sum_{j=1}^{N} R_j \tau_{g_{j}} f_{j_x} (x) r_{j} (t) \right)^2 \, dt \, dx$$

$$\leq (A(2))^{-2} \int_{x \in \mathbb{T}} \sum_{j=1}^{N} |R_j \tau_{g_{j}} f_{j_x} (x)|^2 \, dx.$$. 

\[
= (A(2))^{-2} \sum_{j=1}^{N} \| R_j \tau_{g_j} f_j (x) \|^2_{L^2(T)}
= (A(2))^{-2} \sum_{j=1}^{N} \| R_j f_j (x) \|^2_{L^2(T)}
\leq (A(2))^{-1} \cdot D < \infty.
\]

For our notational convenience, if \( L^\Phi \) is a normed space on \([0, 1]\) and \( L^\Psi \) is a normed space on \( \mathbb{T} \), we define the mixed norm \( \| \cdot \|_{L^\Phi(L^\Psi)} \) on functions on \([0, 1] \times \mathbb{T}\) by

\[
\| f(x, t) \|_{L^\Phi(L^\Psi)} = \left\| \left\| f(\cdot, t) \right\|_{L^\Psi(\mathbb{T})} \right\|_{L^\Phi([0, 1])}.
\]

Now note that

\[
\| h(x, t) \|_{L^1(L^{\log L})} \leq 10 \| h(x, t) \|_{L^1(L^2)}
\leq 100 \| h(x, t) \|_{L^2(\mathbb{T} \times [0, 1])}
\leq 100 \cdot \left( (A(2))^{-1} \cdot D \right)^{1/2} = C' < \infty.
\]

By Theorem 1 we then see that

\[
\left| \left\{ (x, t) : \left| T h(x, t) \right| > \alpha \right\} \right| \leq \int_{t=0}^{1} \frac{c \| h(\cdot, t) \|_{L^{\log L}(\mathbb{T})}}{\alpha} \leq \frac{c \cdot C'}{\alpha}.
\]

This however is in contradiction to Equation 3, which holds for arbitrary large values of \( M \).

\[\square\]

We now have that \( T \) is of restricted weak-type \((1, 1)\) and of restricted weak-type \((2, 2)\). By the extension of the Marcinkiewicz theorem to the case of restricted-weak endpoints (see [6] for details) we have that for \( 1 < p < \frac{3}{2} \) and for all simple functions \( f \),

\[
\| T f \|_{L^p(\mathbb{T})} \leq \frac{1}{p-1} \| f \|_{L^p(\mathbb{T})}.
\]

Applying the Yano extrapolation theorem [8], we see then that

\[
\| T f \|_{L^1(\mathbb{T})} \lesssim \| f \|_{L^{\log L}(\mathbb{T})}
\]

for all simple functions \( f \) supported on \( \mathbb{T} \), as desired.

\[\square\]
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DEPARTMENT OF MATHEMATICS, Baylor University, Waco, Texas 76798
E-mail address: paul_hagelstein@baylor.edu