MATH4426
Robert Gross
Homework 8
Due November 6, 2020

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EST on November 6. Please name your file hw08-lastname-firstname.pdf. My solution file is hw08-gross-robert.pdf.

I will try to acknowledge receipt of each e-mail.

1. Remember that the probability density function for the *gamma distribution* with parameters α and λ (two positive real numbers) was defined by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy.$$

(a) Show that this is indeed a probability density function by showing that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

(b) Show that $E[X] = \dfrac{\alpha}{\lambda}$.

(c) Show that $\text{Var}(X) = \dfrac{\alpha}{\lambda^2}$.

2. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, by making the change of variables $z = \sqrt{2y}$ in the definition of the Γ-function.

3. Suppose that the probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} cxe^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) Compute c.

(b) What is $E[X]$?

(c) What is $\text{Var}(X)$?

4. Suppose that the probability density function for a random variable X is given by

$$f(x) = \begin{cases} ax^2 + b & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose as well that $E[X] = 0.4$. Is this enough information to determine a and b? If so, compute their values. If not, explain what additional information is needed to determine a and b.

5. Suppose that you are waiting for the BC shuttle, and you know that the bus runs every thirty minutes, uniformly distributed.

(a) What is the probability that you will wait for more than 5 minutes?
(b) Suppose that you have waited for 5 minutes, and the bus has not arrived. What is the probability that you will wait at least another 5 minutes?

6. Suppose that X is an exponential random variable with $\lambda = 2$. Suppose that $Y = \log X$. What is the probability density function for Y?

7. Suppose that X is a continuous random variable taking values between 0 and c (where c is a positive constant). In other words, $P\{0 \le x \le c\} = 1$. Show that $\text{Var}(X) \le \frac{c^2}{4}$. *Hint*: You can start by showing that $E[X^2] \le cE[X]$, and then argue that $\text{Var}(X) \le c^2(\alpha - \alpha^2)$, with $\alpha = E[X]/c$.

8. Suppose that Z is a standard normal random variable, and let $g(x)$ be a differentiable function so that $\lim_{x \to \pm\infty} g(x)e^{-x^2/2} = 0$.
(a) Show that $E[g'(Z)] = E[Zg(Z)]$. *Hint*: Integrate by parts.
(b) Show that $E[Z^{n+1}] = nE[Z^{n-1}]$
(c) Use this to compute $E[Z^4]$.