Incentivized Kidney Exchange: Online Appendices

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Appendix A Main Analytical Results

In this appendix, we provide the omitted proof of Theorem 1, derivation of waiting times, and additional comparative results. To do that, we first provide a formal model of tissue-type incompatibility: Each patient has a type, depending on against which subset of HLA tissue proteins he has preformed antibodies. We study the limit as the number of types goes to infinity. First, fix the number of types to a finite k. The probability that a patient is of type i is $m_{i,k} > 0$, so that $\sum_i m_{i,k} = 1$. Let $\theta_{i,k}$ be the tissue-type incompatibility probability between any donor and patient of type i. If a donor is tissue-type compatible with a type i patient, then the donor is tissue-type compatible with all patients of type i. We take the number of types, k, to infinity and make some regularity assumptions on the growth of $m_{i,k}$ and $\theta_{i,k}$ in the limit. See Appendix F for details. These assumptions hold for the special case when $\theta_{i,k} = \theta < 1$ and $m_{i,k} \to 0$ for every patient type i as $k \to \infty$ (Lemma A-7). In the current appendix, as well as for the results in the main text, we use this special case.

We also define steady states formally: A **state** is defined through the measures of type $X-Y$ pairs who have waited t years in the patient pool, denoted by $(X-Y,t)$, and blood-type X unpaired patients who have waited t years in the patient pool, denoted by (X,t), for all blood types X and Y and waiting time t. We say that the population under a given policy of transplantation is at a **steady state** when the measures of all $(X-Y,t)$ and (X,t) are constant through time, i.e., the state does not change over time.

A.1 Optimal Regular and Incentivized Exchange

We first formally categorize pair types in the following three classes. The naming of these classes is based on the comparison of the flows of the type and its reciprocal type for ABO-i optimal exchange (Assumptions 1 and 2). The types $X-Y$ with a weakly lower flow than that of $Y-X$ are overdemanded, while the ones with the weakly higher flow are underdemanded. Remaining types are referred to as self-demanded, as each such pair is matched with another pair of the same type in ABO-i optimal exchange:

- **Overdemanded Types**: These are pair types $X-Y$ such that $Y \triangleright X$ and $Y \neq X$ and pair type $A-B$. There are six of these types, $A-O$, $A-B$, $B-O$, $AB-O$, $AB-A$ and $AB-B$.

- **Self-demanded Types**: These are pair types $X-X$. There are four of these types, $O-O$, $A-A$, $B-B$, and $AB-AB$.
- **Underdemanded Types**: These are pair types $X-Y$ such that $X \triangleright Y$ and $X \neq Y$ and pair type $B-A$. There are six of these types, $O-A$, $O-B$, $O-AB$, $A-AB$, $B-A$, and $B-AB$.

The following lemma characterizes feasible exchanges. It is useful in the proof of Theorem 1. Similar results also appear in Roth, Sönmez, and Ünver (2007) and Ünver (2010), so we skip its proof.

Lemma A-1 (Exchange blood-type feasibility) *An underdemanded-type pair can be matched only with an overdemanded-type pair in an exchange. An overdemanded-type pair can be matched with pairs with types from each of the three categories. A self-demanded-type pair can be matched with a same-type or overdemanded-type pair. In particular, the following results hold:*

- *An underdemanded-type $O-A$ (or $O-B$) pair can be matched only with a pair from overdemanded types $A-O$ (or $B-O$) or $AB-O$. An underdemanded-type $A-AB$ (or $B-AB$) pair can be matched only with a pair from overdemanded types $AB-A$ (or $AB-B$) or $AB-O$. An underdemanded-type $O-AB$ pair can be matched only with an overdemanded-type $AB-O$ pair.*
- *An overdemanded-type $A-B$ (or underdemanded-type $B-A$) pair can be matched only with a pair from its reciprocal type $B-A$ (or $A-B$); or from overdemanded types $B-O$ (or $A-O$), $AB-A$ (or $AB-B$), or $AB-O$.*
- *A self-demanded-type $X-X$ pair can be matched with a same-type pair. Additionally, a type $O-O$ pair can be matched only with a pair from overdemanded types $A-O$, $B-O$, or $AB-O$; a type $A-A$ (or $B-B$) pair can be matched only with a pair from overdemanded types $AB-A$ (or $AB-B$) or $AB-O$; and a type $AB-AB$ pair can be matched only with a pair from overdemanded types $AB-A$, $AB-B$, or $AB-O$.*

Proof of Theorem 1. Suppose Assumptions 1 and 2 hold. Under the proposed policy, by Lemma A-6 in Appendix F, all self-demanded-type pairs can be matched with their own-type pairs as soon as they arrive.

Similarly, type $A-B$ pairs, which have a weakly lower flow rate than that of type $B-A$ by Assumption 2, can be matched as soon as they arrive with type $B-A$ pairs (Lemma A-4 in Appendix F). Hence, under this policy some type $B-A$ pairs will remain in the exchange pool. These pairs can only be matched with some overdemanded-type pairs by Lemma A-1, as type $A-B$ pairs are already committed to other type $B-A$ pairs.

Next consider underdemanded-type pairs except those of $B-A$. These are type $Y-X$ pairs such that $Y \neq X$ and $Y \triangleright X$. By Assumption 1, we have $\theta p_Y \alpha_X \boldsymbol{\pi}_X \leq p_X \alpha_Y \boldsymbol{\pi}_Y$. By Lemma A-1, they can only be matched with overdemanded-type pairs. Recall that the flow of each type $Y-X$ pair to the exchange pool is $p_X \alpha_Y \boldsymbol{\pi}_Y$. Their reciprocal type $X-Y$, which is overdemanded, has flow $\theta p_Y \alpha_X \boldsymbol{\pi}_X \leq p_X \alpha_Y \boldsymbol{\pi}_Y$. Hence, we can match all such overdemanded-type $X-Y$ pairs as soon as they enter the pool with their reciprocal-type pairs (Lemma A-5 in Appendix F). As all overdemanded- and self-demanded-type pairs are matched as soon as they arrive, by Lemma A-1, the proposed policy achieves the maximum measure of pairs matched. At steady state, as no incompatible overdemanded-type or self-demanded-type pair waits in the pool (and moreover, get

immediately matched and help one additional pair), the maximum mass of possible exchanges is also conducted in this manner in any closed time interval.

On the other hand, if we do not conduct the exchanges immediately whenever they become available but only after some time interval, then some of the patients will not survive. Hence, when we do not conduct the exchanges as soon as possible, we will match a strictly smaller mass of pairs than we would have matched under the proposed policy. ■

A.2 Finding Waiting Times for Transplantation

In this subsection of this appendix, we explain how we find waiting times for deceased-donor and living-donor transplants at the steady state using our dynamic continuum model.

A.2.1 Only Deceased-Donor Transplantation

We start when the only available transplantation regime is deceased donation. In this case, at any time the longest-waiting cohort of blood-type X patients receive the incoming blood-type X deceased-donor kidneys. Let this cohort have arrived $t_X^{\mathbf{d}}$ years before the current time. Assuming deceased-donor kidneys are the only source of transplants, at steady state we have

$$[\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X]S(t_X^{\mathbf{d}}) = \boldsymbol{\delta}_X.$$

Hence, the time spent on the blood-type X deceased-donor queue at steady state, or equivalently the transplant waiting time for blood-type X patients, can be found as

$$t_X^{\mathbf{d}} = S^{-1}\left(\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X}\right) = S^{-1}\left(s_X^{\mathbf{d},dec}\right).$$

A.2.2 Deceased-Donor/Direct Living-Donor Transplantation

When additionally direct living-donor transplantation is available, at any time the longest-waiting cohort of blood-type X patients without compatible donors receive the incoming blood-type X deceased-donor kidneys. Let this cohort have arrived $t_X^{\mathbf{l}}$ years before the current time. At steady state, we have $\boldsymbol{\pi}_X^{\mathbf{l}}S(t_X^{\mathbf{l}}) = \boldsymbol{\delta}_X$, and therefore the time spent on the blood-type X deceased-donor queue by the receiving cohort can be found as

$$t_X^{\mathbf{l}} = S^{-1}\left(\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X^{\mathbf{l}}}\right) = S^{-1}\left(\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X - (1-\phi^{\mathbf{l}})\boldsymbol{\lambda}_X}\right) = S^{-1}\left(s_X^{\mathbf{l},dec}\right).$$

All living-donor transplants are carried out instantaneously; thus, their waiting time is zero.

A.2.3 Adding Regular or Incentivized Exchange

Next, we derive waiting times for transplantations when regular or incentivized exchange is also feasible in addition to deceased-donor and direct living-donor transplantation.

Recall that for all incentivized-exchange-eligible pairs, i.e., of all types $X-Y$ such that $Y \triangleright X$, $Y \neq X$, and the patient and donor have no tissue-type incompatibility, $\rho_{X-Y} \in [0,1]$ is the fraction that participate in incentivized exchange. Let $\rho = (\rho_{X-Y})_{Y \triangleright X, Y \neq X}$ be the vector of such fractions. We use the terms "regular exchange" and "incentivized exchange with $\rho = 0\%$" interchangeably. To determine the steady-state outcomes, we introduce certain flow rates.[1] For each blood type X

[1]Some of these were previously defined throughout Section 4.

and each $Y \neq X$, let

$$\boldsymbol{\pi}_{X-Y} = \begin{cases} [\theta + \rho_{X-Y}(1-\theta)]p_Y \alpha_X \boldsymbol{\pi}_X & \text{if } Y \triangleright X \\ p_Y \alpha_X \boldsymbol{\pi}_X & \text{otherwise} \end{cases} \tag{1}$$

refer to the pair-type $X-Y$ flow to the exchange pool. Let the **incentivized pair flow** relevant for blood type X be given by

$$\boldsymbol{\kappa}_X = \Big(\sum_{Y \,:\, Y \triangleright X \;\&\; Y \neq X} \rho_{X-Y}(1-\theta)p_Y \Big) \alpha_X \boldsymbol{\pi}_X. \tag{2}$$

Observe that $\phi^{\mathbf{l}}\boldsymbol{\kappa}_X$ is the reentry flow of previously incentivized blood-type X patients to the patient pool. These patients will be prioritized in the deceased-donor queue of blood type X and will not wait upon reentry. Thus, the effective flow rate of deceased-donor kidneys for nonprioritized blood-type X patients is $\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X$. We also have

$$\boldsymbol{\pi}_X^{np\&u} = \underbrace{(1-\alpha_X)\boldsymbol{\pi}_X}_{\text{new unpaired}} + \underbrace{\phi^{\mathbf{d}}\boldsymbol{\delta}_X}_{\text{reentry / deceased}} + \underbrace{\phi^{\mathbf{l}}[\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X - \boldsymbol{\kappa}_X]}_{\text{reentry / all live minus incentivized}} \tag{3}$$

as the total **nonprioritized and unpaired blood-type X patient flow**.

We define the following ratios:

1. The ratio of the deceased-donor effective flow for nonprioritized patients to the nonprioritized and unpaired patient flow is
$$r_X = \frac{\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X}{\boldsymbol{\pi}_X^{np\&u}}. \tag{4}$$
2. For each underdemanded type $X-Y$ except $B-A$ (i.e., $X \neq Y$ and $X \triangleright Y$), the ratio of the flow of incompatible or incentivized type $Y-X$ pairs to the total flow of type $X-Y$ pairs is
$$r_{X-Y} = \frac{\boldsymbol{\pi}_{Y-X}}{\boldsymbol{\pi}_{X-Y}} = \frac{[\theta + \rho_{Y-X}(1-\theta)]p_X \alpha_Y \boldsymbol{\pi}_Y}{p_Y \alpha_X \boldsymbol{\pi}_X}.$$
3. For the remaining underdemanded type $B-A$, the ratio of type $A-B$ flow to type $B-A$ flow is
$$r_{B-A} = \frac{\boldsymbol{\pi}_{A-B}}{\boldsymbol{\pi}_{B-A}} = \frac{p_B \alpha_A \boldsymbol{\pi}_A}{p_A \alpha_B \boldsymbol{\pi}_B}.$$

The first ratio, r_X, is less than one because of our assumption that there is a shortage of deceased-donor kidneys for unpaired new entrants, i.e., $(1-\alpha_X)\boldsymbol{\pi}_X > \boldsymbol{\delta}_X$. The second ratio, r_{X-Y}, is less than one by Assumption 3. Finally, the last ratio, r_{B-A}, is less than or equal to one by Assumption 2. Ratio r_X would be a service rate if we wanted to allocate all blood-type X deceased donors that are reserved for nonprioritized patients to nonprioritized and unpaired blood-type X patients. For an underdemanded type $X-Y$, ratio r_{X-Y} would be a service rate for living-donor transplants if type $X-Y$ pairs did not receive any deceased-donor transplants but only participated in exchange. In these cases, the waiting time for a deceased-donor transplant for nonprioritized and unpaired blood-type X patients would be

$$t_X = S^{-1}\Big(\frac{\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X}{\boldsymbol{\pi}_X^{np\&u}}\Big),$$

and the waiting time of underdemanded-type $X-Y$ pairs would be

$$t_{X-Y} = S^{-1}\Big(\frac{\boldsymbol{\pi}_{Y-X}}{\boldsymbol{\pi}_{X-Y}}\Big).$$

However, underdemanded-type pairs have another option besides exchange. If deceased donors become available earlier than the exchange option, they will receive deceased-donor transplants. As we mentioned in the main text, we assume that patients accept the first donor who is offered to them, either through deceased-donor allocation or exchange. Hence, the patient of a type $X-Y$ pair will never wait for a type $Y-X$ pair for exchange if a deceased donor becomes available first, i.e., if $t_{X-Y} > t_X$. As waiting times are strictly decreasing functions of the r ratios defined above, we need to compare these ratios in an iterative manner to decide whether pairs of one or more underdemanded types will also receive deceased-donor transplants.

To this end, we first define $X-Y_1,\ldots,X-Y_{k(X)}$ as the ordered list of underdemanded types according to ascending r_{X-Y} ratios, where we have $k(O)=3$, $k(B)=2$, $k(A)=1$, and $k(AB)=0$ as the respective numbers of underdemanded pair types whose patients have blood types O, B, A, and AB. We define the following potential pooling ratio for each $\ell = 0,\ldots,k(X)$:

$$r_{X,X-Y_1,\ldots,X-Y_\ell} \;=\; \frac{\boldsymbol{\delta}_X - \phi^{\mathbf{1}}\boldsymbol{\kappa}_X + \boldsymbol{\pi}_{Y_1-X} + \ldots + \boldsymbol{\pi}_{Y_\ell-X}}{\boldsymbol{\pi}_X^{np\&u} + \boldsymbol{\pi}_{X-Y_1} + \ldots + \boldsymbol{\pi}_{X-Y_\ell}}. \tag{5}$$

Exchange regime iterative pooling procedure for unpaired and paired patients:

Fix a blood type X. We iteratively consider the following procedure starting with $\ell = 0$.

Step ℓ: Suppose types $X-Y_1,\ldots,X-Y_\ell$ have already been deemed to be receiving both deceased-donor and exchange transplants.

- If $r_{X-Y_{\ell+1}} < r_{X,X-Y_1,\ldots,X-Y_\ell}$ then type $X-Y_{\ell+1}$ pairs receive exchange and deceased-donor transplants at the same time together with the nonprioritized and unpaired blood-type X patients and type $X-Y_1,\ldots,X-Y_\ell$ pairs. We continue with Step $\ell+1$.
- If $r_{X-Y_{\ell+1}} \geq r_{X,X-Y_1,\ldots,X-Y_\ell}$ then all type $X-Y_{\ell+1},\ldots,X-Y_{k(X)}$ pairs only receive exchange transplants but no transplants from deceased donors. We terminate the procedure.

Based on this procedure, we state the following theorem:

Theorem A-1 (Waiting times under regular and incentivized exchange) *Suppose Assumptions 2 and 3 hold. Consider the ABO-i deceased-donor allocation and incentivized-exchange policies with a given incentivized-exchange participation-rate vector $\rho = (\rho_{X-Y})_{Y\triangleright X, Y\neq X}$ (which can possibly be zero). Consider a blood type X. Then the following statements hold:*

1. *Blood-type X patients, who are in overdemanded-type or self-demanded-type pairs and who have either incompatible donors or are eligible and willing to participate in incentivized exchange, participate in exchange immediately upon their arrival to the patient pool.*
2. *Suppose the patients in pairs of underdemanded types $X-Y_1,\ldots,X-Y_{\ell(X)}$ receive exchange and deceased-donor transplants, while the patients in pairs of underdemanded types*

$X - Y_{\ell(X)+1}, \ldots, X - Y_{k(X)}$ *receive only exchange transplants for some* $\ell(X) \in \{0, \ldots, k(X)\}$.[2] *Then*

- *nonprioritized and unpaired blood-type* X *patients and the patients of type* $X-Y_1, \ldots, X-Y_{\ell(X)}$ *pairs wait for a deceased-donor (or exchange) transplant for the duration*

$$t_X^{\mathbf{i}} = S^{-1}\left(\frac{\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X + \boldsymbol{\pi}_{Y_1-X} + \ldots + \boldsymbol{\pi}_{Y_{\ell(X)}-X}}{\boldsymbol{\pi}_X^{np\&u} + \boldsymbol{\pi}_{X-Y_1} + \ldots + \boldsymbol{\pi}_{X-Y_{\ell(X)}}}\right), \text{ and,} \tag{6}$$

- *for all* $\ell \in \{\ell(X)+1, \ldots, k(X)\}$, *type* $X - Y_\ell$ *pairs are exclusively matched through exchange and wait for an exchange transplant for the duration*

$$t_{X-Y_\ell}^{\mathbf{i}} = S^{-1}\left(\frac{\boldsymbol{\pi}_{Y_\ell-X}}{\boldsymbol{\pi}_{X-Y_\ell}}\right) \leq t_X^{\mathbf{i}}. \tag{7}$$

The average waiting time to a transplant for all blood-type X *patients is*

$$t_X^{\mathbf{i},all} = \frac{\left[\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X + \sum_{\ell=1}^{\ell(X)} \boldsymbol{\pi}_{Y_\ell-X}\right] t_X^{\mathbf{i}} + \sum_{\ell=\ell(X)+1}^{k(X)} \left[\boldsymbol{\pi}_{Y_\ell-X}\, t_{X-Y_\ell}^{\mathbf{i}}\right]}{\boldsymbol{\delta}_X + \boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X} \tag{8}$$

Proof. The proof follows from the procedure discussed before the statement of the theorem. Since blood-type X patients with compatible paired donors and blood-type X patients with incompatible but blood-type-compatible donors have zero waiting time, Equation 8 is established. ■

When $\rho = 0$, we will refer to $t_X^{\mathbf{i}}$ as $t_X^{\mathbf{e}}$ and $t_X^{\mathbf{i},all}$ as $t_X^{\mathbf{e},all}$.

A.3 Welfare Consequences of Transplant Regimes on Access of Patients to Living-Donor Transplantation

We next present a result, which formulates how access to living-donor transplantation differs across blood types with the introduction of each transplantation modality. For this analytical result, we consider a baseline scenario where no blood type has an advantage over another for access to transplantation beyond the asymmetry induced by blood-type compatibility and the impact of the transplantation modalities analyzed. We present a corresponding result for access to deceased-donor transplantation, formulated through waiting times, in the last subsection of this appendix.

Theorem A-2 *Suppose Assumptions 2 and 3 hold. Let* $p_A > p_B$. *Suppose that* $\alpha_X = \alpha$ *for any blood type* X, *and* $\frac{\boldsymbol{\pi}_X}{p_X} = \frac{\boldsymbol{\pi}_Y}{p_Y}$ *for any two blood types* X *and* Y. *Suppose also that the fraction of pairs taking the incentivized-exchange option is uniform at a fixed* $\rho < 1$ *for any eligible type. Then:*

1. *For direct living-donor transplantation, the access to living donation is ranked as*

$$\frac{\boldsymbol{\lambda}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_A}{\boldsymbol{\pi}_A} < \frac{\boldsymbol{\lambda}_{AB}}{\boldsymbol{\pi}_{AB}}.$$

2. *Kidney exchange, in addition to direct living-donor transplantation, by itself increases access to living-donor transplantation for patients of blood type* B *the most, patients of blood type* A *next,*

[2] $\ell(X) = 0$ refers to the case where no underdemanded type with blood-type X patient receives deceased-donor transplant, and $\ell(X) = k(X)$ refers to the case where all underdemanded types with blood-type X patients receive both exchange and deceased-donor transplants.

and patients of blood types AB and O equally and last: $\frac{\boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} > \frac{\boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} > \frac{\boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} = \frac{\boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O}$. *With the inclusion of kidney exchange, overall access to living donation is ranked as*

$$\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} = \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} = \alpha.$$

3. *Incentivized exchange, in addition to regular exchange and direct living-donor transplantation, by itself increases access to living-donor transplantation for patients of blood type O the most, patients of blood types A and B equally and next, and does not increase access for patients of blood type AB:* $\frac{\boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} > \frac{\boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} > \frac{\boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} = 0$. *With the inclusion of either version of incentivized exchange, overall access to living donation is ranked as*

$$\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O + \boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} = \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_A} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB} + \boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} = \alpha.$$

Proof of Theorem A-2. Let $p_A > p_B$. Suppose that $\alpha_X = \alpha$ for any blood type X, and $\frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{p_X}{p_Y}$ for any two blood types X and Y. Also assume that the fraction of pairs taking the incentivized-exchange option is uniform at a fixed $\rho < 1$ for any eligible type.

1. We consider $\boldsymbol{\lambda}_X$, the overall flows of pairs with blood-type X patients participating in direct living-donor-transplantation regime for each blood type X:

$$\begin{aligned}
\frac{\boldsymbol{\lambda}_O}{\boldsymbol{\pi}_O} &= \frac{(1-\theta)p_O\alpha\boldsymbol{\pi}_O}{\boldsymbol{\pi}_O} = (1-\theta)p_O\alpha, \\
\frac{\boldsymbol{\lambda}_A}{\boldsymbol{\pi}_A} &= \frac{(1-\theta)(p_O+p_A)\alpha\boldsymbol{\pi}_A}{\boldsymbol{\pi}_A} = (1-\theta)(p_O+p_A)\alpha, \\
\frac{\boldsymbol{\lambda}_B}{\boldsymbol{\pi}_B} &= \frac{(1-\theta)(p_O+p_B)\alpha\boldsymbol{\pi}_B}{\boldsymbol{\pi}_B} = (1-\theta)(p_O+p_B)\alpha, \text{ and} \\
\frac{\boldsymbol{\lambda}_{AB}}{\boldsymbol{\pi}_{AB}} &= \frac{(1-\theta)\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_{AB}} = (1-\theta)\alpha.
\end{aligned}$$

Thus,

$$\frac{\boldsymbol{\lambda}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_A}{\boldsymbol{\pi}_A}, \frac{\boldsymbol{\lambda}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_{AB}}{\boldsymbol{\pi}_{AB}}.$$

Moreover, since $p_B < p_A$, we have $\frac{\boldsymbol{\lambda}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_A}{\boldsymbol{\pi}_A}$.

2. We consider $\boldsymbol{\epsilon}_X$, the overall flows of pairs that have blood-type X patients and participate in regular exchange, for each X:

$$\begin{aligned}
\frac{\boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} &= \frac{\theta p_O\alpha(\boldsymbol{\pi}_O+\boldsymbol{\pi}_A+\boldsymbol{\pi}_B+\boldsymbol{\pi}_{AB})}{\boldsymbol{\pi}_O} = (\theta p_O + \theta p_A + \theta p_B + \theta p_{AB})\,\alpha = \theta\alpha, \\
\frac{\boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} &= \frac{\theta p_O\alpha\boldsymbol{\pi}_A + \theta p_A\alpha\boldsymbol{\pi}_A + p_B\alpha\boldsymbol{\pi}_A + \theta p_A\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_A} = (\theta p_O + \theta p_A + p_B + \theta p_{AB})\,\alpha, \\
\frac{\boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} &= \frac{\theta p_O\alpha\boldsymbol{\pi}_B + p_B\alpha\boldsymbol{\pi}_A + \theta p_B\alpha\boldsymbol{\pi}_B + \theta p_B\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_B} = (\theta p_O + p_A + \theta p_B + \theta p_{AB})\alpha, \text{ and} \\
\frac{\boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} &= \frac{\theta p_O\alpha\boldsymbol{\pi}_{AB} + \theta p_A\alpha\boldsymbol{\pi}_{AB} + \theta p_B\alpha\boldsymbol{\pi}_{AB} + \theta p_{AB}\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_{AB}} = (\theta p_O + \theta p_A + \theta p_B + \theta p_{AB})\,\alpha = \theta\alpha,
\end{aligned}$$

where the second equality in each line (except the last) follows from the assumption that $\frac{p_X}{\boldsymbol{\pi}_X}$ is constant among all X. Since $\theta < 1$ and $p_A, p_B > 0$, we have

$$\frac{\boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} = \frac{\boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} < \frac{\boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A}, \frac{\boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B}.$$

With the additional assumption $p_A > p_B$, we obtain $\frac{\boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} < \frac{\boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B}$.

We consider each $\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X$, the flow of direct living-donor and exchange transplants in total. We have

$$\begin{aligned}
\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} &= (1-\theta)p_O\alpha + \theta(p_O + p_A + p_B + p_{AB})\alpha = (p_O + \theta p_A + \theta p_B + \theta p_{AB})\alpha, \\
\frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} &= (1-\theta)(p_O + p_A)\alpha + (\theta p_O + \theta p_A + p_B + \theta p_{AB})\alpha = (p_O + p_A + p_B + \theta p_{AB})\alpha, \\
\frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} &= (1-\theta)(p_O + p_B)\alpha + (\theta p_O + p_A + \theta p_B + \theta p_{AB})\alpha = (p_O + p_A + p_B + \theta p_{AB})\alpha, \text{ and} \\
\frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} &= (1-\theta)\alpha + \theta\alpha = \alpha.
\end{aligned}$$

Since $\theta < 1$ and $p_A, p_B, p_{AB} > 0$,

$$\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} = \alpha.$$

3. Next we consider $\boldsymbol{\iota}_X$, the overall flow of pairs with blood-type X patients benefitting from incentivized exchange for each blood type X:

$$\begin{aligned}
\frac{\boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} &= \frac{\rho(1-\theta)p_O\alpha\boldsymbol{\pi}_A + \rho(1-\theta)p_O\alpha\boldsymbol{\pi}_B + \rho(1-\theta)p_O\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_O} = \rho(1-\theta)(p_A + p_B + p_{AB})\alpha, \\
\frac{\boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} &= \frac{\rho(1-\theta)p_A\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_A} = \rho(1-\theta)p_{AB}\alpha, \\
\frac{\boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} &= \frac{\rho(1-\theta)p_B\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_B} = \rho(1-\theta)p_{AB}\alpha, \text{ and} \\
\frac{\boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} &= 0,
\end{aligned}$$

where the second equality in each line (except the last) follows from the assumption that $\frac{p_X}{\boldsymbol{\pi}_X}$ is constant among all X. Since $\rho > 0$, $\theta < 1$, and $p_A, p_B, p_{AB} > 0$,

$$0 = \frac{\boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} < \frac{\boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\iota}_O}{\boldsymbol{\pi}_O}.$$

We consider each $\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X$, i.e., direct living-donor, regular-exchange, and incentivized-exchange transplants in total. We have

$$\begin{aligned}
\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O + \boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} &= p_O\alpha + [\theta + \rho(1-\theta)](p_A + p_B + p_{AB})\alpha, \\
\frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A + \boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} &= (p_O + p_A + p_B)\alpha + [\theta + \rho(1-\theta)]p_{AB}\alpha, \\
\frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} &= (p_O + p_A + p_B)\alpha + [\theta + \rho(1-\theta)]p_{AB}\alpha, \text{ and} \\
\frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB} + \boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} &= \alpha.
\end{aligned}$$

Since $\rho, p_{AB}, \alpha > 0$,

$$\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O + \boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A + \boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB} + \boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} = \alpha,$$

and they are all equal if and only if $\rho = 1$, because $\theta < 1$. ■

A.4 Consequences of Different Transplantation Regimes on Transplant Waiting Times

In this subsection, we state and prove a lemma that formalizes the marginal effects of living-donor exchange policies on the transplant waiting times of the following 29 groups of patients under some reasonable assumptions. These 29 groups are nonprioritized and unpaired patients of each blood type (4 groups), compatible pairs of overdemanded and self-demanded types (5 groups for overdemanded types and 4 groups for self-demanded types; recall that the overdemanded type $A-B$ pairs are never compatible), incompatible pairs of overdemanded and self-demanded types (6 groups for overdemanded types and 4 groups for self-demanded types), and pairs of underdemanded types (6 groups).

In addition to Assumptions 2 and 3, we also assume that the tissue-type incompatibility probability θ and the reentry rate of living-donor-transplant recipients $\phi^{\mathbf{l}}$ are sufficiently small. Formally, "for a vector of **sufficiently small** parameters x, some claim holds" means that "there exists some vector $\overline{x} \gg 0$ (i.e., all entries of the vector are larger than 0) such that for all x, $0 \leq x \leq \overline{x}$, that claim holds." These assumptions guarantee that all underdemanded-type pairs, except possibly type $B-A$, are pooled with their respective nonprioritized and unpaired patients for deceased-donor transplantation under the regular-exchange regime. Furthermore, we also assume that the difference between flows of pair types $B-A$ and $A-B$ is sufficiently small. This guarantees that $B-A$ pairs only participate in exchange and are never pooled for deceased-donor transplantation in all of exchange regimes we consider. This lemma will be used to prove Theorem A-3, our last result of this appendix in the next subsection.

Lemma A-2 (Consequences of regular and incentivized exchange) *Suppose Assumptions 2 and 3 hold for a given incentivized-exchange participation-fraction profile $\rho^* = (\rho^*_{X-Y})_{Y \triangleright X \,\&\, Y \neq X}$. Suppose also that reentry fraction of living-donor-transplant recipients $\phi^{\mathbf{l}}$, inflow difference between types $B-A$ and $A-B$ given as $p_A \alpha_B \boldsymbol{\pi}_B - p_B \alpha_A \boldsymbol{\pi}_A$, and tissue-type incompatibility probability θ are sufficiently small. Then the following results hold:*

- *With respect to deceased-donor/direct living-donor transplantation regime, regular-exchange regime causes steady-state transplant waiting times of all nonprioritized and unpaired patient groups and all incompatible pair groups to decrease. In particular, in addition to compatible pair groups, all incompatible overdemanded and self-demanded pair groups no longer wait for a transplant and receive exchange transplants immediately upon entry to the patient pool.*
- *With respect to regular-exchange regime, incentivized-exchange regime causes the transplant waiting times of*
  - *all overdemanded- and self-demanded-type pair groups to stay at zero,*
  - *all underdemanded-type pair groups except type $B-A$ pairs to decrease,*
  - *type $B-A$ pairs not to change,*
  - *nonprioritized and unpaired blood-type O, A, and B patient groups to decrease, and*
  - *nonprioritized and unpaired blood-type AB patient group to increase.*

Proof of Lemma A-2. Suppose we fix an incentivized-exchange participation-fraction vector $\rho^* = (\rho^*_{X-Y})_{Y \triangleright X \,\&\, Y \neq X}$ such that Assumptions 2 and 3 hold. Then under any of the exchange policies (i.e., regular with $\rho = 0$ or incentivized with $\rho = \rho^*$) the flow of underdemanded-type $X - Y$ pairs and their reciprocal-type $Y - X$ pairs (from Equation 1) satisfy:

$$\boldsymbol{\pi}_{X-Y} = p_Y \alpha_X \boldsymbol{\pi}_X \geq \boldsymbol{\pi}_{Y-X} = \begin{cases} [\theta + \rho_{Y-X}(1-\theta)] p_X \alpha_Y \boldsymbol{\pi}_Y & \text{if } Y - X \neq A - B \\ p_B \alpha_A \boldsymbol{\pi}_A & \text{otherwise} \end{cases}.$$

As we have established before, in the optimal, ABO-i exchange regime for regular and incentivized exchange, none of the pairs of incompatible overdemanded and self-demanded types wait for a transplant, as they immediately receive transplant through exchange.

In the rest of the proof, we focus on the other patient groups: underdemanded-type pairs and nonprioritized and unpaired patients.

Suppose also that the tissue-type incompatibility probability, θ, and the inflow difference between types $B - A$ and $A - B$, $p_A \alpha_B \boldsymbol{\pi}_B - p_B \alpha_A \boldsymbol{\pi}_A$, are sufficiently small.

We prove the following claim first:

Claim 1. Under regular exchange, patients of all underdemanded-types pairs except that of $B - A$ are pooled with nonprioritized and unpaired deceased-donor-transplantation recipients of the same blood type, while patients of type $B - A$ pairs are never pooled with nonprioritized and unpaired blood-type B patients under any exchange regime.

Proof of Claim 1. For a blood type $X \in \{O, A, B\}$ (note that blood-type AB patients are not in any underdemanded-type pairs), under regular exchange we have $\boldsymbol{\kappa}_X\big|_{\rho=0} = 0$. We also have

$$r_X\big|_{\rho=0} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X^{np\&u}\big|_{\rho=0}} > r_{X-Y}\big|_{\rho=0} = \frac{\theta p_X \alpha_Y \boldsymbol{\pi}_Y}{p_Y \alpha_X \boldsymbol{\pi}_X} \tag{9}$$

for any underdemanded type $X - Y \neq B - A$, where the inequality follows from sufficiently small θ assumption.

Recall that $k(A) = 1$ and $k(O) = 3$ are the numbers of underdemanded pair types with blood-type A and O patients, respectively.

Thus, pairs of the only underdemanded type with blood-type A patient, $A - AB$, are pooled with nonprioritized and unpaired blood-type A patients under regular exchange by Equation 9.

We order underdemanded pair types with patient blood type O according to the ascending order of their r ratios as $O - Y_1$, $O - Y_2$, and $O - Y_3$. Then, for $\ell = 1, 2$,

$$r_{O,O-Y_1,\ldots,O-Y_\ell}\big|_{\rho=0} = \frac{\boldsymbol{\delta}_O + \sum_{m=1}^{\ell} \theta p_O \alpha_{Y_m} \boldsymbol{\pi}_{Y_m}}{\boldsymbol{\pi}_O^{np\&u}\big|_{\rho=0} + \sum_{m=1}^{\ell} p_{Y_m} \alpha_O \boldsymbol{\pi}_O} > \frac{\theta p_O \alpha_{Y_{\ell+1}} \boldsymbol{\pi}_{Y_{\ell+1}}}{p_{Y_{\ell+1}} \alpha_O \boldsymbol{\pi}_O} = r_{O-Y_{\ell+1}}\big|_{\rho=0} \tag{10}$$

because of the assumption that θ is sufficiently small.

Thus, under regular exchange, underdemanded-type pairs with blood-type O patients are pooled for deceased-donor transplantation with nonprioritized and unpaired blood-type O patients.

On the other hand, for the underdemanded pair type $B - A$, we have

$$r_{B-A} = \frac{p_B \alpha_A \boldsymbol{\pi}_A}{p_A \alpha_B \boldsymbol{\pi}_B} > r_B \tag{11}$$

for any ρ because of the assumption that the difference $p_A\alpha_B\boldsymbol{\pi}_B - p_B\alpha_A\boldsymbol{\pi}_A$ is sufficiently small. Thus, pairs of type $B-A$ are never pooled with nonprioritized and unpaired blood-type B patients under regular or incentivized exchange for any ρ.

Equation 9 with $X = B$ and Equation 11 imply that pairs of type $B - AB$ are pooled with nonprioritized and unpaired blood-type B patients under regular exchange. $\square$

We also assume that the fraction of living-donor-transplant recipients reentering the patient pool, $\phi^{\mathbf{l}}$, is also sufficiently small in the rest of the proof.

Transition to Regular Exchange:

Consider a blood type X. The flow of pairs that benefit from direct living-donor transplant regime is given by $\boldsymbol{\lambda}_X = \sum_{Y:Y\triangleright X}(1-\theta)p_Y\alpha_X\boldsymbol{\pi}_X$. The flow of pairs that benefit from regular exchange satisfies $\boldsymbol{\epsilon}_X = \sum_{Y:Y\triangleright X}\theta p_Y\alpha_X\boldsymbol{\pi}_X + \sum_{Y:X\triangleright Y, Y\neq X}\theta p_X\alpha_Y\boldsymbol{\pi}_Y + \mathbf{1}_{\left\{X\in\{A,B\}\right\}}p_B\alpha_A\boldsymbol{\pi}_A > 0$.[3] This is also the flow of patients that fall out of competition from the blood-type X deceased-donor queue with respect to the deceased-donor/direct living-donor transplantation regime.

We consider the ratio of the available deceased-donor flow to the flow of patients who cannot receive direct living donation, which we refer to as $r^{\mathbf{l}}_X$, and $r_{X,X-Y_1,\ldots,X-Y_{\ell(X)|_{\rho=0}}}\big|_{\rho=0}$ when pairs of underdemanded types $X-Y_1,\ldots,X-Y_{\ell(X)|_{\rho=0}}$ are pooled for deceased-donor transplantation under regular exchange. We have

$$r^{\mathbf{l}}_X = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X - \underbrace{\sum_{Y:Y\triangleright X}(1-\theta)p_Y\alpha_X\boldsymbol{\pi}_X}_{=\boldsymbol{\lambda}_X} + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}\boldsymbol{\lambda}_X} \quad \text{and} \tag{12}$$

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0} = \frac{\boldsymbol{\delta}_X + \sum_{m=1}^{\ell(X)|_{\rho=0}}(\theta p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m})}{\underbrace{\boldsymbol{\pi}_X - \alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}\boldsymbol{\lambda}_X + \phi^{\mathbf{l}}\boldsymbol{\epsilon}_X}_{=\boldsymbol{\pi}^{np\&u}_X\big|_{\rho=0}} + \sum_{m=1}^{\ell(X)|_{\rho=0}}(p_{Y_m}\alpha_X\boldsymbol{\pi}_X)}. \tag{13}$$

Claim 2. The transplant waiting time decreases with the addition of regular exchange to deceased-donor/direct living-donor transplantation for unpaired patients and underdemanded-type pairs.

Proof of Claim 2. For all X, we have from Equations 12 and 13 that, when $\phi^{\mathbf{l}} = 0$

$$r^{\mathbf{l}}_X\big|_{\phi^{\mathbf{l}}=0} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X - (1-\theta)\sum_{Y:Y\triangleright X}p_Y\alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X} \text{ and}$$

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0,\phi^{\mathbf{l}}=0} = \frac{\boldsymbol{\delta}_X + \sum_{m=1}^{\ell(X)}\big|_{\rho=0,\phi^{\mathbf{l}}=0}(\theta p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m})}{\boldsymbol{\pi}_X - (1-\sum_{m=1}^{\ell(X)}\big|_{\rho=0,\phi^{\mathbf{l}}=0}p_{Y_m})\alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X}.$$

Since $B-A$ pairs are not pooled for deceased-donor transplantation by Claim 1, we have $B-A \neq X-Y_m$ for any X and m. Thus, for each Y_m, $X\triangleright Y_m$ and $Y_m \neq X$. Thus, we obtain

[3]The indicator function $\mathbf{1}_{\{Z\}}$gets value 1 if the event Z is true and value 0 if the event Z is false.

$1-\sum_{m=1}^{\ell(X)}\big|_{\rho=0,\phi^{\mathbf{l}}=0} p_{Y_m} \geq 1-\sum_{Y:X\triangleright Y, Y\neq X} p_Y$. We also have $1-\sum_{Y:X\triangleright Y,Y\neq X} p_Y = \sum_{Y:Y\triangleright X} p_Y > (1-\theta)\sum_{Y:Y\triangleright X} p_Y$, as $\theta>0$. Thus,

$$r^{\mathbf{l}}_X\big|_{\phi^{\mathbf{l}}=0} < r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0,\phi^{\mathbf{l}}=0}.$$

By the continuity of the r ratios in $\phi^{\mathbf{l}}$, for sufficiently small $\phi^{\mathbf{l}}$ we have $r^{\mathbf{l}}_X < r_X\big|_{\rho=0}$, implying that

$$t^{\mathbf{l}}_X = S^{-1}\left(r^{\mathbf{l}}_X\right) > S^{-1}\left(r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0}\right) = t^{\mathbf{e}}_X. \tag{14}$$

Since by Claim 1 pairs of underdemanded types except $B-A$ are pooled with deceased-donor-transplant recipients under regular exchange, their transplant waiting times also decrease. Moreover, the transplant waiting time of type $B-A$ pairs decreases even more, as it is not pooled with deceased-donor-transplant recipients by Claim 1.□

Transition to Incentivized Exchange:

Consider a blood type $X\in\{A,B,O\}$. Suppose ρ^* is the participation profile for incentivized exchange. The flow of pairs who benefit from incentivized exchange with any ρ in addition to regular exchange satisfies

$$\boldsymbol{\iota}_X = \sum_{Y:X\triangleright Y,Y\neq X} \rho_{Y-X}(1-\theta)p_X\alpha_Y\boldsymbol{\pi}_Y,$$

while the flow of prioritized reentrants satisfies

$$\phi^{\mathbf{l}}\boldsymbol{\kappa}_X = \phi^{\mathbf{l}}\left(\sum_{Y:Y\triangleright X,Y\neq X} \rho_{X-Y}(1-\theta)p_Y\alpha_X\boldsymbol{\pi}_X\right).$$

As a result, for some $\ell(X)\in\{0,\ldots,k(X)\}$, pairs of underdemanded types $X-Y_1,\ldots,X-Y_{\ell(X)}$ are pooled for deceased-donor transplantation at ρ, and thus, we have

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}} = \frac{\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X + \sum_{m=1}^{\ell(X)}\left([\theta+\rho_{Y_m-X}(1-\theta)]p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}\right)}{\underbrace{\boldsymbol{\pi}_X - \alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}\boldsymbol{\lambda}_X + \phi^{\mathbf{l}}\boldsymbol{\epsilon}_X + \phi^{\mathbf{l}}\boldsymbol{\iota}_X}_{=\boldsymbol{\pi}_X^{np\&u}} + \sum_{m=1}^{\ell(X)} p_{Y_m}\alpha_X\boldsymbol{\pi}_X}. \tag{15}$$

Claim 3. The transplant waiting times decrease under incentivized exchange with respect to regular exchange for nonprioritized and unpaired blood-type X patients and all underdemanded-type pairs with blood-type X patients—except type $B-A$.

Proof of Claim 3. We will show that all ratios r_{X-Y_m} for all $m=1,\ldots,k(X)$, such that $X-Y_m\neq B-A$, and ratio $r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}$ increase from $\rho=0$ to $\rho=\rho^*$, and thus, the related transplant waiting time decreases. We have $\ell(X)=\ell(X)\big|_{\rho=0}$ (i.e., the number of pooled types at $\rho=0$) for sufficiently small ρ profiles, since r ratios are continuous around $\rho=0$ and there are no sudden jumps in pooling by Claim 1. Thus, for sufficiently small ρ, when $\phi^{\mathbf{l}}=0$

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\phi^{\mathbf{l}}=0} = \frac{\boldsymbol{\delta}_X + \sum_{m=1}^{\ell(X)|_{\phi^{\mathbf{l}}=0}}\left([\theta+\rho_{Y_m-X}(1-\theta)]p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}\right)}{\boldsymbol{\pi}_X - \alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \sum_{m=1}^{\ell(X)|_{\phi^{\mathbf{l}}=0}} p_{Y_m}\alpha_X\boldsymbol{\pi}_X}$$

and

$$r_{X-Y_m} = \frac{[\theta + \rho_{Y_m-X}(1-\theta)]p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}}{p_{Y_m}\alpha_X\boldsymbol{\pi}_X}$$

are increasing in ρ. Suppose that we increase each ρ_{W-Z} from 0 to ρ^*_{W-Z} in a steady speed equal to ρ^*_{W-Z} throughout so that ρ reaches ρ^* at time $t = 1$. We can compare the rates of change in both entities along this line as the inner product of their gradient vector and speed vector:

$$\begin{aligned}\left(\nabla_\rho r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\right)\cdot\rho^*\big|_{\phi^{\mathbf{l}}=0} &= \frac{\sum_{m=1}^{\ell(X)|_{\phi^{\mathbf{l}}=0}}\rho^*_{Y_m-X}(1-\theta)p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}}{\boldsymbol{\pi}_X-\alpha_X\boldsymbol{\pi}_X+\phi^{\mathbf{d}}\boldsymbol{\delta}_X+\sum_{m=1}^{\ell(X)|_{\phi^{\mathbf{l}}=0}}p_{Y_m}\alpha_X\boldsymbol{\pi}_X}\\ &< \frac{\rho^*_{Y_m-X}(1-\theta)p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}}{p_{Y_m}\alpha_X\boldsymbol{\pi}_X} = (\nabla_\rho r_{X-Y_m})\cdot\rho^*,\end{aligned}$$

for $m = \ell(X)|_{\phi^{\mathbf{l}}=0}$, i.e., the r ratio for the pooled nonprioritized and unpaired patients and underdemanded-type pairs changes slower than the largest of the r ratios of the underdemanded types that are pooled when $\phi^{\mathbf{l}} = 0$. Thus, as ρ increases to ρ^*, either ρ reaches ρ^* without $\ell(X)|_{\phi^{\mathbf{l}}=0}$ changing or there will be a profile ρ^1 such that $0 \ll \rho^1 < \rho^*$, at which $\ell(X)|_{\phi^{\mathbf{l}}=0}$ decreases to $\ell(X)\big|_{\rho=0,\phi^{\mathbf{l}}=0} - 1$ so that pairs of the underdemanded type with the highest r ratio are no longer pooled with the rest. Similarly, the resulting new r value relevant for the pool of nonprioritized and unpaired patients and remaining underdemanded-type pairs will be increasing in ρ until ρ reaches a new cutoff $\rho^2 \leq \rho^*$. At this new cutoff $\ell(X)\big|_{\rho=\rho^2,\phi^{\mathbf{l}}=0} = \ell(X)\big|_{\rho=0,\phi^{\mathbf{l}}=0} - 2$, and so on, so forth. Possibly, no underdemanded pairs may remain pooled at sufficiently high ρ, implying that $\ell(X)|_{\phi^{\mathbf{l}}=0} = 0$, and thus, $\left(\nabla_\rho r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\right)\cdot\rho^*\big|_{\phi^{\mathbf{l}}=0} = 0$. Except after this last iteration, all r ratios strictly increase at each iteration until $t = 1$ at different speeds when $\phi^{\mathbf{l}} = 0$.

In the end, for sufficiently small $\phi^{\mathbf{l}}$, by the continuity of the r ratios (and their gradients) in $\phi^{\mathbf{l}}$ and by the fact that all underdemanded-type pairs were pooled initially at $\rho = 0$, all gradients are strictly positive at least for small ρ. Thus, we obtain that the r ratios strictly increase from $\rho = 0$ to $\rho = \rho^*$. As the transplant waiting time is decreasing in its relevant r ratio for each patient group, for all underdemanded types with blood-type X patient blood type—except type $B-A$—and nonprioritized and unpaired blood-type X patients, the transplant waiting times strictly decrease with respect to their levels at $\rho = 0$. □

On the other hand, all paired blood-type AB patients receive direct or exchange living-donor transplants without waiting when $\rho = 0$. This fact does not change when $\rho = \rho^*$. Thus, in both cases the flow of blood-type patients that enter the deceased-donor queue is the same. In particular, as there are no underdemanded pair types with blood-type AB patients, Equation 15 implies

$$t^{\mathbf{i}}_{AB}|_\rho = S^{-1}\left(\frac{\boldsymbol{\delta}-\phi^{\mathbf{l}}\boldsymbol{\kappa}_{AB}|_\rho}{\boldsymbol{\pi}_X-(1-\phi^{\mathbf{l}})\alpha_X\boldsymbol{\pi}_X+\phi^{\mathbf{d}}\boldsymbol{\delta}_X}\right).$$

Since $\boldsymbol{\kappa}_{AB}|_{\rho=\rho^*} > \boldsymbol{\kappa}_{AB}|_{\rho=0} = 0$,

$$t^{\mathbf{i}}_{AB}|_{\rho=\rho^*} > t^{\mathbf{i}}_{AB}|_{\rho=0},$$

i.e., the transplant waiting time of nonprioritized and unpaired blood-type AB patients strictly increases from $\rho = 0$ to $\rho = \rho^*$ regardless of $\phi^{\mathbf{l}}$ and θ.

For sufficiently small type $B-A$ and type $A-B$ flow difference, since pairs of type $B-A$ are not

pooled with nonprioritized and unpaired B patients regardless of ρ^* by Claim 1, their transplant waiting time remains unaffected for any ρ, including $\rho = 0$ and $\rho = \rho^*$. ■

A.5 Welfare Consequences of Different Transplant Regimes on Deceased-Donor Queues

Our last result of this section formulates how access to deceased-donor transplantation differs with the successive introduction of deceased-donor transplantation, living-donor transplantation, kidney exchange, and incentivized exchange.

Theorem A-3 *Suppose Assumptions 2 and 3 hold. Let $p_A > p_B$. Suppose that $\alpha_X = \alpha$ for any blood type X, and $\frac{\boldsymbol{\delta}_X}{\boldsymbol{\delta}_Y} = \frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{p_X}{p_Y}$ for any two blood types X and Y. Suppose also that the fraction of pairs taking the incentivized-exchange option is uniform at a fixed $\rho < 1$ for any eligible type. Then:*

1. *With deceased-donor transplantation only, the transplant waiting time at each deceased-donor queue is the same for any blood type X:*

$$t^{\mathbf{d}}_O = t^{\mathbf{d}}_A = t^{\mathbf{d}}_B = t^{\mathbf{d}}_{AB}.$$

2. *Introduction of direct living-donor transplantation reduces the transplant waiting time at each deceased-donor queue. The changes in transplant waiting times and the transplant waiting times are ranked as follows:*

$$(t^{\mathbf{d}}_{AB} - t^{\mathbf{l}}_{AB}) > (t^{\mathbf{d}}_A - t^{\mathbf{l}}_A) > (t^{\mathbf{d}}_B - t^{\mathbf{l}}_B) > (t^{\mathbf{d}}_O - t^{\mathbf{l}}_O),$$
$$t^{\mathbf{l}}_{max} = t^{\mathbf{l}}_O > t^{\mathbf{l}}_B > t^{\mathbf{l}}_A > t^{\mathbf{l}}_{AB} = t^{\mathbf{l}}_{min}.$$

Further suppose that θ and $\phi^{\mathbf{l}}$ are sufficiently small. Then:

3. *Introduction of kidney exchange in addition to deceased-donor/direct living-donor transplantation further reduces the transplant waiting time at each deceased-donor queue, but more for blood type B than blood type A equalizing the deceased-donor queue transplant waiting times for these two blood types. The combination of kidney exchange and living-donor transplantation reduces the transplant waiting time at the blood-type AB deceased-donor queue the most, at the blood-type A and B deceased-donor queues equally next, and at the blood-type O deceased-donor queue the least:*

$$(t^{\mathbf{d}}_{AB} - t^{\mathbf{e}}_{AB}) > (t^{\mathbf{d}}_A - t^{\mathbf{e}}_A) = (t^{\mathbf{d}}_B - t^{\mathbf{e}}_B) > (t^{\mathbf{d}}_O - t^{\mathbf{e}}_O).$$

   The inclusion of kidney exchange with deceased-donor/direct living-donor transplantation results in the following ranking of the transplant waiting times:

$$t^{\mathbf{e}}_{max} = t^{\mathbf{e}}_O > t^{\mathbf{e}}_B = t^{\mathbf{e}}_A > t^{\mathbf{e}}_{AB} = t^{\mathbf{e}}_{min}.$$

4. *Inclusion of incentivized exchange with regular exchange and deceased-donor/direct living-donor transplantation decreases the transplant waiting times at the blood-type O, A, and B deceased-donor queues but increases it at the blood-type AB deceased-donor queue. The waits at the blood-type A and B deceased-donor queues continue to be equal:*

$$t^{\mathbf{i}}_O < t^{\mathbf{e}}_O, \qquad t^{\mathbf{i}}_A = t^{\mathbf{i}}_B < t^{\mathbf{e}}_A = t^{\mathbf{e}}_B, \qquad t^{\mathbf{i}}_{AB} > t^{\mathbf{e}}_{AB}.$$

Proof of Theorem A-3. Suppose Assumptions 2 and 3 hold. Let $p_A > p_B$. Suppose that $\alpha_X = \alpha$ for any blood type X, and $\frac{\boldsymbol{\delta}_X}{\boldsymbol{\delta}_Y} = \frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{p_X}{p_Y}$ for any two blood types X and Y. Also assume that the fraction of pairs taking the incentivized-exchange option is uniform at a fixed $\rho < 1$ for any eligible pair type.

1. ***With deceased-donor transplantation only***, the transplant waiting time at each deceased-donor queue is $t^{\mathbf{d}}_X = S^{-1}\left(\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X}\right) = S^{-1}\left(\frac{\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}{1+\phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}\right)$ for any blood type X. Since $\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X} = \frac{\boldsymbol{\delta}_Y}{\boldsymbol{\pi}_Y}$ for any two blood types X and Y, we have $t^{\mathbf{d}}_X = t^{\mathbf{d}}_Y$.
2. ***Introduction of direct living-donor transplantation*** reduces the transplant waiting time at each deceased-donor queue X, since $t^{\mathbf{l}}_X = S^{-1}\left(\frac{\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}{1+\phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X} - (1-\phi^{\mathbf{l}})p^{\mathbf{l}}_X\alpha}\right) < S^{-1}\left(\frac{\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}{1+\phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}\right) = t^{\mathbf{d}}_X$. Since the probability of being compatible with the paired donor conditional on having a living donor satisfies for each blood type

$$p^{\mathbf{l}}_O = (1-\theta)p_O, \qquad p^{\mathbf{l}}_B = (1-\theta)(p_O + p_B),$$
$$p^{\mathbf{l}}_A = (1-\theta)(p_O + p_A), \qquad p^{\mathbf{l}}_{AB} = (1-\theta)(p_O + p_A + p_B + p_{AB}) = (1-\theta),$$

and $p_A > p_B$, we have $p^{\mathbf{l}}_O < p^{\mathbf{l}}_B < p^{\mathbf{l}}_A < p^{\mathbf{l}}_{AB}$. Thus, as $t^{\mathbf{l}}_X$ is decreasing in $p^{\mathbf{l}}_X$ and $\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}$ is constant among blood types, we have

$$t^{\mathbf{l}}_{AB} < t^{\mathbf{l}}_A < t^{\mathbf{l}}_B < t^{\mathbf{l}}_O.$$

Moreover, Part 1 implies that

$$\left(t^{\mathbf{d}}_{AB} - t^{\mathbf{l}}_{AB}\right) > \left(t^{\mathbf{d}}_A - t^{\mathbf{l}}_A\right) > \left(t^{\mathbf{d}}_B - t^{\mathbf{l}}_B\right) > \left(t^{\mathbf{d}}_O - t^{\mathbf{l}}_O\right).$$

Further assume that θ and $\phi^{\mathbf{l}}$ are sufficiently small in the rest of the proof. We also have the flow difference between type $B-A$ and type $A-B$ as $p_B\alpha\boldsymbol{\pi}_A - p_A\alpha\boldsymbol{\pi}_B = 0$ since $\frac{p_A}{p_B} = \frac{\boldsymbol{\pi}_A}{\boldsymbol{\pi}_B}$. Thus, hypothesis of Lemma A-2 holds.

3. ***Introduction of regular exchange***, in addition to deceased-donor/direct living-donor transplantation, causes the deceased-donor waiting times for all blood types to decrease by Lemma A-2. By Claim 1 in the proof of the same lemma, pairs of all underdemanded types except $B-A$ are pooled for deceased-donor transplantation with unpaired patients of their patients' blood types. By Equation 13 and $\frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{p_X}{p_Y}$ for any two blood types X, Y, we obtain

$$\begin{aligned} r_{O,O-A,O-B,O-AB}\big|_{\rho=0} &= \frac{\boldsymbol{\delta}_O + (\theta p_O\alpha\boldsymbol{\pi}_A + \theta p_O\alpha\boldsymbol{\pi}_B + \theta p_O\alpha\boldsymbol{\pi}_{AB})}{\boldsymbol{\pi}_O - \alpha\boldsymbol{\pi}_O + \phi^{\mathbf{d}}\boldsymbol{\delta}_O + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O) + (p_A\alpha\boldsymbol{\pi}_O + p_B\alpha\boldsymbol{\pi}_O + p_{AB}\alpha\boldsymbol{\pi}_O)} \\ &= \frac{\frac{\boldsymbol{\delta}_O}{\boldsymbol{\pi}_O} + (\theta p_A\alpha + \theta p_B\alpha + \theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_O}{\boldsymbol{\pi}_O} + \phi^{\mathbf{l}}\left(\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O}\right) + (p_A\alpha + p_B\alpha + p_{AB}\alpha)}, \end{aligned}$$

$$r_{A,A-AB}\big|_{\rho=0} = \frac{\boldsymbol{\delta}_A + (\theta p_A \alpha \boldsymbol{\pi}_{AB})}{\boldsymbol{\pi}_A - \alpha\boldsymbol{\pi}_A + \phi^{\mathbf{d}}\boldsymbol{\delta}_A + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A) + (p_{AB}\alpha\boldsymbol{\pi}_A)} = \frac{\frac{\boldsymbol{\delta}_A}{\boldsymbol{\pi}_A} + (\theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_A}{\boldsymbol{\pi}_A} + \phi^{\mathbf{l}}\left(\frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A}\right) + (p_{AB}\alpha)},$$

$$r_{B,B-AB}\big|_{\rho=0} = \frac{\boldsymbol{\delta}_B + (\theta p_B \alpha \boldsymbol{\pi}_{AB})}{\boldsymbol{\pi}_B - \alpha\boldsymbol{\pi}_B + \phi^{\mathbf{d}}\boldsymbol{\delta}_B + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B) + (p_{AB}\alpha\boldsymbol{\pi}_B)} = \frac{\frac{\boldsymbol{\delta}_B}{\boldsymbol{\pi}_B} + (\theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_B}{\boldsymbol{\pi}_B} + \phi^{\mathbf{l}}\left(\frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B}\right) + (p_{AB}\alpha)},$$

$$r_{AB} = \frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_B - \alpha\boldsymbol{\pi}_{AB} + \phi^{\mathbf{d}}\boldsymbol{\delta}_{AB} + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB})} = \frac{\frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_{AB}}}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_{AB}} + \phi^{\mathbf{l}}\left(\frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}}\right)}.$$

Since $\frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\delta}_Y}$ for any two blood types X and Y, we have by Theorem A-2, $\frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B}$, and thus, $r_{A,A-AB}\big|_{\rho=0} = r_{B,B-AB}\big|_{\rho=0}$ implying that

$$t^{\mathbf{e}}_A = S^{-1}\left(r_{A,A-AB}\big|_{\rho=0}\right) = S^{-1}\left(r_{B,B-AB}\big|_{\rho=0}\right) = t^{\mathbf{e}}_B.$$

Suppose $\phi^{\mathbf{l}} = 0$. Then,

$$r_{O,O-A,O-B,O-AB}\big|_{\rho=0,\phi^{\mathbf{l}}=0} = \frac{\frac{\boldsymbol{\delta}_O}{\boldsymbol{\pi}_O} + (\theta p_A\alpha + \theta p_B\alpha + \theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_O}{\boldsymbol{\pi}_O} + (p_A\alpha + p_B\alpha + p_{AB}\alpha)},$$

$$r_{A,A-AB}\big|_{\rho=0,\phi^{\mathbf{l}}=0} = \frac{\frac{\boldsymbol{\delta}_A}{\boldsymbol{\pi}_A} + (\theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_A}{\boldsymbol{\pi}_A} + (p_{AB}\alpha)}, \text{ and}$$

$$r_{AB}\big|_{\phi^{\mathbf{l}}=0} = \frac{\frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_{AB}}}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_{AB}}}.$$

Since for sufficiently small θ, $\frac{\boldsymbol{\delta}_X/\boldsymbol{\pi}_X}{1-\alpha+\phi^{\mathbf{d}}\boldsymbol{\delta}_X/\boldsymbol{\pi}_X} > \theta$ for any X, we have that $\frac{\boldsymbol{\delta}_X/\boldsymbol{\pi}_X + \theta q(X)}{1-\alpha+\phi^{\mathbf{d}}\boldsymbol{\delta}_X/\boldsymbol{\pi}_X + q(X)}$ is decreasing in $q(X)$ for any $q(X) \geq 0$. Thus, we can rank the above entities as $r_{O,O-A,O-B,O-AB}\big|_{\rho=0,\phi^{\mathbf{l}}=0} < r_{A,A-AB}\big|_{\rho=0,\phi^{\mathbf{l}}=0} < r_{AB}\big|_{\phi^{\mathbf{l}}=0}$. By the continuity of these r ratios in $\phi^{\mathbf{l}}$, for sufficiently small $\phi^{\mathbf{l}}$ we still have $r_{O,O-A,O-B,O-AB}\big|_{\rho=0} < r_{A,A-AB}\big|_{\rho=0} < r_{AB}$. As the generic transplant waiting time $t = S^{-1}(r)$ is decreasing in r, we can rank the waiting times for deceased-donor transplantation in the queue under regular exchange as

$$t^{\mathbf{e}}_{AB} < t^{\mathbf{e}}_A = t^{\mathbf{e}}_B < t^{\mathbf{e}}_O,$$

and thus, by Part 1,

$$\left(t^{\mathbf{d}}_{AB} - t^{\mathbf{e}}_{AB}\right) > \left(t^{\mathbf{d}}_A - t^{\mathbf{e}}_A\right) = \left(t^{\mathbf{d}}_B - t^{\mathbf{e}}_B\right) > \left(t^{\mathbf{d}}_O - t^{\mathbf{e}}_O\right).$$

4. ***Introduction of incentivized exchange***, in addition to deceased-donor/direct living-donor transplantation and regular exchange, causes the waiting time for a deceased-donor transplant to decrease for all blood types except AB, for which it increases by Lemma A-2. Since $\frac{p_X}{p_Y} = \frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y}$ for any two blood types X and Y, the relevant r ratios for transplant waiting times in the deceased-donor queue satisfy for each $k = 0, \ldots, k(X)$, such that $X - Y_k \neq B - A$,

$$r_{X,X-Y_1,\ldots,X-Y_k} = \frac{\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X} - \phi^{\mathbf{l}}\frac{\boldsymbol{\kappa}_X}{\boldsymbol{\pi}_X} + \sum_{m=1}^{k}\left([\theta + \rho(1-\theta)]p_{Y_m}\alpha\right)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X} + \phi^{\mathbf{l}}\frac{\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X}{\boldsymbol{\pi}_X} + \sum_{m=1}^{k} p_{Y_m}\alpha},$$

where $\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O + \boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A + \boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_{AB}} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB} + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} = \alpha$ by Theorem A-2, and

$$\frac{\boldsymbol{\kappa}_O}{\boldsymbol{\pi}_O} = 0 \quad < \frac{\boldsymbol{\kappa}_A}{\boldsymbol{\pi}_A} = \rho(1-\theta)p_O\alpha \quad = \frac{\boldsymbol{\kappa}_B}{\boldsymbol{\pi}_B} = \rho(1-\theta)p_O\alpha \quad < \frac{\boldsymbol{\kappa}_{AB}}{\boldsymbol{\pi}_{AB}} = \rho(1-\theta)(p_O + p_A + p_B)\alpha.$$

Moreover, we have that for all underdemanded types $X-Y$ except type $B-A$, the r ratio

$$r_{X-Y}=\frac{[\theta+\rho(1-\theta)]p_X\alpha\boldsymbol{\pi}_Y}{p_Y\alpha\boldsymbol{\pi}_X}=\theta+\rho(1-\theta) \tag{16}$$

is uniform. Define $\hat{r}_X := r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}$. Thus, type $A-AB$ pairs will be pooled with nonprioritized and unpaired blood-type A patients if and only if type $B-AB$ pairs will be pooled with nonprioritized and unpaired blood-type B patients. This implies $\hat{r}_A=\hat{r}_B$ and

$$t_A^{\mathbf{i}}=S^{-1}\left(\hat{r}_A\right)=S^{-1}\left(\hat{r}_B\right)=t_B^{\mathbf{i}}.$$

■

Appendix B Construction of Calibration Parameters for Numerical Predictions of the Model

In this appendix, we explain how the calibration parameters, reported in Table 2 in Section 5 and used to generate the numerical model predictions, are constructed.

In Table A-1, we report the blood-type distribution for different ethnicities and fractions of these ethnicities in the US population. Using these, we calculate an overall US blood-type distribution (the last row of this table). We use this as the blood-type distribution of living donors, (p_X), in our model.

US Blood Type and Ethnicity Distribution Data					
Ethnicities	**Blood Types**				**Pop. %**
	O	*A*	*B*	*AB*	
African American	0.490	0.270	0.200	0.040	12.4%
Asian American	0.400	0.280	0.270	0.050	3.3%
Native American	0.790	0.160	0.040	0.010	0.8%
White American	0.450	0.400	0.110	0.040	83.4%
US population	0.456	0.378	0.126	0.040	

Table A-1: The US ethnical blood type distribution and US ethnicity distribution are from Bloodbook.com (2018a,b). The blood-type distribution for the overall US population is constructed using the ethnicity distribution and could be slightly different from the general distributions reported in other sources.

In Table A-2, we report the OPTN and SRTR data for average of deceased-donor queue additions and deceased- and living-donor transplants between 2009-2017 (OPTN and SRTR, 2009-2018a,b,c,d,e,f). We measure time in one year units and calculate the flows using the annual numbers reported in this data. First, we observe that on average $\frac{2\times7936}{11714}=1.4761$ kidneys are harvested from each deceased donor, since a total of 7936 deceased donors arrive while 11714 deceased-donor transplants are conducted. The deceased-donor flows, $(\boldsymbol{\delta}_X)$, are constructed by multiplying each entry in the second to last row of the table with 1.4761. The row above, deceased-donation recipients, is used as the de-facto deceased-donor flows, $(\boldsymbol{\delta}'_X)$, in the numerical calculations.

The US OPTN and SRTR Kidney Data						
		O	*A*	*B*	*AB*	All
Patient Arrivals	Total Additions to the Queue	17,010	11,507	5,156	1,337	35,009
	Living-Donor-Transplant Recipients not on Queue	179	173	52	17	420
	Reentrants	1,973	1,481	592	181	4,227
Total Transplants	Direct Living-Donor Transplants	2,127	1,978	667	208	4,979
	Other Living-Donor Transplants	421	283	112	27	842
	Deceased-Donor Transplants	5,357	4,188	1,548	621	11,714
Deceased-Donor Arrivals		3,786	2,942	939	268	7,936
Average CPRA (for the year 2017)						6.79%

Table A-2: Arrival and transplant averages per year to and from the kidney deceased-donor queue for 2009-2017 entrants. Data is obtained from OPTN and SRTR using the "advanced report" option from `http://optn.transplant.hrsa.gov` (on 10/30/2018) (OPTN and SRTR, 2009-2018a,b,c,d,e,f).

New patient arrival flows, $(\boldsymbol{\pi}_X)$, are calculated as follows: We know the annual additions to the deceased-donor queue (the first row of the table). However, some patients receive living-donor transplants without even registering in the queue (the second row of the table). We add these two numbers and subtract the number of reentrants (the third row of the table) from them to find $\boldsymbol{\pi}_X$ for each blood type X.

Reentry fractions, $\phi^{\mathbf{l}}$ and $\phi^{\mathbf{d}}$, are assumed to be the same, as the data from OPTN and SRTR (2009-2018e) do not distinguish reentrants based on their previous transplantation type. We divide the total number of reentrants (the last cell of the third row of the table) by the total number of transplants (the sum of the last cells of the fourth-sixth rows of the table).

The tissue-type incompatibility probability, θ, is taken as the average calculated panel reactive antibody (CPRA), 0.0679, for the 2017 entrants (see Table A-6 in Appendix D for its calculation using data from OPTN and SRTR, 2009-2018d). CPRA measures the percentage of the population with which the patient is tissue-type incompatible. We chose year 2017 because the entry flow CPRA has been increasing over time since 2009. We consider different θ values in our robustness analyses as explained in in Section 5.2.

The calculation of paired-donor fractions, (α_X), requires the knowledge of the total number of patients who arrive with paired donors. However, this information is not available since only the realized living-donor transplants are recorded in this database. Most of these transplants are direct transplants, i.e., those from the compatible paired donor of a patient. A smaller percentage of those are from exchanges or from non-directed altruistic living donors. In the fourth row of Table A-2, we report the numbers of direct living-donor transplants conducted (i.e., each entry is $\boldsymbol{\lambda}_X$ in our model). Assuming patients and living donors are paired initially as in our model, we calculate the probability of having a compatible donor conditional on being paired with a living donor. These probabilities are calculated as follows using the living-donor blood-type distribution, (p_X), reported

in the last row of Table A-1:

$$p_O^{\mathbf{l}} = (1-\theta)p_O = 0.4251, \qquad p_B^{\mathbf{l}} = (1-\theta)(p_O + p_B) = 0.5424,$$
$$p_A^{\mathbf{l}} = (1-\theta)(p_O + p_A) = 0.7773, \text{ and} \qquad p_{AB}^{\mathbf{l}} = (1-\theta) = 0.9321.$$

Then, we calculate $\alpha_X = \frac{\boldsymbol{\lambda}_X}{p_X^{\mathbf{l}} \boldsymbol{\pi}_X}$ for each blood type X. These values are stated in Table 2.

The incentivized-exchange participation fraction for a compatible pair type $X - Y$ with $Y \triangleright X$ and $Y \neq X$, ρ_{X-Y}, is our free calibration variable. We assume that this fraction is uniform for each type, and we denote it as ρ. We consider five regimes with $\rho = 10$, 20, 30, 50, and 100 percent.

The calibration of the survival function is explained in Appendix C.

Appendix C Calibrating Transplant Waiting Times

In this appendix, we give the model calibration results under benchmark parameters regarding transplant waiting times, using the analytical derivations in Appendix A. We start with the survival function and then give the results using this survival function.

C.1 Calibration of the Survival Function

Survival rate function $S(t)$ is obtained from Hart et al. (2018) for deceased-donor queue departures. We fit a piecewise linear function (for t measured in years) as

$$S(t) = \sum_{k=1}^{K} \mathbf{1}_{\{t_{k-1} \leq t < t_k\}} \cdot \left(\frac{t_k - t}{t_k - t_{k-1}} S_{k-1} + \frac{t - t_{k-1}}{t_k - t_{k-1}} S_k \right)$$

with indicator function $\mathbf{1}_{\{Z\}}$ getting value one if Z is a true event and value zero otherwise. We used the anchor survival rates $S_1, \ldots, S_6$ and times $t_1, \ldots, t_6$ reported in Table A-3.

Survival Rates in the Deceased-Donor Queue						
Time in years (t_k):	0.5	1	1.5	2	2.5	3
Surviving Fraction (S_k):	97.1%	94.5%	89.8%	83.0%	78.1%	70.0%

Table A-3: For 2013 entrants obtained from Figure K16 in OPTN and SRTR "2016 Annual Data Report: Kidney" (Hart et al., 2018) through the following calculation: the ratio of the patients on the deceased-donor queue to the total number of patients who did not receive transplant at the end of each of the time periods reported above.

We use $t_0 = 0$ and $S_0 = 100\%$ for the initial anchors. We only have data for three years. For the final interval at between $t_6 = 3$ and t_7, we assume the same slope as in the previous interval from $t_5 = 2.5$ to $t_6 = 3$ continues (which is a slope of -15% per year). Thus, we obtain $S_7 = 0\%$ at $t_7 = 7.32$ years.

C.2 Numerical Predictions of the Model: Transplant Waiting Times

Table A-4 reports the numerical predictions of our model for waiting times for nonprioritized deceased-donor transplantation across all regimes. These waiting times are calculated conditional on receiving a transplant.

A more standard waiting time measure used by OPTN and SRTR in the US is the **median transplant waiting time**, which is the time at which half of the patients of a given cohort have received a transplant (for example see Hart et al., 2018). Some of the reported waiting times in Table A-4 also correspond to the median transplant waiting time; those are the ones reported in boldface. Other regimes do not have well-defined median transplant waiting times because more than half of the patients of a steady-state cohort die without a transplant. As seen in the last column of Table 6 in Section 5.1, always less than 50 percent of B blood-type patients receive transplants under any regime. Therefore, there is no median transplant waiting time defined for them. For the general patient population, median transplant time is well defined starting with the availability of regular exchange. At this regime, the overall service rate is 50.1 percent of all new and reentering patients. As noted before, this finding is consistent with the current situation in the US, in which since 2005, no yearly cohort has an assigned overall median transplant waiting time empirically. The median patient of 2005 cohort is still in the deceased-donor queue as of December 2018. Regular exchange technologies are not currently fully utilized in their full extent in the US (see Agarwal et al., 2019). Our model predicts in such cases median transplant waiting time is not well defined.

Numerical Predictions of the Model:
Time to Nonprioritized Deceased-Donor Transplant
Conditional on Receiving One / Unconditional
Median Time to Any Kind of Transplant (only in boldface)

	O	A	B	AB	All	O	A	B	AB	All
Deceased-Donor Transplantation Only						**All plus $\rho = 20\%$-Incentivized E.**				
ABO-i	5.24	4.94	5.60	5.40	5.18	4.83	**4.51**	5.24	5.16	**4.77**
De-facto	5.32	5.02	5.41	4.43	5.18	4.91	**4.60**	5.01	**4.06**	**4.77**
Deceased/Direct Living						**All plus $\rho = 30\%$-Incentivized E.**				
ABO-i	5.02	**4.58**	5.40	5.13	4.90	4.76	**4.52**	5.25	5.19	**4.75**
De-facto	5.11	**4.66**	5.20	**4.04**	4.90	4.85	**4.61**	5.03	**4.09**	**4.75**
Deceased/Direct Living & Exchange						**All plus $\rho = 50\%$-Incentivized E.**				
ABO-i	4.95	**4.47**	5.20	5.11	**4.81**	**4.65**	**4.56**	5.29	5.24	**4.71**
De-facto	5.04	**4.56**	4.97	**4.01**	**4.81**	4.74	**4.62**	5.07	**4.14**	**4.71**
All plus $\rho = 10\%$-Incentivized E.						**All plus $\rho = 100\%$-Incentivized E.**				
ABO-i	4.89	**4.49**	5.22	5.14	**4.79**	**4.58**	**4.62**	5.37	5.38	**4.71**
De-facto	4.98	**4.58**	4.99	**4.03**	**4.79**	**4.68**	**4.71**	5.14	**4.27**	**4.73**

Table A-4: Numerical predictions of the model for waiting times for nonprioritized deceased-donor transplantation *conditional* on receiving this type of a transplant (measured in years) under the benchmark parameters. The transplant waiting times in boldface also refer the (*unconditional*) **median waiting time** to either deceased-donor or living-donor transplant, i.e., the time at which half of a cohort have received transplants. If an entry is not bold, it means that the median patient dies, i.e., less than half of a cohort receive transplants, and thus, a median transplant waiting time cannot be calculated.

Nonprioritized deceased-donor-transplant waiting times (and thus, overall median transplant waiting times) decrease with increasing participation rates of pairs to incentivized exchange. For example, for each additional 10 percent participation increase, the overall transplant waiting time

decreases about 0.04 years or two weeks up to $\rho = 50$ percent.

The largest waiting-time gap in only-deceased-donor transplantation regime with de-facto allocation is between types B and AB, as 0.98 years (see Table A-4). This gap further increases to 1.16 years in the deceased-donor/direct living-donor transplantation regime. Addition of regular exchange decreases the largest gap to 1.03 years (though for this regime the largest gap is between types O and AB). For each $\Delta\rho = 10$ percent increase in participation in incentivized exchange further decreases the largest gap by about 6 days (i.e., 0.016 years) (which is between types B and AB when incentivized exchange becomes available). Thus, besides its welfare improving effects, incentivized exchange seems to alleviate also the transplant waiting time inequality across blood types.

Waiting times for living-donor-transplant recipients conditional on receiving a transplant are reported in Table A-5. For overdemanded and self-demanded pair types, waiting time for a living-donor transplant is always zero. For underdemanded pair types, with increasing participation to incentivized exchange, the transplant waiting times weakly decrease. For low participation rates, most types are pooled with nonprioritized and unpaired patients and receive transplants at the same time with nonprioritized deceased-donor-transplant recipients. On the other hand, they are no longer pooled with nonprioritized and unpaired patients under full participation. An exception is $B - A$. Pairs of this type get matched exclusively with $A - B$ pairs as long as some form of exchange is feasible. They wait for only 3.06 years with the availability of exchange.

Numerical Predictions of the Model*: *Time to Transplant for Blood-Type-Incompatible Pairs Conditional on Receiving a Transplant

		$O-A$	$O-B$	$O-AB$	$A-B$	$A-AB$	$B-A$	$B-AB$
Deceased/Direct Living		pooled	pooled	pooled	pooled	pooled	pooled	pooled
Deceased/Direct Living & Exchange		pooled	pooled	pooled	0	pooled	3.06	pooled
All	$\rho = 10\%$	pooled	pooled	pooled	0	pooled	3.06	pooled
plus	$\rho = 20\%$	pooled	pooled	pooled	0	pooled	3.06	pooled
Incentivized	$\rho = 30\%$	pooled	pooled	pooled	0	pooled	3.06	pooled
Exchange	$\rho = 50\%$	pooled	4.39	pooled	0	pooled/4.60	3.06	pooled
	$\rho = 100\%$	3.53	1.56	4.20	0	2.05	3.06	3.81

Table A-5: Numerical predictions of the model for time to transplant for blood-type-incompatible pair types (measured in years) *conditional* on receiving a transplant. "Pooled" means type $X - Y$ pairs are pooled with nonprioritized and unpaired blood-type X patients under both ABO-i and de-facto deceased-donor allocation policies. Thus, these paired patients wait as long as their nonprioritizded and unpaired counterparts and receive either a deceased-donor transplant or a living-donor transplant through exchange (only if exchange is available) at this time. One exception is noted: Type $A - AB$ pairs are pooled with nonprioritizded and unpaired blood-type A patients under ABO-i policy. On the other hand, after waiting for 4.60 years they receive entirely living-donor transplants, as long as they can live that long, under de-facto policy. Note that the "pooled" transplant waiting times are the same times as the ones reported in Table A-4 for the respective patient blood types.

Appendix D Simulations

In addition to the numerical model predictions in Section 5, we also conduct simulations emulating the discrete paired- and unpaired-patient and deceased-donor arrival processes in real life. Our goal in conducting these simulations is to assess the welfare and equity consequences of our policy proposal, incentivized exchange, more accurately. Moreover, the simulations give us a chance to assess the validity of our continuum model in conducting policy analysis. We also assess the impact of alternative exchange technologies, such as three-way exchange in addition to two-way.

D.1 Simulation Methodology

In the simulations, we allocate deceased-donor kidneys according to the de-facto allocation policy on a FIFO basis to a compatible patient. If no compatible patient is found in the queue, the kidney immediately perishes. We evaluate our proposal under two exchange-size restrictions, two-way exchange and two-and-three-way exchange: Each arriving eligible type $X - Y$ pair waits to match in the next run of the kidney-exchange mechanism. The exchange is run once in every month, and, hence, 12 times a year. As most real-life kidney-exchange programs do, the exchange mechanism myopically maximizes the number of transplants among the available pairs using the considered exchange-size policy. It chooses one arbitrary maximum matching and implements it. If a compatible pair that is participating in incentivized exchange cannot be matched after one exchange run, then it is taken out of the exchange pool. In this case, the patient of the pair receives a direct transplant. Nevertheless, the patient of such a pair is eligible for a prioritized deceased-donor transplant if he reenters the patient pool.

We assume that patients are heterogenous in their tissue-type incompatibility probabilities. We use the entrant CPRA distribution reported in Table A-6 to generate the tissue-type incompatibility probability θ_i for each patient i. The mean of this distribution gives us the value of θ we use in the benchmark numerical model predictions, 0.068 (or 6.8 percentage points in the CPRA reporting metric), in Section 5. This table gives the fraction of entrants in five different CPRA intervals. We assume that all patients uniformly and randomly take CPRA values in their assigned CPRA intervals. For example, this table reports that 4.25 percent of all entrants have CPRA points between 0 percent and 20 percent (the second column of this table). We first assume that a simulated patient i is assigned to this group with probability 0.0425. Then his exact tissue-type incompatibility probability θ_i is determined uniformly randomly from the interval $(0, 0.2)$.

The US OPTN and SRTR Data for CPRA Distribution for Entrants					
	CPRA intervals (in % points)				
	0	**(0,20)**	**[20,80)**	**[80,98)**	**[98,100)**
Fraction of Entrants	86.35%	4.25%	5.53%	2.22%	1.64%

Table A-6: Data obtained from OPTN and SRTR (2009-2018d) for the year 2017 from http://optn.transplant.hrsa.gov (on 10/30/2018).

Our simulations use a scaled-down version of the calibrated inflow rates for new patients and

deceased donors. The US consists of 13 transplant regions of various sizes. Deceased-donor kidneys are first offered to patients within their arrival regions. If a suitable match cannot be found in the region, then they are offered nationally. Our simulation roughly maps to one small region that comprises 1/20 of the population of the US and reflects the same patient and donor characteristics as the overall US population does. Thus, we obtain deceased-donor and new-patient arrival flows by dividing the population flows $\boldsymbol{\delta}'_X$ and $\boldsymbol{\pi}_X$ reported in Table 2 by 20. For the other parameters of the simulation, (p_X), (α_X), $\phi^{\mathbf{l}}$, $\phi^{\mathbf{d}}$, and $S(t)$, we use the same parameters reported in this table and Table A-3 in Appendix C, respectively.

In each iteration, we simulate the evolution of the kidney-allocation process in such a region for 15 years.[4] Each year is divided into finite periods so that in each period either only one new patient, reentrant, or deceased donor arrives. Thus, the number of periods in each year equals the sum of the total flow of new patients, $\sum_X \boldsymbol{\pi}_X/20$, the total flow of deceased-donor kidneys, $\sum_X \boldsymbol{\delta}'_X/20$, and the total number of reentrants. The number of reentrants in a year is calculated as the minimum of (a) the reentry fraction ϕ multiplied by the number of total transplants in the previous year and (b) the total number of patients who previously received a transplant and are still alive. The numbers of patients waiting in the queue, periods per year, and reentrants per year stabilize after a number of years passes. We report the averages of the last three years (years $13-15$). We run a total of 100 simulations and report their averages and standard errors.

The new-patient, deceased-donor, and reentrant generation processes are as follows: Each new patient is generated independently and randomly with the underlying blood-type, tissue-type incompatibility probability, and living-donor pairing probability distributions. We also randomly determine his survival time while waiting for a transplant so that the population probability of remaining alive after waiting for t years is $S(t)$. Once a patient is deemed paired, his paired donor's blood type is also independently and randomly generated in a similar fashion using the living-donor blood-type distribution. We determine whether they are compatible using their blood types and the patient's tissue-type incompatibility probability with a random donor. For a deceased-donor kidney, we only generate its blood type according to the distribution dictated by $(\boldsymbol{\delta}'_X/20)$. A reentrant to the patient pool is determined according to the reentry probability among the living transplanted patients with uniform distribution. We use the following transplanted patient survival functions to determine how long each patient lives after receiving a transplant:

- The living-donor-transplant recipient survival-probability function is $S^{\mathbf{l}}(t) = 1.00e^{-0.033t}$.[5]
- The deceased-donor-transplant recipient survival-probability function is $S^{\mathbf{d}}(t) = 0.99e^{-0.050t}$.[6]

They are obtained by non-linear least squares using the survival probabilities reported in Table A-7 including the survival rate 100 percent for $t = 0$.

[4]Note that according to our survival function a patient can remain alive at most for 7.7 years without receiving a transplant, as reported in Appendix C.

[5]Using NLLS, the coefficients' 95% confidence intervals are $(0.9893, 1.011)$ for 1.00 and $(-0.0378, -0.0286)$ for -0.033. We also have $R^2 = 0.9905$.

[6]Using NLLS, the coefficients' 95% confidence intervals are $(0.9773, 1.008)$ for 0.99 and $(-0.05654, -0.04286)$ for -0.050. We also have $R^2 = 0.9909$.

US Transplant Survival Rates					
Time:	3 mo.	1 yr.	2 yr.	3 yr.	5 yr.
Living-Donor Transplant Recipient	98.9%	96.3%	94.3%	91.2%	84.1%
Deceased-Donor Transplant Recipient	97.1%	93.9%	90.4%	86.4%	76.8%

Table A-7: The reported survival rate are for patients who received transplants in 2011 and obtained from the "2018 USRDS Annual Data Report" (United States Renal Data System, 2018), Volume 2, Table 5.3 of Chapter 5.

For a reentrant, we use his original tissue-type incompatibility probability and blood type. We assume that he is now unpaired. We also randomize his new survival time using the same overall survival probability function $S(t)$.

We consider 13 regimes in our simulations. The first eight regimes are (1) only deceased-donor transplantation, (2) deceased-donor/direct living-donor transplantation, (3) deceased-donor/direct living-donor transplantation and regular exchange, and (4, 5, 6, 7, 8) deceased-donor/direct living-donor transplantation, regular and incentivized exchange for uniform participation rates $\rho = 10$, 20, 30, 50, 100 percent. These are also used in our numerical model predictions. We also consider five additional incentivized-exchange regimes in which compatible type $X-X$ pairs are also incentivized. In our continuum model, this incentivization scheme does not have additional welfare benefits, as all incompatible type $X - X$ pairs are matched with each other in regular exchange as soon as they arrive. On the other hand, in our simulations, as pair arrivals are discrete and patients are heterogenous in their tissue-type compatibility probabilities, there could be potential welfare gains from the participation of compatible type $X - X$ pairs in exchange with incompatible $X - X$ pairs already in the queue.

D.2 Simulation Results

The simulation results for service rates regarding two-way exchange are slightly lower than or comparable to those of the calibrated-model predictions for the de-facto deceased-donor allocation policy. Service rates are reported in Table A-8. The new regimes, incentivized exchange with compatible type $X - X$ pairs, fare better than the incentivized regimes without compatible-type-$X - X$-pair participation. When compared with Table 6 in Section 5, the corresponding percentages are slightly lower than calibrated-model results in all exchange regimes. The simulation and calibration results are similar to each other for only deceased-donor transplantation regime and deceased-donor/direct living-donor transplantation regime. This can be attributed to the fact that overdemanded pairs with high-CPRA patients do not necessarily participate in an exchange in the simulations while they do under the continuum-model assumptions.

The simulation results regarding two-and-three-way exchange are reported in Table A-9 for service rates. We also plot the comparison of service rates for paired patients to receive a living-donor transplant between two-way exchange and two-and-three-way exchange in Figure A-1 and for all transplants in Figure A-2. In the first figure (as well as Tables A-8 and A-9), we observe that 67.7 percent of living donation candidates are served through two-and-three-way exchanges in addition

to direct donation when $X-X$ pairs are not incentivized. This rate is 66.5 percent under two-way exchange. Under $\rho = 30$ percent participation in incentivized exchange, these rates go up to 75.5 percent and 72.1 percent, respectively (resulting in a 3.4 percent difference). There is a further increase in the difference of the service rates at $\rho = 50$ percent. The marginal impact of three-way exchange technology slightly decreases at $\rho = 100$ percent. This is expected as most gains from exchange are utilized through higher incentivized participation rates. There is one contribution of three-way exchange that higher incentivized participation rates cannot compensate under two-way exchange: According to our model calibration, annually more $B-A$ pairs arrive than $A-B$ pairs do. Thus, all type $B-A$ pairs cannot be matched in our optimal two-way exchange policy. As ρ increases, all remaining $B-A$ pairs can be matched through three-way exchanges consisting of pairs of types $A-O$, $O-B$, $B-A$ or $AB-B$, $B-A$, $A-AB$ (see Roth, Sönmez, and Ünver, 2007 for details). The figure also shows that once all the remaining $B-A$ pairs can be matched through three-way exchanges, even if ρ increases further, the marginal gains from three-way exchange no longer increases. About 98.5 percent or more of all paired A, B, and AB patients are matched under two-and-three-way exchange policy when $\rho = 100$ percent (see the last row of Table A-9), while this rate is lower for B under two-way exchange (around 93.9 percent).

When compatible pairs of types $X-X$ are also incentivized, we observe almost no difference under two-and-three-way exchange. However, compatible-$X-X$-pair participation has a much higher impact when three-way exchange technology is unavailable. For example, for $\rho = 20$ percent, 71.1 percent of all paired patients are matched when compatible $X-X$ pairs are incentivized. When they are not incentivized, the service rate is 70.3 percent. Differences stand for different ρ values. The reason for this disparity is the flexibility provided by three-way exchanges. An incompatible type $X-X$ pair can potentially be inserted in a two-way exchange that includes a blood-type X patient. For example, a two-way exchange of pair types $(X-Y, Y-X)$ can be extended to a three-way exchange as $(X-X, X-Y, Y-X)$.[7] Thus, such a couple of $X-Y$ and $Y-X$ pairs plays a role similar to a single compatible $X-X$ pair; they both help an incompatible $X-X$ pair to be matched through an exchange. Thus, the availability of three-way exchange almost perfectly substitutes for compatible-$X-X$-pair participation in matching incompatible pairs.

We also give some absolute numbers from our simulations that are multiplied by 20, the simulation scale parameter, to compare them with the continuum model's predictions. We observe that the simulations lead to an average of $4,994$ (with a standard error of 16.8) direct-living annual donor transplants compared to the real-life number of $4,979$, which is our calibration parameter for the continuum model. Two-way exchange and two-and-three-way exchange add, respectively, average 998 and $1,100$ transplants annually opposed to $1,135$ of the calibrated continuum model, which exclusively uses two-way exchange. At each $\Delta\rho = 10$ percent participation increase in incentivized exchange, additional averages of 172/184 (depending on $X-X$ pairs are incentivized or not) and 188 annual transplants are conducted under two-way and two-and-three-way exchange simulations,

[7] Even when the patient of the type $X-Y$ pair is tissue-type incompatible with the donor of the type $Y-X$ pair and a two-way exchange is not feasible between these two pairs, the $X-X$ pair potentially can be inserted to create a three-way exchange benefitting three additional patients.

respectively. Recall that this number was 180 in the continuum model calibrations. Thus, while finite market simulations lead to slightly less regular exchange transplants due to the frictions in its more realistic discrete setup, incentivized exchange protocols seem to overcome these frictions such that simulations lead to at least as well or more added transplants than the continuum model's predictions. We explained its reasons in Section 5.2 in the main text before.

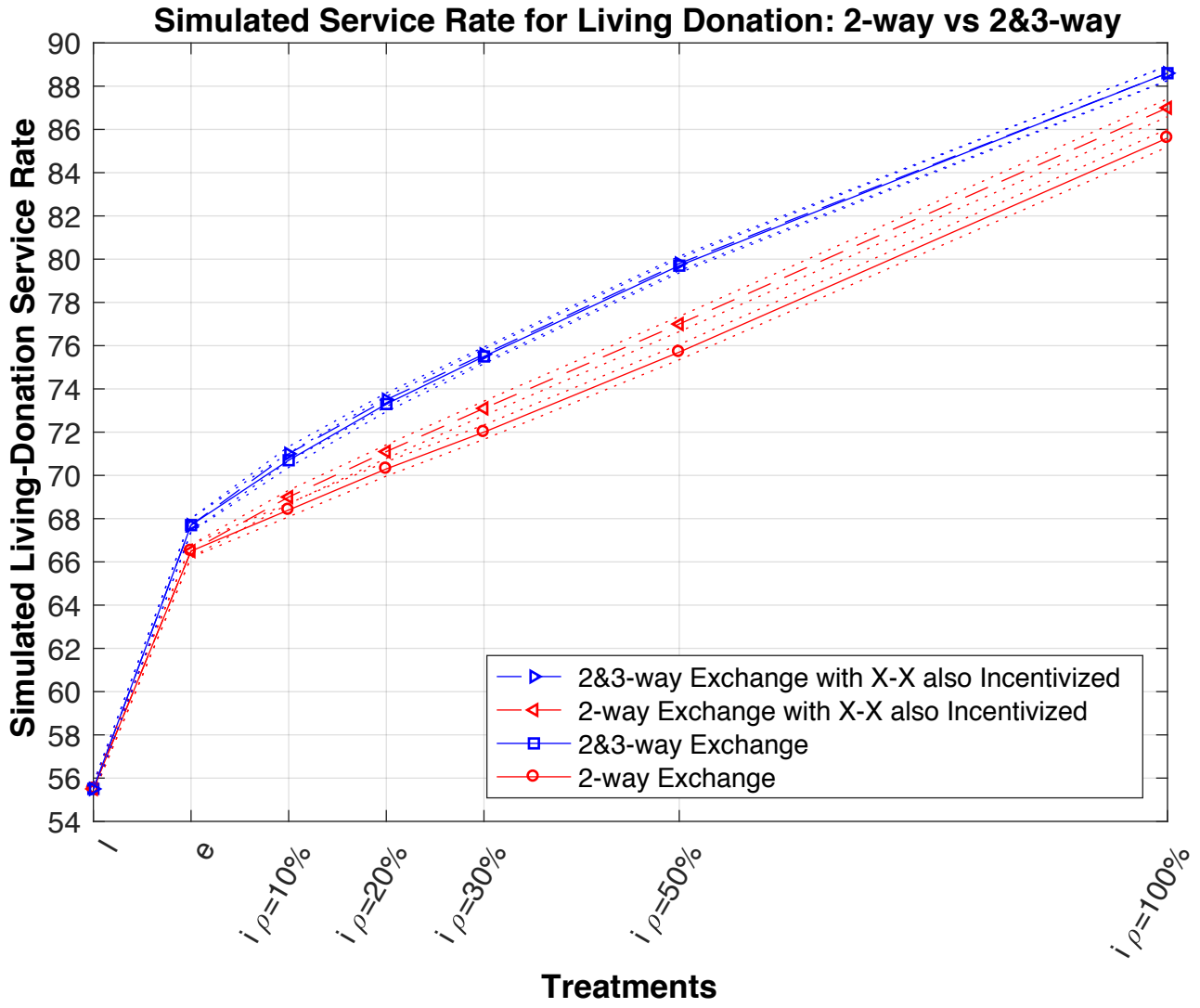

Figure A-1: Simulation results for service rates for paired patients to receive living-donor transplants under two-way vs two-and-three-way exchange. Dotted lines are 95% confidence intervals for the averages.

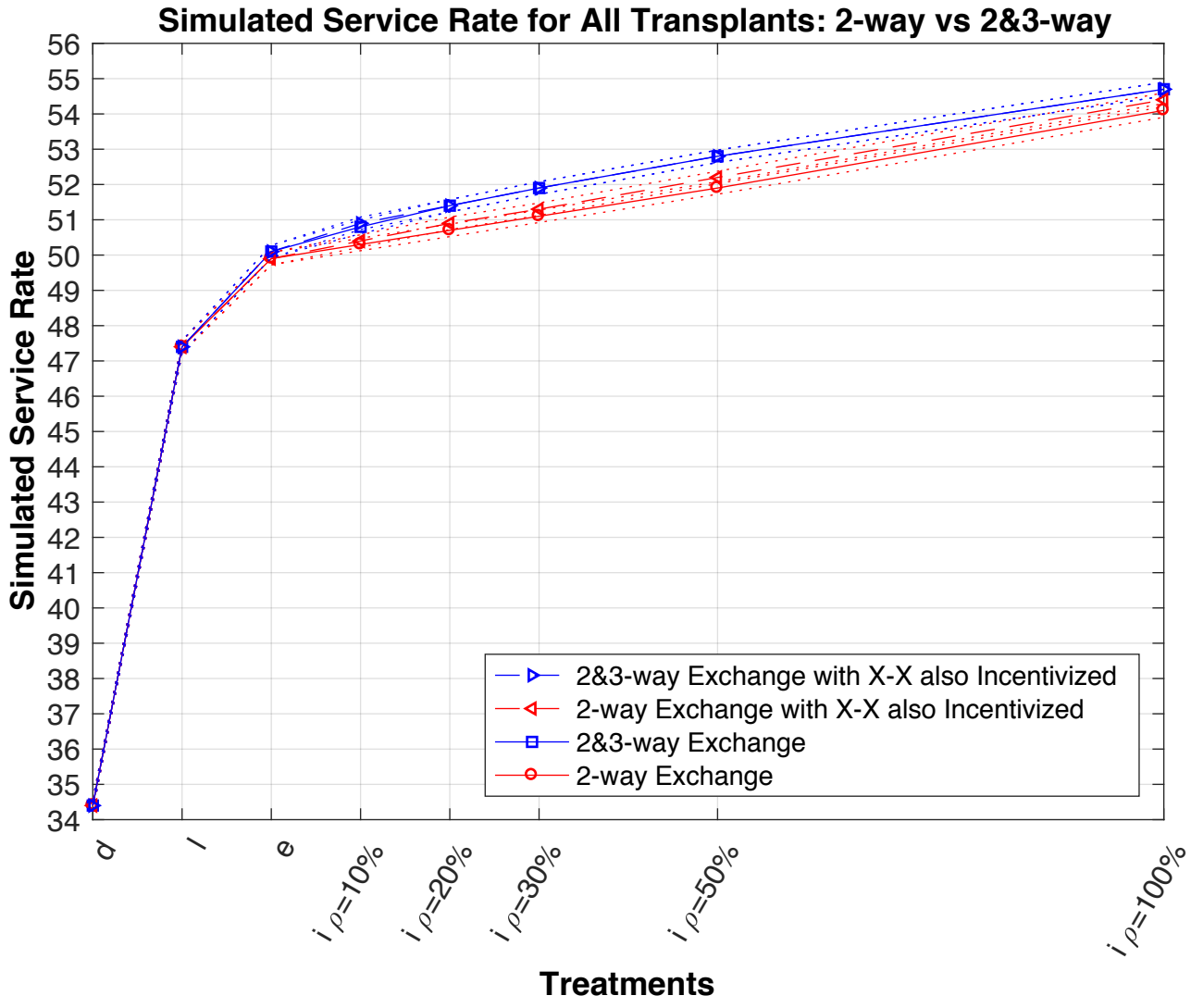

Figure A-2: Simulation results for service rates for all transplants under two-way vs two-and-three-way exchange. Dotted lines are 95% confidence intervals for the averages.

Simulation Results: Service Rate for Transplantation in % under De-facto Deceased-Donor Allocation and 2-way Exchange															
Living-Donor Trans.					**Deceased-Donor Trans.**					**All Transplants**					
O	*A*	*B*	*AB*	All	*O*	*A*	*B*	*AB*	All	*O*	*A*	*B*	*AB*	All	
Deceased-Donor Transplantation Only (d)															
					32.5	37.6	30.8	46.5	34.4	32.5	37.6	30.8	46.5	34.4	
					(1.2)	(1.6)	(2.1)	(5.0)	(0.8)	(1.2)	(1.6)	(2.1)	(5.0)	(0.8)	
Deceased-/Direct Living-Donor Transplantation (l)															
42.7	78.0	54.2	92.9	55.5	36.0	43.4	34.4	52.7	38.7	44.0	53.0	42.9	59.6	47.4	
(1.9)	(1.9)	(3.7)	(5.0)	(1.3)	(1.3)	(1.9)	(2.3)	(5.6)	(0.9)	(1.3)	(1.8)	(2.2)	(5.2)	(0.9)	
Deceased-/Direct Living-Donor Transplantation & Regular Exchange (e)															
46.8	94.7	82.8	99.0	66.5	36.4	44.8	36.5	53.4	39.7	45.1	56.0	48.9	60.7	49.9	
(2.0)	(1.2)	(5.3)	(1.9)	(1.5)	(1.4)	(2.1)	(2.4)	(6.0)	(0.9)	(1.4)	(1.9)	(2.2)	(5.6)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 10\%$-Incentivized Exchange (i $\rho = 10\%$)															
49.9	95.1	83.3	99.2	68.4	36.7	44.9	36.5	53.1	39.9	45.9	56.1	49.0	60.5	50.3	
(2.1)	(1.2)	(5.3)	(2.0)	(1.6)	(1.4)	(2.0)	(2.6)	(5.7)	(0.9)	(1.3)	(1.9)	(2.5)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 20\%$-Incentivized Exchange (i $\rho = 20\%$)															
52.9	95.5	83.8	99.1	70.3	36.9	44.9	36.6	53.2	40.1	46.7	56.2	49.1	60.5	50.7	
(2.2)	(1.3)	(5.1)	(2.1)	(1.7)	(1.3)	(2.1)	(2.4)	(5.7)	(0.9)	(1.3)	(1.9)	(2.3)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 30\%$-Incentivized Exchange (i $\rho = 30\%$)															
55.5	96.2	84.7	99.2	72.0	37.2	45.0	36.6	53.4	40.2	47.3	56.4	49.2	60.8	51.1	
(2.2)	(1.3)	(5.0)	(1.8)	(1.7)	(1.4)	(2.0)	(2.6)	(5.7)	(0.9)	(1.3)	(1.9)	(2.4)	(5.2)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 50\%$-Incentivized Exchange (i $\rho = 50\%$)															
60.8	97.3	87.6	99.1	75.7	37.7	45.1	36.9	53.2	40.6	48.6	56.6	50.0	60.5	51.9	
(2.2)	(1.3)	(5.2)	(2.7)	(1.8)	(1.4)	(2.0)	(2.6)	(5.8)	(0.9)	(1.3)	(1.8)	(2.4)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 100\%$-Incentivized Exchange (i $\rho = 100\%$)															
76.2	98.8	94.3	99.2	85.6	39.3	45.2	37.5	53.4	41.5	52.6	56.8	51.3	60.8	54.1	
(2.8)	(1.3)	(5.2)	(2.2)	(2.1)	(1.4)	(2.0)	(2.6)	(6.1)	(1.0)	(1.3)	(1.8)	(2.4)	(5.6)	(1.0)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 10\%$-Incentivized Exchange with $X-X$ (i $\rho = 10\%$)															
50.6	95.7	83.7	99.3	69.0	36.7	44.9	36.7	53.1	39.9	46.1	56.2	49.2	60.5	50.4	
(2.2)	(1.2)	(5.2)	(1.8)	(1.6)	(1.4)	(2.0)	(2.5)	(5.7)	(0.9)	(1.4)	(1.8)	(2.4)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 20\%$-Incentivized Exchange with $X-X$ (i $\rho = 20\%$)															
54.0	96.1	84.3	99.2	71.1	37.1	44.9	36.6	53.4	40.1	47.0	56.3	49.2	60.7	50.9	
(2.2)	(1.3)	(5.3)	(2.1)	(1.6)	(1.4)	(2.1)	(2.6)	(6.1)	(0.9)	(1.3)	(1.9)	(2.4)	(5.5)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 30\%$-Incentivized Exchange with $X-X$ (i $\rho = 30\%$)															
56.9	96.7	85.7	99.3	73.1	37.3	45.0	36.8	53.6	40.3	47.7	56.4	49.5	60.9	51.3	
(2.2)	(1.2)	(5.3)	(1.8)	(1.7)	(1.4)	(2.0)	(2.5)	(5.9)	(0.9)	(1.3)	(1.8)	(2.4)	(5.4)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 50\%$-Incentivized Exchange with $X-X$ (i $\rho = 50\%$)															
62.8	97.6	88.6	99.3	77.0	37.9	45.1	36.9	53.5	40.7	49.2	56.6	50.1	60.8	52.2	
(2.2)	(1.3)	(5.3)	(2.3)	(1.8)	(1.4)	(2.0)	(2.7)	(6.1)	(0.9)	(1.3)	(1.9)	(2.5)	(5.7)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 100\%$-Incentivized Exchange with $X-X$ (i $\rho = 100\%$)															
78.7	99.0	93.9	99.2	87.0	39.6	45.3	37.4	53.2	41.7	53.2	56.9	51.2	60.5	54.4	
(2.7)	(1.4)	(5.2)	(2.1)	(2.1)	(1.5)	(2.0)	(2.5)	(5.9)	(1.0)	(1.3)	(1.8)	(2.4)	(5.4)	(1.0)	

Table A-8: Simulation results for service rates for paired patients to receive living-donor transplants, service rates for deceased-donor-queue participants, and overall service rates for patients to receive any kind of transplant (all measured in %) under de-facto deceased-donor allocation policy and two-way exchange when compatible $X-X$ pairs are not incentivized (middle four rows) and incentivized (last four rows).

Simulation Results: Service Rate for Transplantation in % under De-facto Deceased-Donor Allocation and 2&3-way Exchange															
Living-Donor Trans.					**Deceased-Donor Trans.**					**All Transplants**					
O	*A*	*B*	*AB*	All	*O*	*A*	*B*	*AB*	All	*O*	*A*	*B*	*AB*	All	
Deceased-Donor Transplantation Only (d)															
					32.5	37.6	30.8	46.5	34.4	32.5	37.6	30.8	46.5	34.4	
					(1.2)	(1.6)	(2.1)	(5.0)	(0.8)	(1.2)	(1.6)	(2.1)	(5.0)	(0.8)	
Deceased-/Direct Living-Donor Transplantation (l)															
42.7	78.0	54.2	92.9	55.5	36.0	43.4	34.4	52.7	38.7	44.0	53.0	42.9	59.6	47.4	
(1.9)	(1.9)	(3.7)	(5.0)	(1.3)	(1.3)	(1.9)	(2.3)	(5.6)	(0.9)	(1.3)	(1.8)	(2.2)	(5.2)	(0.9)	
Deceased-/Direct Living-Donor Transplantation & Regular Exchange (e)															
47.3	96.1	86.3	99.5	67.7	36.4	44.9	36.8	53.3	39.8	45.2	56.3	49.6	60.7	50.1	
(2.1)	(1.1)	(5.2)	(1.5)	(1.6)	(1.4)	(2.0)	(2.6)	(6.3)	(0.9)	(1.4)	(1.9)	(2.4)	(5.8)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 10\%$-Incentivized Exchange (i $\rho = 10\%$)															
50.8	96.9	92.4	99.6	70.7	36.7	45.0	37.5	53.4	40.1	46.1	56.4	51.1	60.7	50.8	
(2.1)	(1.2)	(6.0)	(1.3)	(1.7)	(1.4)	(2.0)	(2.7)	(5.8)	(0.9)	(1.4)	(1.8)	(2.6)	(5.3)	(1.0)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 20\%$-Incentivized Exchange (i $\rho = 20\%$)															
54.4	97.4	95.8	99.8	73.3	37.1	45.1	37.5	53.5	40.3	47.1	56.5	51.5	60.9	51.4	
(2.3)	(1.3)	(4.9)	(0.9)	(1.8)	(1.4)	(2.0)	(2.7)	(6.0)	(0.9)	(1.4)	(1.9)	(2.5)	(5.5)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 30\%$-Incentivized Exchange (i $\rho = 30\%$)															
57.9	97.9	96.2	99.7	75.5	37.4	45.1	37.7	53.6	40.6	47.9	56.6	51.8	60.9	51.9	
(2.3)	(1.4)	(3.2)	(1.3)	(1.7)	(1.4)	(2.1)	(2.6)	(5.9)	(0.9)	(1.4)	(1.9)	(2.4)	(5.4)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 50\%$-Incentivized Exchange (i $\rho = 50\%$)															
64.7	98.7	97.6	99.5	79.7	38.1	45.2	37.7	53.4	41.0	49.7	56.8	51.9	60.8	52.8	
(2.3)	(1.3)	(2.5)	(1.4)	(1.7)	(1.4)	(2.0)	(2.8)	(6.1)	(0.9)	(1.3)	(1.8)	(2.5)	(5.6)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 100\%$-Incentivized Exchange (i $\rho = 100\%$)															
80.0	99.7	98.9	99.6	88.6	39.7	45.3	37.9	53.4	41.8	53.5	57.0	52.2	60.7	54.7	
(2.9)	(0.7)	(2.4)	(1.5)	(1.8)	(1.5)	(2.0)	(2.6)	(5.9)	(1.0)	(1.4)	(1.8)	(2.4)	(5.4)	(1.0)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 10\%$-Incentivized Exchange with $X-X$ (i $\rho = 10\%$)															
51.0	96.9	93.1	99.7	71.0	36.8	45.0	37.3	53.3	40.1	46.2	56.5	51.0	60.7	50.9	
(2.1)	(1.1)	(5.7)	(1.4)	(1.7)	(1.3)	(2.0)	(2.6)	(5.7)	(0.9)	(1.3)	(1.9)	(2.5)	(5.2)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 20\%$-Incentivized Exchange with $X-X$ (i $\rho = 20\%$)															
54.7	97.4	95.9	99.8	73.5	37.1	45.1	37.6	53.5	40.4	47.1	56.6	51.6	60.9	51.4	
(2.1)	(1.2)	(3.9)	(1.1)	(1.6)	(1.3)	(2.1)	(2.6)	(5.8)	(0.9)	(1.3)	(1.9)	(2.5)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 30\%$-Incentivized Exchange with $X-X$ (i $\rho = 30\%$)															
58.1	97.9	96.5	99.7	75.6	37.5	45.1	37.6	53.5	40.6	48.0	56.6	51.7	60.9	51.9	
(2.3)	(1.3)	(2.6)	(1.3)	(1.7)	(1.4)	(2.1)	(2.6)	(6.0)	(0.9)	(1.4)	(1.9)	(2.4)	(5.5)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 50\%$-Incentivized Exchange with $X-X$ (i $\rho = 50\%$)															
64.8	98.8	97.8	99.6	79.8	38.1	45.2	37.7	53.4	41.0	49.7	56.8	52.0	60.7	52.8	
(2.4)	(1.3)	(2.0)	(1.5)	(1.7)	(1.4)	(2.1)	(2.7)	(6.0)	(0.9)	(1.3)	(1.9)	(2.5)	(5.5)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 100\%$-Incentivized Exchange with $X-X$ (i $\rho = 100\%$)															
80.1	99.6	98.7	99.6	88.6	39.8	45.3	37.9	53.3	41.9	53.5	57.0	52.2	60.7	54.7	
(2.8)	(0.7)	(2.7)	(1.7)	(1.7)	(1.5)	(2.0)	(2.6)	(5.8)	(1.0)	(1.5)	(1.8)	(2.4)	(5.3)	(1.0)	

Table A-9: Simulation results for service rates for paired patients to receive living-donor transplants, service rates for deceased-donor-queue participants, and overall service rates for patients to receive any kind of transplant (all measured in %) under de-facto deceased-donor allocation policy and two-and-three-way exchange when compatible $X-X$ pairs are not incentivized (middle four rows) and incentivized (last four rows).

Appendix E Remaining Stress Tests

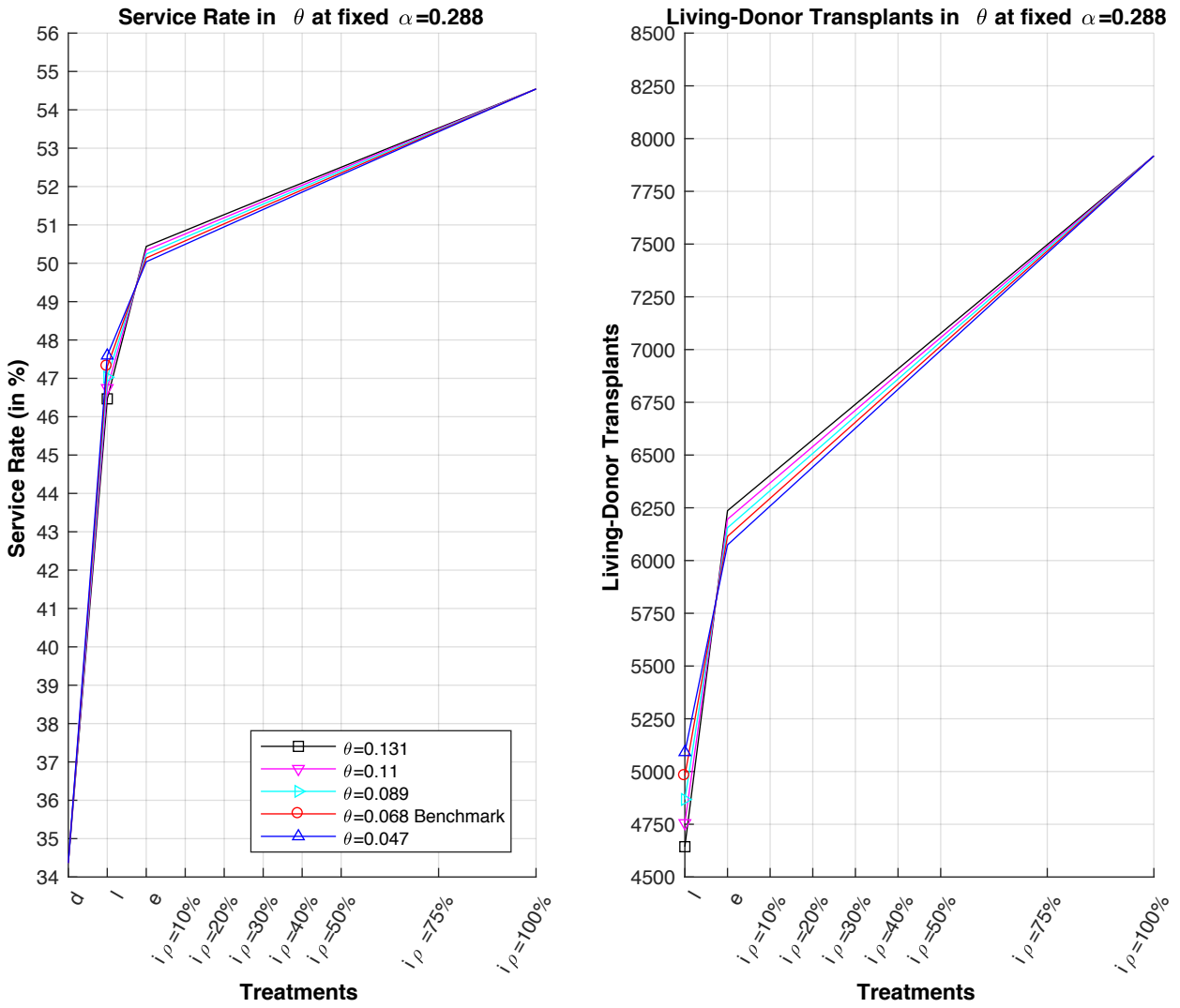

Figure A-3: Stress tests for the numerical predictions of the model discussed in Section 5.2: Overall service rate and service rate of paired patients to receive a living-donor transplant in changing θ assuming (α_X) is fixed at its benchmark level with the average 0.288.

We report the results of the remaining stress test discussed in Section 5.2. Note that Figure A-3 is already discussed in detail in Section 5.2.

Next we assume that θ is not precisely known. For each given θ, we find what levels of (α_X) will be necessary to support the observed direct living-donor transplant numbers $(\boldsymbol{\lambda}_X)$ in the data given in Table A-2. The set of corresponding (θ, α) pairs is $\{(0.047, 0.282), (0.068, 0.288), (0.089, 0.295), (0.11, 0.301), (0.131, 0.309)\}$, where α is the mean probability of a random patient to have a paired donor for the calibrated (α_X) values.

Increasing θ means that a lower number of patients can receive direct transplants from their own donors. Since $(\boldsymbol{\lambda}_X)$ are kept constant, an increasing θ corresponds to higher (α_X) values. As

a result, the service rates and number of living transplants increase uniformly for all ρ values with increasing θ and (α_X) (see Figure A-4). Each 0.021 probability increase in θ that corresponds to 0.007 increase in pairing rate leads to 180 additional transplants per year, 3.6 percent of direct living-donor transplants. The comparative static results regarding changes in ρ that we reported in Section 5 remain intact for different (θ, α) pairs.

Thus, changes in θ accompanied with induced changes in (α_X) to keep the number of direct living-donor transplants constant at its observed level has no effect on the impact of incentivized exchange, it only effects the number of patients that benefit from regular exchange.

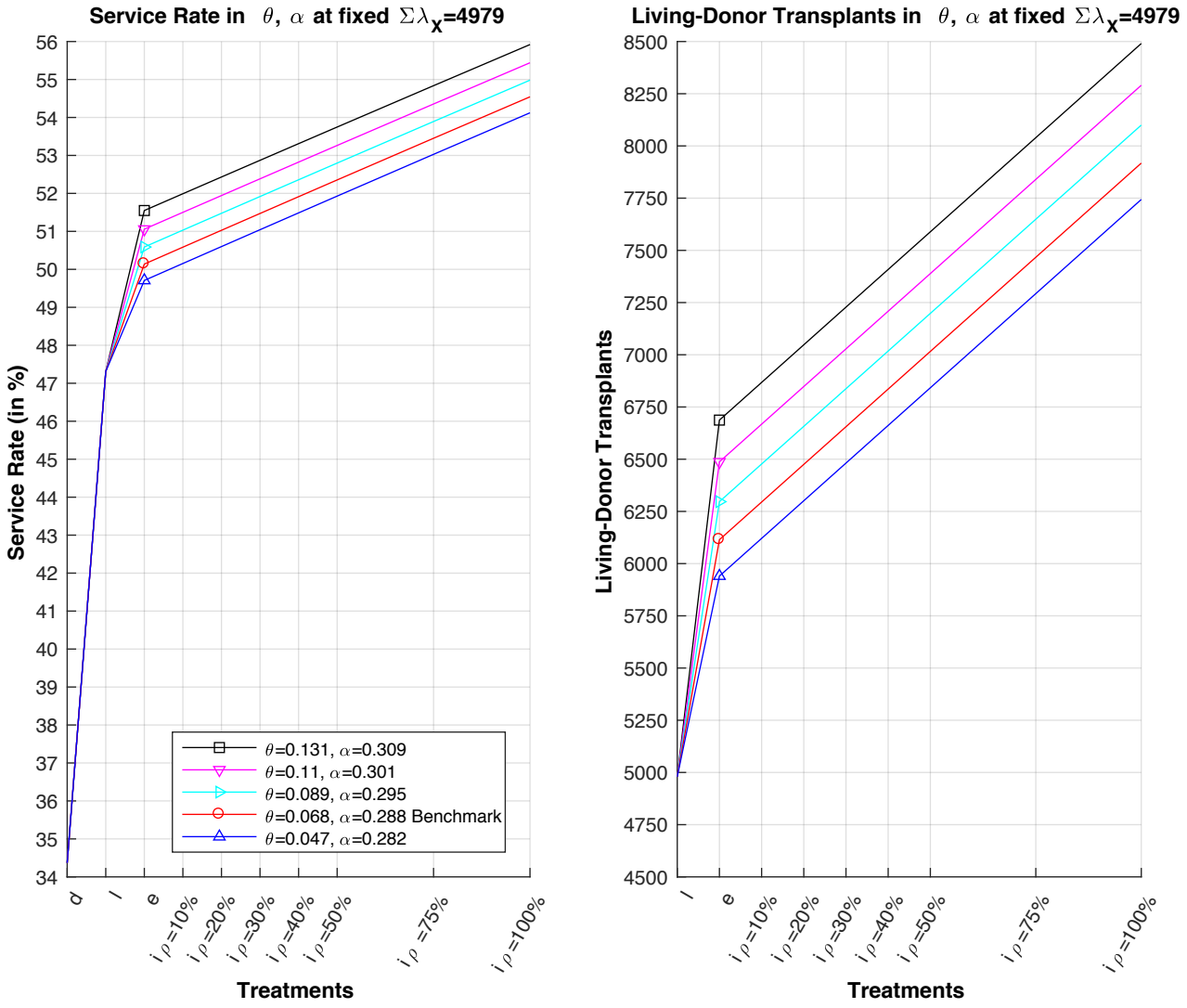

Figure A-4: Stress tests for the numerical predictions of the model discussed in Appendix E: Stress tests of total service rate and total living-donor transplants in changing θ and (α_X) assuming $(\boldsymbol{\lambda}_X)$ is fixed at its benchmark level with the total $\sum \boldsymbol{\lambda}_X = 4979$.

Appendix F Perfect Matching with (Heterogenous) Tissue-Type Incompatibilities

In this appendix, we study the limit assumptions on the patient types under which different populations of pairs can be matched or patients can be assigned deceased-donor kidneys. The lemmas that we establish below are used in all results regarding steady states of the transplantation policies.

F.1 Matching Deceased-Donor Kidneys

We first consider the case when deceased-donor kidneys are matched with patients. We make the following regularity assumption on the frequency and incompatibility probability of patient types.

Assumption A-1 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for every $k > k_0$ and $l \leq k$ and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1 - \prod_{i=1}^{l} \theta_{\sigma(i),k}.$$

When $\epsilon \to 0$, the regularity assumption can be rewritten as $\sum_{i=l+1}^{k} m_{\sigma(i),k} \geq \prod_{i=1}^{l} \theta_{\sigma(i),k}$. It implies that if you take a set of patients and a set of kidneys with the same measure, then for any set of patient types the measure of patients with those types is greater than or equal to the measure of the set of kidneys that are tissue-type incompatible with all the other patient types.

Under this assumption, we get the following result.

Lemma A-3 *Suppose Assumption A-1 holds. Consider a measurable set of patients and deceased-donor kidneys that are blood-type compatible with all the patients such that both sets have the same measure. Suppose these sets are formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every patient can be matched with a compatible kidney.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Fix a small $\epsilon > 0$. By Assumption A-1, there exists k_0 such that, for every $k > k_0$, $l \leq k$, and permutation σ, $\frac{1-\prod_{i=1}^{l} \theta_{\sigma(i),k}}{\sum_{i=1}^{l} m_{\sigma(i),k}} \geq 1 - \epsilon$. Consider any such k.

We use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the kidneys can be matched with compatible patients. Consider a random measurable subset of patients with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of patients with the kidneys can still be formed randomly using the governing population. We need to show that for any subset of patients, the measure of kidneys that are compatible with at least one patient is weakly greater than the measure of patients. Without loss of generality, instead of considering any set of patients we can consider the set of all patients who have types from any given set. Let the set of patient types be $\{\sigma(1), \ldots, \sigma(l)\}$.

The measure of patients that have a type in this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$. The measure of kidneys that are incompatible with all such types is $\prod_{i=1}^{l} \theta_{\sigma(i),k}$ because the measure of kidneys is

one. Therefore, the measure of kidneys that are compatible with at least one patient in the set is $1 - \prod_{i=1}^{n} \theta_{\sigma(i),k}$. The desired inequality holds by Assumption A-1. The claim of the lemma follows by taking the limit as $\epsilon \to 0$ and $k \to \infty$. ■

F.2 Matching Type $A-B$ Pairs with Type $B-A$ Pairs

We next consider the case when we match reciprocal pairs, $A-B$ with $B-A$. For any such pair, tissue-type compatibility is not known because the pair is blood-type incompatible. Therefore, for any such pair, tissue-type incompatibility is determined randomly as in the overall population.

We make the following assumption on how the market grows, which guarantees that we can match almost every patient in two measurable sets of $A-B$ pairs and $B-A$ pairs that have the same measure.

Assumption A-2 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k}[1 - \prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

Consider two measurable sets of $A-B$ and $B-A$ pairs with the same measure. As $\epsilon \to 0$, the assumption guarantees that for any measurable set of reciprocal-type pairs, say $B-A$, the measure of this set is smaller than the measure of $A-B$ pairs that are compatible with at least one $B-A$ pair in this set.

Lemma A-4 *Suppose Assumption A-2 holds. Consider two measurable sets of $A-B$ and $B-A$ pairs that have the same measure. Suppose these sets are formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every pair can be matched with a compatible pair.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Fix a small $\epsilon > 0$. By Assumption A-2, there exists k_0 such that, for every $k > k_0$, $l \leq k$, and permutation σ, $\frac{\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]}{\sum_{i=1}^{l} m_{\sigma(i),k}} \geq 1-\epsilon$. Consider any such k.

Like before, we use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the $B-A$ pairs can be matched with compatible $A-B$ pairs. Consider a random measurable subset of $B-A$ pairs with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of donors with patients can still be formed randomly using the governing population. We need to show that for any subset of $B-A$ pairs, the measure of $A-B$ pairs who are compatible with at least one $B-A$ pair in the chosen set is weakly greater than the measure of the chosen set of $B-A$ pairs. Without loss of generality, instead of considering any set of $B-A$ pairs, we can consider the set of all $B-A$ pairs with patients that have types from any given set. Let the set of patient types be $\{\sigma(1), \ldots, \sigma(l)\}$.

The measure of $B-A$ pairs with patients that have a type in this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$. The measure of $A-B$ pairs with patient type $\sigma(i)$ who are incompatible with all such pairs is

$m_{\sigma(i),k} \prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Therefore, the measure of $A-B$ pairs with patient type $\sigma(i)$ who are compatible with at least one $B-A$ pair from the chosen set is $m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. Hence, the measure of $A-B$ pairs that are compatible with at least one $B-A$ pair in the chosen set is $\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of chosen $B-A$ pairs, $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$, by Assumption A-2.

Therefore, $1-\epsilon$ measure of $B-A$ pairs can be matched with compatible $A-B$ pairs. The lemma follows by taking $\epsilon \to 0$ and $k \to \infty$. ■

F.3 Matching Overdemanded-Type Pairs Except $A-B$ Pairs with Underdemanded-Type Pairs Except $B-A$ Pairs

We next consider the case when we match overdemanded-type pairs except $A-B$ pairs with underdemanded-type pairs except $B-A$ pairs. In the rest of this subsection, when we mention overdemanded-type pairs we exclude $A-B$ pairs, and similarly, when we mention underdemanded-type pairs we exclude $B-A$ pairs.

We make the following assumption on the frequency and incompatibility probability of patient types.

Assumption A-3 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, $0 \leq \rho \leq 1$, and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l} \frac{m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))}{M} \leq \sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))],$$

where $M = \sum_{i=1}^{k} m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))$.

For overdemanded-type pairs, only tissue-type-incompatible ones participate in the regular exchange. However, in the incentivized exchange, compatible pairs also participate. As a result, a fraction of the overdemanded pairs are compatible, while the rest are incompatible. Here, ρ is the participation rate of compatible pairs. The assumption guarantees that, for any set of overdemanded-type pairs, the set of underdemanded pairs that are compatible with at least one pair in the set has a greater measure as $\epsilon \to 0$.

Lemma A-5 *Suppose Assumption A-3 holds. Consider two measurable sets of overdemanded $X-Y$ pairs and underdemanded $Y-X$ pairs with the same measure. Suppose that a fraction of overdemanded $X-Y$ pairs are known to be tissue-type incompatible and the rest are known to be tissue-type compatible, but otherwise these sets are formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every pair can be matched with a compatible pair.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Then, for underdemanded $Y-X$ pairs, $m_{i,k}$ measure of the patients have type i for every i. For overdemanded $X-Y$ pairs, some are known to be tissue-type compatible

while others are tissue-type incompatible. The measure of compatible pairs is proportional to $\rho m_{i,k}(1-\theta_{i,k})$ and the measure of incompatible pairs is proportional to $m_{i,k}\theta_{i,k}$. Therefore, the measure of overdemanded $X-Y$ pairs with patient type i is $\frac{m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))}{M}$ where $M = \sum_{i=1}^{k} m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))$.

Fix a small $\epsilon > 0$. Consider any k that satisfies Assumption A-3. Like before, we use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the overdemanded $X-Y$ pairs can be matched with compatible underdemanded $Y-X$ pairs. Consider a random measurable subset of overdemanded $X-Y$ pairs with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of pairs can still be formed randomly using the governing population. We need to show that, for any subset of overdemanded $X-Y$ pairs, the measure of underdemanded $Y-X$ pairs who are compatible with at least one overdemanded $X-Y$ pair is weakly greater than the measure of overdemanded $X-Y$ pairs. In this calculation, we use a lower bound for the measure of such underdemanded $Y-X$ pairs by assuming that if their patient has type i, then they are incompatible with overdemanded $X-Y$ pairs with patient type i. Without loss of generality, instead of considering any set of overdemanded $X-Y$ pairs, we can consider the set of all overdemanded $X-Y$ pairs with patients who have tissue types from any given set. Let the set of patient types be $\{\sigma(1),\ldots,\sigma(l)\}$.

The measure of overdemanded $X-Y$ pairs with patients who have types in the set is $(1-\epsilon)\sum_{i=1}^{l}\frac{m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))}{M}$. The measure of underdemanded $Y-X$ pairs with patient type $\sigma(i)$ for $i \leq l$ who are incompatible with all such pairs is $m_{\sigma(i),k}\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Note that we are assuming that these pairs are incompatible with overdemanded $X-Y$ pairs with patient of type $\sigma(i)$. On the other hand, if $i > l$, then the measure of underdemanded $Y-X$ pairs with patient type $\sigma(i)$ who are incompatible with all such pairs is $m_{\sigma(i),k}\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Hence, the measure of underdemanded $Y-X$ pairs that are compatible with at least one overdemanded $X-Y$ pair in the chosen set is at least $\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of chosen overdemanded $X-Y$ pairs Assumption A-3.

The proof that $1-\epsilon$ measure of overdemanded $X-Y$ pairs can be matched follows. The lemma follows by taking $k\to\infty$ and $\epsilon\to 0$. ■

F.4 Matching Self-Demanded-Type Pairs

In this section, we consider the case when we match self-demanded type pairs. Fix any self-demanded-type pair $X-X$ for some blood type X. Any such pair in the exchange pool is tissue-type incompatible. We match these pairs with each other. Therefore, in contrast with the previous sections, this is a one-sided matching problem.

We make the following assumption to show that almost every pair can be matched in the limit.

Assumption A-4 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

Our next result shows that under this assumption almost all self-demanded pairs can be matched.

Lemma A-6 *Suppose Assumption A-4 holds. Consider a set of self-demanded-type pairs $X-X$ that are tissue-type incompatible. Assume that this set is formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every pair can be matched with a compatible pair.*

Proof. Since the pairs are tissue-type incompatible, but otherwise formed randomly using the governing population distributions, for each patient type i, the measure of pairs with patient type i is proportional to $m_i\theta_i$.

Fix a small $\epsilon > 0$. Consider any k that satisfies Assumption A-4.

We use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ fraction of the self-demanded $X-X$ pairs can be matched with compatible self-demanded $X-X$ pairs. To show this, we first construct a two-sided matching problem with these pairs. For any patient type i, we split the set of pairs with patient type i into two sets with equal measure. These sets are then added to different sides of the market. As a result, we get a two-sided matching problem where each side has $X-X$ pairs where those with patient type i have a measure proportional to $m_i\theta_i$. For ease of exposition, suppose that the measure is exactly $m_i\theta_i$.

Consider one side of the market. To apply Gale's Supply-Demand Theorem, take a random measurable subset of pairs on this side of the market that has measure $1-\epsilon$ fraction of all pairs on this side. Since the subset is chosen randomly, the compatibility of patients can still be formed randomly using the governing population. We need to show that for any subset of pairs, the measure of pairs on the other side of the market that are compatible with at least one pair in the set is weakly greater than the measure of chosen pairs. Without loss of generality, instead of considering any set of patient types, we can consider the set of all patients that have types from any given set. Let this set be $\{\sigma(1),\ldots,\sigma(l)\}$.

The measure of the set of pairs that have patient types from this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k}$. The measure of pairs that have patient type $\sigma(i)$ on the other side that are incompatible with all such types is $m_{\sigma(i),k}\theta_{\sigma(i),k}\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. The measure of pairs that have patient type $\sigma(i)$ on the other side that are compatible with at least one type in the set is $m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. Therefore, the measure of pairs on the other side that are compatible with at least one pair in the chosen set is $\sum_{i=1}^{k} m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of pairs that are chosen, which is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k}$ by Assumption A-4.

Therefore, $1-\epsilon$ fraction of pairs on both sides of the market can be matched. As we take $\epsilon\to 0$ and $k\to\infty$, we establish the desired result that almost every pair is matched with a compatible pair. ■

F.5 Sufficient Limit Conditions

In the next lemma, we provide sufficient conditions under which all of the limit assumptions hold.

Lemma A-7 *Suppose that* $\theta_{i,k} = \theta < 1$ *and* $m_{i,k} \to 0$ *for every* $i \leq k$ *as* $k \to \infty$. *Then Assumptions A-1, A-2, A-3, and A-4 hold.*

Proof. When $\theta_{i,k} = \theta$ for every $i \leq k$, Assumption A-1 reduces to

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1-\theta^l$$

under the same conditions as stated therein. Likewise, Assumption A-2 reduces to

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1-(1-(1-\theta)^2)^l,$$

and Assumptions A-3 and A-4 reduce to

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{l} m_{\sigma(i),k}[1-(1-(1-\theta)^2)^{l-1}] + \sum_{i=l+1}^{k} m_{\sigma(i),k}[1-(1-(1-\theta)^2)^l].$$

If we show that $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{l} m_{\sigma(i),k}[1-\beta^{l-1}] + \sum_{i=l+1}^{k} m_{\sigma(i),k}[1-\beta^l]$ for every $\beta < 1$ under the conditions stated in these assumptions, then we will be done. This inequality can be rewritten as

$$(\beta^{l-1}-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=l+1}^{k} m_{\sigma(i),k}[1-\beta^l]. \tag{17}$$

For a fixed ϵ such that $1 > \epsilon > 0$, there exists a natural number n such that $\beta^{n-1} \geq \epsilon > \beta^n$. Then Inequality 17 holds for $l > n$ for every k because the left side of the inequality is negative whereas the right side is positive. Furthermore, as $k \to \infty$ Inequality 17 holds also for every $l \leq n$ because $m_{i,k} \to 0$ for every i and n is a fixed natural number which does not depend on k. In this case, the left side converges to zero and the right side is always positive. ■

References

Agarwal, Nikhil, Itai Ashlagi, Eduardo Azevedo, Clayton Featherstone, and Ömer Karaduman (2019). "Market failure in kidney exchange." *American Economic Review*, 109 (11), 4026–4070.

Bloodbook.com (2018a). "Race and ethnic blood type analysis." http://bloodbook.com/race-eth.html (retrieved on 03/05/2018).

Bloodbook.com (2018b). "Racial and ethnic distribution of ABO blood types." http://bloodbook.com/world-abo.html (retrieved on 03/05/2018).

Gale, David (1957). "A theorem on flows in networks." *Pacific Journal of Mathematics*, 7 (2), 1073–1082.

Hart, A., J. M. Smith, M. A. Skeans, S. K. Gustafson, A. R. Wilk, A. Robinson, J. L. Wainright, C. R. Haynes, J. J. Snyder, B. L. Kasiske, and A. K. Israni (2018). "OPTN/SRTR 2016 annual data report: Kidney." *American Journal of Transplantation*, 18 (S1), 18–113.

OPTN and SRTR (2009-2018a). "Data advanced build, donor: Donation year by donor ABO, living donor relation to recipient, donors recovered: January 1, 1988 - October 30, 2018, for count = kidney." Department of Health and Human Services, Health Resources and Services Administration, Healthcare Systems Bureau, Division of Transplantation, Rockville, MD; United Network for Organ Sharing, Richmond, VA; University Renal Research and Education Association, Ann Arbor, MI. https://optn.transplant.hrsa.gov/data/view-data-reports/build-advanced/ (retrieved on 10/30/2018).

OPTN and SRTR (2009-2018b). "Data advanced build, transplant: Transplant year by recipient ABO, donor type, U.S. transplants performed: January 1, 1988 - October 30, 2018, for organ = kidney." Department of Health and Human Services, Health Resources and Services Administration, Healthcare Systems Bureau, Division of Transplantation, Rockville, MD; United Network for Organ Sharing, Richmond, VA; University Renal Research and Education Association, Ann Arbor, MI. https://optn.transplant.hrsa.gov/data/view-data-reports/build-advanced/ (retrieved on 10/30/2018).

OPTN and SRTR (2009-2018c). "Data advanced build, transplant: Transplant year by recipient ABO, living donor relation to donor, U.S. transplants performed: January 1, 1988 - October 30, 2018, for organ = kidney." Department of Health and Human Services, Health Resources and Services Administration, Healthcare Systems Bureau, Division of Transplantation, Rockville, MD; United Network for Organ Sharing, Richmond, VA; University Renal Research and Education Association, Ann Arbor, MI. https://optn.transplant.hrsa.gov/data/view-data-reports/build-advanced/ (retrieved on 10/30/2018).

OPTN and SRTR (2009-2018d). "Data advanced build, waitlist additions: List year by ABO, CPRA at listing, added to the waiting list: January, 1995 - October 30, 2018, for organ = kidney, count = candidates." Department of Health and Human Services, Health Resources and Services Administration, Healthcare Systems Bureau, Division of Transplantation, Rockville, MD; United Network for Organ Sharing, Richmond, VA; University Renal Research and Education Association,

Ann Arbor, MI. https://optn.transplant.hrsa.gov/data/view-data-reports/build-advanced/ (retrieved on 10/30/2018).

OPTN and SRTR (2009-2018e). "Data advanced build, waitlist additions: List year by ABO, previous transplant, added to the waiting list: January, 1995 - October 30, 2018, for organ = kidney, count = candidates." Department of Health and Human Services, Health Resources and Services Administration, Healthcare Systems Bureau, Division of Transplantation, Rockville, MD; United Network for Organ Sharing, Richmond, VA; University Renal Research and Education Association, Ann Arbor, MI. https://optn.transplant.hrsa.gov/data/view-data-reports/build-advanced/ (retrieved on 10/30/2018).

OPTN and SRTR (2009-2018f). "Data advanced build, waitlist removals: Removal year by ABO, removed from the waiting list : January, 1995 - October 30, 2018, for organ = kidney, count = candidates." Department of Health and Human Services, Health Resources and Services Administration, Healthcare Systems Bureau, Division of Transplantation, Rockville, MD; United Network for Organ Sharing, Richmond, VA; University Renal Research and Education Association, Ann Arbor, MI. https://optn.transplant.hrsa.gov/data/view-data-reports/build-advanced/ (retrieved on 10/30/2018).

Roth, Alvin E., Tayfun Sönmez, and M. Utku Ünver (2007). "Efficient kidney exchange: coincidence of wants in markets with compatibility-based preferences." *American Economic Review*, 97 (3), 828–851.

United States Renal Data System (2018). *2018 USRDS annual data report: Epidemiology of kidney disease in the United States*. National Institutes of Health, National Institute of Diabetes, Digestive, and Kidney Diseases.

Ünver, M. Utku (2010). "Dynamic kidney exchange." *Review of Economic Studies*, 77 (1), 372–414.