MATH4426
Robert Gross
Homework 10
Due December 4, 2020

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EST on December 4. Please name your file hw10-lastname-firstname.pdf. My solution file is hw10-gross-robert.pdf.

I will try to acknowledge receipt of each e-mail.

1. Let

$$f(x,y) = 24xy, \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le x+y \le 1.$$

with $f(x,y) = 0$ otherwise.
(a) Show that $f(x,y)$ is a density function.
(b) Find $E[X]$.
(c) Find $E[Y]$.
(d) Find $E[XY]$.
(e) Are X and Y independent?

2. Suppose that the joint density of X and Y is given by

$$f(x,y) = C(x-y)e^{-x} \qquad 0 \le x \le \infty, \quad |y| < x$$

and $f(x,y) = 0$ otherwise.
(a) Compute C.
(b) Find the marginal density of X.
(c) Find the marginal density of Y.
(d) Find $E[X]$.
(e) Find $E[Y]$.
(f) Compute $\mathrm{Cov}(X,Y)$.

3. A fair die is rolled. Let X be the number of rolls until a 6 appears, and Y the number of rolls until a 5 appears. Compute
(a) $E[X]$.
(b) $E[X|Y=1]$
(c) $E[X|Y=4]$.

4. Suppose that X_1, X_2, $\ldots$, X_6 are independent identically distributed continuous random variables. Compute
(a) $P\{X_6 > X_1 | X_1 = \max(X_1, X_2, \ldots, X_5)\}$.
(b) $P\{X_6 > X_2 | X_1 = \max(X_1, X_2, \ldots, X_5)\}$.

5. Suppose that X is a nonnegative continuous random variable, and $g(x)$ is a differentiable function with $g(0) = 0$. Show that

$$E[g(X)] = \int_0^\infty P\{X > t\} g'(t)\, dt.$$

6. Suppose that X is a random variable, $E[X] = \mu$, $\text{Var}(X) = \sigma^2$, and $g(x)$ is a function that can be differentiated at least 3 times. Show that

$$E[g(X)] \approx g(\mu) + \frac{g''(\mu)}{2}\sigma^2.$$

Hint: Approximate $g(x)$ as a Taylor polynomial at $x = \mu$.

7. Suppose that X is a Poisson random variable, $E[X] = \lambda > 0$. Show that with appropriate assumptions on λ,

$$\text{Var}(\sqrt{X}) \approx 0.25$$

Hint: Apply the previous problem.