WEAK $L^1$ NORMS OF RANDOM SUMS

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Abstract. Let \( \{g_j\} \) denote a sequence of measurable functions on \( \mathbb{R}^n \), and let \( \|\|_{W^1} \) denote the weak \( L^1 \) norm. It is shown that
\[
\|E \left( \left| \sum_{j=1}^{N} \varepsilon_j g_j \right| \right) \|_{W^1} \lesssim \sum_{j=1}^{N} \|g_j\|_{W^1},
\]
where \( \{\varepsilon_j\} \) is a sequence of independent random variables taking on values +1 and -1 with equal probability. Moreover, it is shown that
\[
\left\| E \left( \left| \sum_{j=1}^{N} \varepsilon_j g_j \right| \right) \right\|_{W^1} \lesssim E \left( \left\| \sum_{j=1}^{N} \varepsilon_j g_j \right\|_{W^1} \right).
\]

The paper concludes by providing an example indicating that, if \( \|g_1\|_{W^1} = \cdots = \|g_N\|_{W^1} = 1 \), the estimate \( E \left( \left\| \sum_{j=1}^{N} \varepsilon_j g_j \right\|_{W^1} \right) \lesssim N \log N \) is the best possible.

1. Introduction

The weak \( L^1 \) norm of a measurable function \( f \) supported on \( \mathbb{R}^n \) is defined by
\[
\|f\|_{W^1} = \sup_{\alpha > 0} \alpha \left| \{ x \in \mathbb{R}^n : |f(x)| > \alpha \} \right|.
\]

This norm is of frequent occurrence in modern-day mathematics, finding use in probability theory, harmonic analysis, and ergodic theory. In spite of its many applications, however, the weak \( L^1 \) norm has one decidedly inconvenient drawback: it is not a “norm” in the strictest sense of the word as it does not satisfy the triangle inequality. For example, the functions \( \frac{1}{2} \chi_{(0,1)}(x) \) and \( \frac{1}{2x} \chi_{(0,1)}(x) \) both have weak \( L^1 \) norms of 1, but their sum has a weak \( L^1 \) norm of 4.

The best-known result relating the weak \( L^1 \) norm of a sum of functions to their respective individual weak \( L^1 \) norms is the following one due to E. M. Stein and N. J. Weiss [1].

Theorem 1. Let \( \{g_j\} \) denote a sequence of measurable functions on \( \mathbb{R}^n \) such that \( \|g_j\|_{W^1} = 1 \) for each \( j \). Let \( \{c_j\} \) be a sequence of positive numbers with \( \sum c_j = 1 \), let \( s > 0 \), and set \( K = \sum c_j \log \left( \frac{1}{c_j} \right) \).

Then \( \left| \left\{ x \in \mathbb{R}^n : \sum c_j g_j(x) > s \right\} \right| < \frac{2(K+2)}{s} \).
The bound of $\| \sum c_j g_j \|_{W^1}$ in this theorem is essentially optimal: the log term may not be removed. For example, letting $c_1 = \cdots = c_N = \frac{1}{N}$ and $g_j(x) = \frac{1}{2} \frac{1}{|x - \frac{j}{N}|}$, one may readily check that $\| g_j \|_{W^1} = 1$ for each $j$ and that
\[
\left\| \sum_{j=1}^N c_j g_j(x) \right\|_{W^1} \sim \sum_{j=1}^N c_j \log \left( \frac{1}{c_j} \right) \sim \log N.
\]

Given the prevalence of randomization techniques in modern harmonic analysis, it is natural to conduct the admittedly Baconian experiment of determining the expected value of $\left\| \sum_{j=1}^N \epsilon_j c_j g_j(x) \right\|_{W^1}$, where the $c_j$ and $g_j(x)$ are as above and $\{\epsilon_j\}$ is a sequence of independent random variables taking on the values of $+1$ and $-1$ with equal probability. It turns out that a large majority of the $2^N$ associated weak $L^1$ norms are close to one, exhibiting significant improvement over the log $N$ bound above because of cancellation between the $\epsilon_j c_j g_j$’s. In fact, one can calculate that $\mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j g_j(x) \right\|_{W^1} \right) \sim 1$. Although not difficult, the techniques involved in this calculation are not immediately transparent and are accordingly given below.

**Example 1.** Let $c_1 = \cdots = c_N = \frac{1}{N}$ and $g_j(x) = \frac{1}{2} \frac{1}{|x - \frac{j}{N}|}$. Then
\[
\mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j g_j(x) \right\|_{W^1} \right) \sim \sum_{j=1}^N \frac{1}{N} = 1.
\]

We first observe that
\[
2^{2N} \left\{ x : \sum_{j=1}^N |\epsilon_j c_j g_j(x)| > 2^N \right\} \sim 2^{2N} \cdot N \cdot N^{-1} \cdot 2^{-2N} = 1
\]
for any sequence $\{\epsilon_j\}$ consisting of $-1$’s and $+1$’s, and hence the result that
\[
\mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j g_j(x) \right\|_{W^1} \right) \gtrsim \sum_{j=1}^N \frac{1}{N} = 1
\]
easily holds. We accordingly turn our attention to the opposite inequality.

Let $\tilde{g}_j(x) = g_j(x) \chi_{x:|g_j(x)|>2N}$. Let $h_j(x) = g_j(x) - \tilde{g}_j(x)$. Now,
\[
\left\| \sum_{j=1}^N \epsilon_j c_j g_j \right\|_{W^1} \leq 2 \left[ \left\| \sum_{j=1}^N \epsilon_j c_j \tilde{g}_j \right\|_{W^1} + \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{W^1} \right].
\]
As the \( \tilde{g}_j \) have disjoint supports,
\[
\left\| \sum_{j=1}^N \epsilon_j c_j \tilde{g}_j \right\|_{W^1} \leq \sum_{j=1}^N \|c_j \tilde{g}_j\|_{W^1} \leq 1.
\]
So it suffices to show that \( \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{W^1} \right) \lesssim 1. \)

Note that, letting \( E = [-1, 2] \) and \( \tilde{E} = (-\infty, -1) \cup (2, \infty) \), we have that
\[
\mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{W^1} \right) \leq \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{W^1} \right) + \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{W^1} \right).
\]

Now, for \( x \in \tilde{E} \), \( \sum_{j=1}^N \epsilon_j c_j h_j (x) \sim \left( \sum_{j=1}^N \epsilon_j \right) \cdot \frac{1}{|N|} \leq \frac{1}{|N|} \), and so
\[
\mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{W^1} \right) \lesssim 1. \]

Hence it suffices to show that \( \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{W^1} \right) \lesssim 1. \)

To see this, note that
\[
\mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{W^1} \right) \leq \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{L^1} \right) \]
\[
\leq \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{L^1} \right) \]
\[
= \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j c_j h_j \right\|_{L^1} \right) \]
\[
\lesssim \left( \sum_{j=1}^N |c_j h_j (1)|^2 \right)^{1/2} \]
\[
\lesssim \left( \sum_{j=1}^N \left( \frac{1}{j} \right)^2 \right)^{1/2} \sim 1,
\]
as desired.

It is natural at this point to consider whether or not this example is indicative of more general inequalities involving the expectation operator and the weak \( L^1 \) norm. The purpose of this paper is to show that the techniques involved in the above example may indeed be generalized to prove that, for \( \{g_j\} \) and \( \{c_j\} \) satisfying the hypotheses of Theorem 1, \( \mathbb{E} \left( \left\| \sum c_j g_j \right\|_{W^1} \right) \lesssim \sum c_j \) and \( \mathbb{E} \left( \left\| \sum \epsilon_j c_j g_j \right\|_{W^1} \right) \lesssim \mathbb{E} \left( \left\| \sum \epsilon_j c_j g_j \right\|_{W^1} \right) \). It will also be shown,
however, that the a priori Stein-Weiss estimate $\mathbb{E} \| \sum \epsilon_j c_j g_j \|_{W^1} \lesssim \sum c_j \log \left( \frac{1}{c_j} \right)$ is the best possible.

2. Weak $L^1$ norms of expectations of random sums

This section is devoted to the proof of the following.

Theorem 2. Let $\{g_j\}$ denote a sequence of measurable functions on $\mathbb{R}^n$, and let $\{\epsilon_j\}$ be a sequence of independent random variables taking on values +1 and −1 with equal probability. Then

$$\left\| \mathbb{E} \left( \left| \sum_{j=1}^{N} \epsilon_j g_j(x) \right| \right) \right\|_{W^1} \lesssim \sum_{j=1}^{N} \| g_j \|_{W^1}.$$

Proof. We first note that

$$\mathbb{E} \left( \left| \sum_{j=1}^{N} \epsilon_j g_j(x) \right| \right) \sim \left( \sum_{j=1}^{N} |g_j(x)|^2 \right)^{1/2},$$

as follows from elementary probability theory. So it suffices to show that

$$\left\| \left( \sum_{j=1}^{N} |g_j(x)|^2 \right)^{1/2} \right\|_{W^1} \lesssim \sum_{j=1}^{N} \| g_j \|_{W^1}.$$

Let $\alpha > 0$. Let $g_{j,\alpha}(x) = g_j(x) \chi_{\{x: |g_j(x)| \leq \alpha\}}(x)$. Note that

$$\left\{ x : \left( \sum_{j=1}^{N} |g_j(x)|^2 \right)^{1/2} > \alpha \right\} \subset \left( \bigcup_j \left\{ x : |g_j(x)| > \alpha \right\} \right) \cup \left\{ x : \sum_{j=1}^{N} |g_{j,\alpha}(x)|^2 > \alpha^2 \right\}.$$

By the definition of the weak $L^1$ norm, we have

$$\left| \left\{ x : |g_j(x)| > \alpha \right\} \right| \leq \frac{1}{\alpha} \| g_j \|_{W^1}.$$

Also,

$$\left| \left\{ x : \sum_{j=1}^{N} |g_{j,\alpha}(x)|^2 > \alpha^2 \right\} \right| \leq \frac{1}{\alpha^2} \sum_{j=1}^{N} \int_{\mathbb{R}^n} |g_{j,\alpha}|^2 \, dx,$$

and

$$\int_{\mathbb{R}^n} |g_{j,\alpha}(x)|^2 \, dx \leq \int_{y = \frac{1}{\alpha} \| g_j \|_{W^1}}^{\infty} \left( \frac{\| g_j \|_{W^1}}{y} \right)^2 \, dy \lesssim \| g_j \|_{W^1}^2 \left[ -y^{-1} \right]_{\frac{1}{\alpha} \| g_j \|_{W^1}}^{\infty}.$$
\[ \sim \alpha \|g_j\|_{W^1}. \]

Hence \( \left\{ x : \sum_{j=1}^N |g_{j,\alpha}(x)|^2 > \alpha^2 \right\} \subset \frac{1}{\alpha^2} \alpha \sum_{j=1}^N \|g_j\|_{W^1} = \frac{1}{\alpha} \sum_{j=1}^N \|g_j\|_{W^1}. \)

This inequality together with (1), (2) implies that

\[ \left\| \left\{ x : \left( \sum_{j=1}^N |g_j(x)|^2 \right)^{1/2} > \alpha \right\} \right\|_{W^1} \leq \sum_{j=1}^N \|g_j\|_{W^1}. \]

3. EXPECTATIONS OF WEAK \( L^1 \) NORMS OF RANDOM SUMS

The theory involving the relationship between the expectation operator and the weak \( L^1 \) norm would be very nice indeed if the comparability

\[ \left\| \mathbb{E} \left( \left\{ x : \sum_{j=1}^N g_j(x) \right\} \right) \right\|_{W^1} \sim \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{W^1} \right) \]

held. In this section we shall see that the inequality

\[ \left\| \mathbb{E} \left( \left\{ x : \sum_{j=1}^N g_j(x) \right\} \right) \right\|_{W^1} \leq C \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{W^1} \right) \]

does hold. However, afterwards we shall provide an example indicating that the opposite inequality is false.

Theorem 3. Let \( \{g_j\} \) denote a sequence of measurable functions on \( \mathbb{R}^n \), and let \( \{\epsilon_j\} \) be a sequence of independent random variables taking on values +1 and −1 with equal probability. Then

\[ \left\| \mathbb{E} \left( \sum_{j=1}^N \epsilon_j g_j \right) \right\|_{W^1} \leq \mathbb{E} \left( \left\| \sum_{j=1}^N \epsilon_j g_j \right\|_{W^1} \right). \]

Proof. It suffices to show that there exist universal constants 0 < c, C < \( \infty \) such that, for any \( \alpha > 0 \),

\[ \left\{ x : \mathbb{E} \left( \sum_{j=1}^N \epsilon_j g_j(x) \right) > \alpha \right\} \leq C \mathbb{E} \left( \left\{ x : \sum_{j=1}^N \epsilon_j g_j(x) > c\alpha \right\} \right). \]

To see this, suppose (3) held. Let \( \beta \) be such that \( \beta : \left\{ x : \mathbb{E} \left( \left| \sum_{j=1}^N \epsilon_j g_j(x) \right| \right) > \beta \right\} \sim \left\| \mathbb{E} \left( \left| \sum_{j=1}^N \epsilon_j g_j \right| \right) \right\|_{W^1}. \) (We may assume that \( \left\| \mathbb{E} \left( \left| \sum_{j=1}^N \epsilon_j g_j \right| \right) \right\|_{W^1} < \infty \) without loss of generality.) Then

\[ \left\| \mathbb{E} \left( \sum_{j=1}^N \epsilon_j g_j \right) \right\|_{W^1} \sim \beta \cdot \left\{ x : \mathbb{E} \left( \sum_{j=1}^N \epsilon_j g_j(x) \right) > \beta \right\} \leq C\beta \mathbb{E} \left( \left\{ x : \sum_{j=1}^N \epsilon_j g_j(x) > c\beta \right\} \right) \].
\[ = C \mathbb{E} \left( \beta \left| \left\{ x : \sum_{j=1}^{N} \epsilon_j g_j(x) \geq c \beta \right\} \right| \right) \]
\[ \leq C \mathbb{E} \left( \frac{1}{c} \left\| \sum_{j=1}^{N} \epsilon_j g_j \right\|_{WL^1} \right) \]
\[ \leq \frac{C}{c} \mathbb{E} \left( \left\| \sum_{j=1}^{N} \epsilon_j g_j \right\|_{WL^1} \right), \]
the desired result.

We accordingly turn our attention to (3). To prove (3), it suffices to show that there exist universal constants \(0 < \beta, C < 1\) such that

\[ (4) \quad \left| \left\{ x : \mathbb{E} \left( \left| \sum_{j=1}^{N} \epsilon_j g_j(x) \right| \right) > 1 \right\} \right| \leq C \mathbb{E} \left( \left| \left\{ x : \sum_{j=1}^{N} \epsilon_j g_j(x) > c \right\} \right| \right). \]

The proof of (4) will be facilitated by the use of Rademacher functions.

**Definition 1.** Let \( r_\mathbb{N}(t) \) denote the Rademacher functions on \( \mathbb{R} \), defined by

\[ r_\mathbb{N}(t) = \begin{cases} r_0(2^n t) \quad &0 \leq t \leq \frac{1}{2} \\ -r_0(2^n t) \quad &\frac{1}{2} < t < 1 \\ r_0(t) \quad &t + 1 \end{cases}, \]

where \( r_0(t) = 1 \) if \( 0 \leq t \leq \frac{1}{2} \), \(-1 \) if \( \frac{1}{2} < t < 1 \), and \( r_0(t + 1) = r_0(t) \).

We will also need the following lemma:

**Lemma 1 ([2]).** Let \( \sum_{j=0}^{\infty} |a_j|^2 < \infty \), and let \( F(t) = \sum_{j=0}^{N} a_j r_j(t) \) be a Rademacher series. Let \( 0 < p < \infty \). Then there exist finite, positive constants \( A(p), B(p) \) such that

\[ A(p) \| F \|_{L^p([0,1])} \leq \left( \sum_{j=0}^{N} |a_j|^2 \right)^{1/2} \leq B(p) \| F \|_{L^p([0,1])}. \]

We now fix \( x \) and let \( a_j = g_j(x) \). To prove (4), it suffices to show that

\[ (5) \quad \mathbb{E} \left( \left| \sum_{j=1}^{N} \epsilon_j a_j \right| \right) = 1 \text{ implies that } \left| \left\{ t \in [0,1] : \sum_{j=1}^{N} a_j r_j(t) > \frac{1}{2} \right\} \right| \geq \tilde{c} \]

for some universal constant \( 0 < \tilde{c} < \infty \).
Well, suppose $E \left( \left| \sum_{j=1}^{N} \epsilon_j a_j \right| \right) = 1$. Then

$$\left\| \sum_{j=1}^{N} a_j r_j (t) \right\|_{L^1([0,1])} = 1.$$  \hspace{1cm} (6)

By the above lemma we then realize that for some universal constant $0 < C_2 < \infty$ we have

$$\left\| \sum_{j=1}^{N} a_j r_j (t) \right\|_{L^2([0,1])} \leq C_2.$$  \hspace{1cm} (7)

Equations (6) and (7) imply that $\left| \sum_{j=1}^{N} a_j r_j (t) \right| > \frac{1}{2}$ on a set of measure at least $\tilde{c} > 0$, where $\tilde{c}$ is a universal constant, and so (5) holds, as desired. \hspace{1cm} □

We now provide an example indicating that the inequality opposite to that of Theorem 3 is false. Let $\phi_1, \phi_2, \ldots, \phi_{2^N}$ denote the $2^N$ functions from $\{1, \ldots, N\}$ to $\{-1, +1\}$. For $j = 1, \ldots, N$, let

$$h_j (x) = \sum_{k=1}^{2^N} \phi_k (j) 2^k \chi_{[2^{-(k+1)}, 2^{-k}]} (x).$$

Note that

$$E \left( \left\| \sum_{k=1}^{N} \epsilon_k h_k (x) \right\| \right) \sim N^{1/2} \sum_{k=1}^{2^N} 2^k \chi_{[2^{-(k+1)}, 2^{-k}]} (x),$$

and so $\left\| E \left( \left\| \sum_{k=1}^{N} \epsilon_k h_k \right\| \right) \right\|_{W^{1,1}} \sim N^{1/2}$. However, if $\{\epsilon_j\}_{j=1}^{N}$ is a sequence of $-1$’s and $+1$’s, $\phi_\ell (j) = \epsilon_j$ for some $\ell \in \{1, \ldots, 2^N\}$. Hence $\sum_{j=1}^{N} \epsilon_j h_j (x) = N \cdot 2^\ell$ on $[2^{-(\ell+1)}, 2^{-\ell})$, and so $\left\| \sum_{j=1}^{N} \epsilon_j h_j \right\|_{W^{1,1}} \geq N$. As $\{\epsilon_k\}_{k=1}^{N}$ is an arbitrary sequence of $-1$’s and $+1$’s, we see that $E \left( \left\| \sum_{k=1}^{N} \epsilon_k h_k \right\|_{W^{1,1}} \right) \geq N$. As $N$ may be arbitrarily large, we see that the inequality opposite to that of Theorem 3 cannot hold.

4. A Counterexample regarding expectations of weak $L^1$ norms of random sums

Perhaps the most interesting inequality we could wish for regarding expectations of weak $L^1$ norms of random sums would be that $E \left( \left\| \sum_{j=1}^{N} \epsilon_j g_j \right\|_{W^{1,1}} \right) \lesssim \sum_{j=1}^{N} \|g_j\|_{W^{1,1}}$, providing via Theorem 3 an improvement to Theorem 2. Note that the functions $h_j$ constructed at the end of the previous section,
although not satisfying $E \left( \left\| \sum_{j=1}^{N} \epsilon_j h_j \right\|_{W^{1}} \right) \lesssim E \left( \left\| \sum_{j=1}^{N} \epsilon_j h_j \right\|_{W^{1}} \right)$, do satisfy such an estimate. However, we shall now construct a counterexample indicating that the inequality $E \left( \left\| \sum_{j=1}^{N} \epsilon_j g_j \right\|_{W^{1}} \right) \lesssim \sum_{j=1}^{N} \| g_j \|_{W^{1}}$ in general does not hold, and that, if $\{c_j\}$ is a sequence of positive numbers such that $\sum c_j = 1$ and $\{g_j\}$ is a sequence of functions each of which having a weak $L^1$ norm of 1, then the estimate $E \left( \left\| \sum_{j=1}^{N} \epsilon_j c_j g_j \right\|_{W^{1}} \right) \lesssim \sum c_j \log \left( \frac{1}{c_j} \right)$ following from Theorem 1 is the best possible.

**Theorem 4.** Let $N \geq 2$ be a positive integer. Then there exist functions $g_1, g_2, \ldots, g_N$ supported on $[0, 1]$ such that $\| g_j \|_{W^{1}} \sim 1$ for each $j$ and such that

$$E \left( \left\| \sum_{j=1}^{N} \epsilon_j g_j \right\|_{W^{1}} \right) \sim N \log N.$$ 

**Proof.** Define the function $h_{1,1} (x)$ on $[0, 1]$ by

$$h_{1,1} (x) = \begin{cases} \left( \frac{1}{N} + \left( \frac{1-\frac{1}{N}}{N} \right) \cdot j \right)^{-1} & \text{if } \frac{1}{N} + \frac{1-\frac{1}{N}}{N} \cdot (j-1) < x \leq \frac{1}{N} + \frac{1-\frac{1}{N}}{N} j \\ 0 & \text{for some } j = 1, \ldots, N \end{cases}$$

For $m = 2, \ldots, N$ we define the functions $h_{m,1} (x)$ by

$$h_{m,1} (x) = \begin{cases} \left( \frac{1}{N} + \left( \frac{1-\frac{1}{N}}{N} \right) \cdot (N-m+j+1) \right)^{-1} & \text{if } \frac{1}{N} + \frac{1-\frac{1}{N}}{N} \cdot (j-1) < x \leq \frac{1}{N} + \frac{1-\frac{1}{N}}{N} j \\ \left( \frac{1}{N} + \left( \frac{1-\frac{1}{N}}{N} \right) \cdot (j-m+1) \right)^{-1} & \text{for some } j = m, \ldots, N \end{cases}$$

and $0$ otherwise.

For $k \geq 2$ and $m = 1, \ldots, N$, let $h_{m,k} (x) = N^{k-1} h_{m,1} \left( N^{k-1} x \right)$. Note that $h_{m,k}$ is supported on $\left[ \left( \frac{x}{N} \right)^{k}, \left( \frac{x}{N} \right)^{k-1} \right]$.

Now, we again let $\phi_1, \phi_2, \ldots, \phi_{2N}$ denote the $2^N$ functions from $\{1, \ldots, N\}$ to $\{-1, 1\}$. For $k = 1, \ldots, N$ we define the functions $g_k (x)$ by

$$g_k (x) = \sum_{j=1}^{2N} \phi_j (k) h_{k,j} (x).$$
Note that \( \|g_j\|_{W_L^1} \sim \|\frac{1}{x}\|_{W_L^1} \sim 1 \) for \( j = 1, \ldots, N \). Now, if \( \{\epsilon_j\}_{j=1}^N \) is a sequence of \(-1\)'s and \(+1\)'s, \( \{\epsilon_j\}_{j=1}^N = \{\phi_{\ell}(j)\}_{j=1}^N \) for some \( 1 \leq \ell \leq 2^N \). So for \( x \in (N^{-\ell}, N^{-\ell+1}) \) we have
\[
\sum_{j=1}^{N} \epsilon_j g_j(x) \sim N^\ell \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} \right) \sim N^\ell \log N.
\]
So
\[
\left\| \left( \sum_{j=1}^{N} \epsilon_j g_j \right) \chi_{[N^{-\ell}, N^{-\ell+1}]} \right\|_{W_L^1} \sim N^{-\ell+1} \left( N^\ell \log N \right) = N \log N.
\]
As \( \{\epsilon_j\}_{j=1}^N \) is an arbitrary sequence of \(-1\)'s and \(+1\)'s, we may conclude that
\[
\mathbb{E} \left( \left\| \sum_{j=1}^{N} \epsilon_j g_j \right\|_{W_L^1} \right) \sim N \log N, \text{ as desired.}
\]

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