MATH4426
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Homework 6
Due October 23, 2020

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EDT on October 23. Please name your file hw06-lastname-firstname.pdf. My solution file is hw06-gross-robert.pdf.

I will try to acknowledge receipt of each e-mail.

1. An urn contains 12 balls: 8 black, 4 white. A ball is taken from the urn, its color is recorded, and it is replaced in the urn *along with another ball of the same color.* This process is then repeated (another ball is withdrawn, and 2 are put back), so that there are now 14 balls in the urn. Let X be a random variable that records the number of white balls that were withdrawn. Compute $P\{X = k\}$ for $k = 0$, 1, and 2.

2. A sample of 3 items is chosen at random from a box containing 20 items, of which 4 are defective. Find the expected number of defective items in the sample.

3. Let X be the number of successes that result from $2n$ independent trials, when each trial succeeds with probability p. Show that the quantity $P(X = n)$, considered as a function of n for fixed p, is decreasing.

4. Suppose that the average number of misprints on each page of a book is 0.2. What is the probability that no more than 2 misprints appear in a chapter that is
(a) 4 pages long?
(b) 8 pages long?

5. Suppose that X is a binomial random variable, parameters (n, p). Show that

$$E\left[\frac{1}{X+1}\right] = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

6. Suppose that X is a random variable with cumulative distribution function $F(b)$ given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \frac{1}{3} & 0 \le b < 1 \\ \frac{3}{5} & 1 \le b < 2 \\ \frac{7}{8} & 2 \le b < 3 \\ \frac{9}{10} & 3 \le b < 6 \\ 1 & b \ge 6 \end{cases}$$

Compute the expected value $E[X]$.

7. Doctors want to test a group of 10 people to see who, if any, has coronavirus. Rather than test each person individually, the samples from all 10 people will be pooled and tested in one large test. If the test is negative, the doctors will know that none of those 10 people have the virus. If the large test is positive, then each of the 10 people in the group will be tested individually, so that in all 11 tests will be used.

(a) Assume that the probability that any person in the group has coronavirus is 0.1, independent of any other person's status. What is the expected number of tests that will be needed to find out how many people are sick?
(b) Repeat if the probability that any person is sick is 0.05.
(c) Repeat if the probability that any person is sick is 0.20.
(d) Suppose that the probability that any person in the group is sick is p, independent of any other person. For what value(s) of p will the expected number of tests be 10? You will need to use your calculator to solve the an equation numerically to estimate p.

8. Suppose that X is Poisson, parameter λ. Show that $P(X = k)$ increases monotonically and then decreases monotonically, with maximum value occurring when $k = \lfloor \lambda \rfloor$. HINT: Imitate the similar computation for a binomial random variable.

9. Suppose that X is a geometric random variable, and n and k are positive integers. Show directly from definitions that

$$P\{X = n + k | X > n\} = P\{X = k\}.$$

10. Suppose that X is a Poisson random variable with parameter λ, and n is a positive integer. Show that

$$E[X^n] = \lambda E[(X + 1)^{n-1}].$$

Then use this result to compute $E[X]$, $E[X^2]$, and $E[X^3]$.