MATH4426
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Homework 2
Due September 18, 2020

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EDT on September 18. Please name your file hw02-lastname-firstname-pdf. My solution file is hw02-gross-robert.pdf.

I will try to acknowledge receipt of each e-mail.

1. Combinatorial arguments are often easier than other methods of proof. Here is another identity involving binomial coefficients:

$$\binom{n}{k} = \sum_{j=k}^{n} \binom{j-1}{k-1}$$

for $0 < k < n$.

Give a combinatorial argument by adding in details to this sketch of a proof:

> Consider k-element subsets of $\{1, 2, \ldots, n\}$. Call the largest element in each subset j. Explain why $k \le j \le n$. Count how many subsets there are for each value of j.

2. Twenty-five people, consisting of 15 women and 10 men, are lined up in random order. Find the probability that the ninth woman to appear is in position 17. In other words, find the probability that there are 8 women in positions 1 through 16 and a woman is in position 17.

3. Suppose that a tournament with n teams ends up ranking the n teams, *with ties allowed*. In other words, the outcome of the tournament is a partition of the teams into groups, with the first group consisting of teams that tied for first place, the second group consisting of teams that tied for the next-best place, and so on. Let $N(n)$ count the number of different outcomes. For instance:

- $N(1) = 1$, because a tournament with only one team has only one outcome.
- $N(2) = 3$. A tournament with 2 teams, A and B, has 3 outcomes:
  (1) Team A is ranked ahead of team B.
  (2) Team B is ranked ahead of team A.
  (3) Teams A and B tie.

(a) Compute $N(3)$ by listing all possible outcomes of a tournament with 3 teams.

(b) Define $N(0) = 1$. Show that

$$N(n) = \sum_{i=1}^{n} \binom{n}{i} N(n-i). \tag{1}$$

Hint: How many tournament outcomes are there when i teams tie for last place?

(c) Show that (1) is equivalent to the following slightly simpler formula:

$$N(n) = \sum_{i=0}^{n-1} \binom{n}{i} N(i). \tag{2}$$

(d) Use (2) to compute $N(3)$, $N(4)$, and $N(5)$.

4. How many different linear arrangements are there of the letters A, B, C, D, E, and F are there in which
(a) A and B are adjacent?
(b) A is to the left of B?
(c) A is to the left of B and B is to the left of C?
(d) A is to the left of B and C is to the left of D?
(e) A and B are adjacent and C and D are adjacent?
(f) E is not the last letter in the arrangement?

5. In class, we gave an algebraic verification of the formula

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}.$$

Give a combinatorial argument that verifies the formula.

6. A 3-person basketball team consists of a guard, a forward, and a center.
(a) Suppose that one player is chosen at random from each of 3 different teams. What is the probability that the choices are a guard, a forward, and a center?
(b) Suppose that one player is chosen at random from each of 3 different teams. What is the probability that the 3 players all play the same position?

7. Remember that a bridge hand consists of 13 cards from a standard deck of 52 cards containing 13 cards in each of four suits: $\spadesuit$, $\heartsuit$, $\diamondsuit$, and $\clubsuit$. What is the probability that such a hand contains cards in at most 3 of the 4 suits? Note that the answer is *not*

$$\frac{\binom{4}{1}\binom{39}{13}}{\binom{52}{13}},$$

though the correct answer is in fact very close to this incorrect one.

8. Suppose that E and F are events. Prove that

$$P(EF) \geq P(E) + P(F) - 1.$$