Market Mechanisms for Fair Allocation of Indivisible Objects and Money*

M. Utku Ünver
The University of Pittsburgh†

March 2007

Abstract

This paper studies the problem of fair allocation of indivisible objects and money among agents with quasi-linear preferences. A mechanism determines an allocation for each problem. We introduce a class of mechanisms, namely market mechanisms, which use tâtonnement process. We prove that adjustment process converges to an envy-free, efficient, and individually rational allocation. Some interesting examples of mechanisms in this class are presented.

Keywords: Fair allocation problem, market mechanism, tâtonnement process.

JEL Classification Numbers: D63, D44, C78

*I would like to thank two anonymous referees, Walter Bossert, Corrado Corradi, Lars Ehlers, Fuhito Kojima, Tayfun Sönmez, Yves Sprumont, William Thomson, and seminar participants at Montreal, Ohio State, and Rochester for discussions and comments on preliminary drafts of the paper. Previous version of this paper was named as "Market Mechanisms for Fair Division with Indivisible Objects and Money". I would like to acknowledge the reserach support of NSF through grant SES-0616689. Any errors are my own responsibility.

†Department of Economics, University of Pittsburgh, 4506 W. W. Posvar Hall, 230 S. Bouquet St., Pittsburgh, PA, 15260, USA. E-mail: uunver@pitt.edu, URL: http://www.pitt.edu/~uunver. Phone: (+1)-412-648-1737.

1 Introduction

The market mechanism is considered as the central notion in finding a competitive allocation in an exchange economy. In this paper, we propose a class of market mechanisms for *fair allocation problems with indivisible objects and money.* A fair allocation problem with indivisible objects and money consists of a set of agents, a set of indivisible objects, a fixed amount of money endowment, and utility profiles of agents on objects and money. In an allocation for the problem, each object shall be assigned to an agent and each agent will get a share from the money endowment. We assume that each agent's utility function is quasi-linear in money shares.

In real life, there are several applications of fair allocation problems. One example is parking space and benefit allocation at a workplace. In this problem, each employee shall get a parking space and a share from a fixed benefits package. Another example is allocation of a bequest consisting of houses and money so that each inheritor shall get a house and a share from the money. A third example is job allocation among a group of employees. In this problem, each employee shall be assigned a job and a money compensation. Another application is a room assignment-rent division problem. In this problem, a group of agents shall rent a house. Each agent shall get a room and pay a share of the rent of the house. In this problem, the money endowment is a negative amount, where as in the applications we mentioned above there is a positive money endowment.

An allocation is a matching which assigns each agent an object and a money distribution vector which attaches a money share for each object. An allocation mechanism finds an allocation for each fair allocation problem. In this paper, we propose a class of allocation mechanisms which use the principal ideas of the "tâtonnement process."

As it is well-known, tâtonnement process can be iteratively used to find a competitive allocation in an exchange economy starting from an arbitrary price vector. We keep the sum of the prices of goods constant instead of having a numéraire good. We first formulate *overdemand*, *underdemand*, and *perfect demand* in the domain of fair allocation problems. The dynamic mechanism we propose mimics the market mechanism starting from an initial money distribution vector. After we find the demand of each agent at the initial money distribution, we determine the set of underdemanded objects, the set of overdemanded objects, and the set of perfectly demanded objects using a well-known result in combinatorial optimization theory, Gallai (1963, 1964) -Edmonds(1965) Decomposition Lemma. We then apply the following money adjustment process: (i) money shares attached to the underdemanded objects are increased by an equal small amount; (ii) money shares attached to the overdemanded objects are decreased by an equal small amount; (iii) money shares attached to the perfectly demanded objects are changed by an equal small amount that is no larger than the increment for the underdemanded objects and no smaller than the decrement for the overdemanded objects. We choose the amount of changes such that the sum of money changes for all objects is equal to zero. This adjustment gives a new "feasible" money distribution vector such that the sum of money shares is equal to the money endowment. We determine the demands of all agents, the supplies of all objects, the set of overdemanded objects, the set of underdemanded objects, and the set of perfectly demanded objects at the new money distribution vector. Then we repeat the above money adjustment process. We iteratively continue adjusting money shares until there are no underdemanded objects left. We show that at this point, we can assign each agent a matching in her demand set and such a matching can be constructed by Edmonds' (1965) algorithm.

An allocation is *envy-free* (Foley, 1967) if nobody prefers the object and money share of another agent to her allocation share. An allocation is *efficient* if the summation of indirect utilities of all agents under this allocation is no smaller than the sum under any other allocation. An allocation is *individually rational* if the indirect utility of no agent is less than the reservation utility, which is zero. We show that the outcome of any market mechanism is envy-free, efficient, and individually rational.

Since there are various applications of fair allocation problems, different market mechanisms can be used to solve different applications.

In a room assignment-rent division problem it is important to find an allocation with "non-positive money shares." If an allocation has positive money shares, it involves a compensation of an agent by the other renters. However, the other renters will be better off by keeping this agent out of their coalition and leaving her room empty. However, there may not exist envy-free allocations with non-positive money shares (Maskin, 1987). We propose a market mechanism to solve this problem whenever such an allocation exists. If the change of money shares of the perfectly demanded objects is equal to the increment of money shares of the underdemanded objects, then the induced market mechanism will find an envy-free allocation with non-positive money shares whenever such an allocation exists. We show this result by the equivalence of this particular market mechanism to the Abdulkadiroğlu, Sönmez and Ünver (2004) mechanism, which has the mentioned property.

On the other hand, in a bequest allocation problem it is important to find an allocation with "non-negative money shares." If an allocation has negative money shares, it involves taxation on an agent by the other inheritors. However, there may not be additional money owned by the inheritor to pay the tax share. We propose a market mechanism for solving this problem whenever such an allocation exists. If the change of money shares of the perfectly demanded objects is equal to the decrement of the money shares of the overdemanded objects, then the induced market mechanism will find an envy-free allocation with non-negative money shares whenever such an allocation exists.

Note that the outcome of the first mechanism above can be found using a standard Vickrey-type auction such as Demange, Gale, and Sotomayor's (1986): Find the buyer-optimal competitive price/money distribution and outcome using an auction for the induced auction market and then balance the budget by equally subsidizing/taxing the owner of each object. The dual of this procedure can be used to find the outcome of the second mechanism (another algorithm for this mechanism is suggested by Aragones (1995)). However, the outcome of none of the other market mechanisms we introduce in this paper can be found using an auction already introduced in the literature. An interesting example of these is as follows: The two above mechanisms are "egalitarian" in the sense that they minimize the maximum money share and maximize the minimum money share among all envy-free allocations, respectively. We introduce a compromise between the two above mechanisms: we treat underdemanded and overdemanded objects symmetrically in money adjustments and we do not adjust the prices of perfectly demanded objects in the process. We refer to this mechanism as the "compromised egalitarian mechanism."

1.1 Literature Background

In the literature, there are many studies on fair allocation problems with indivisible objects and money. Studies including Svensson (1983), Quinzii (1984), Maskin (1987) and Alkan, Demange and Gale (1991) derive properties of envy-free allocations and Walrasian equilibrium in fair allocation problems. Tadenuma and Thomson (1991, 1995) and Svensson and Larsson (2002) present axiomatic approaches for fair allocation problems. Aragones (1995), Su (1999), Klijn (2000), Brams and Kilgour (2001), Haake, Raith and Su (2002), Potthoff (2002), Abdulkadiroğlu, Sönmez and Ünver (2004) (ASÜ, from now on) propose different allocation mechanisms for fair allocation problems. In a recent paper, ASÜ formulated an allocation procedure based on the principles of the market mechanism. They propose a natural mechanism for a room assignment-rent division problem. They show that an informal tâtonnement process can be formalized in such a way that the induced market mechanism finds an envy-free allocation with a non-positive money distribution whenever possible. They only formulate the set of overdemanded objects. By decreasing the money share of the objects in this set by a small amount, and by increasing the money shares of the remaining objects by a small amount, they obtain a money adjustment process. This is the key to their market mechanism. We generalize this idea to generate a general class of mechanisms to solve various fair allocation problems. We introduce the set of underdemanded objects and the set of perfectly demanded objects using a well-known result in combinatorial optimization theory, known as Gallai (1963, 1964)-Edmonds (1965) Decomposition Lemma. We then propose the generalized money adjustment process that is outlined in the previous section. It turns out th'at the ASÜ mechanism is one of the mechanisms in this class.

The choice of the tâtonnement process does not usually have an effect on the choice of the competitive allocation in exchange economies. However, in the fair allocation problems, selection of the tâtonnement process bears additional significance. As we show in this paper, the choice of money updating method changes the properties of the outcome extensively. It follows from the definition of our mechanism, ASÜ Theorem 2 and the dual of this theorem that two of the mechanisms in our class find extreme envy-free allocations with the smallest of the money shares maximized or with the largest of the money shares minimized among all envy-free allocations.[1]

In the next section, we introduce our model.

2 Fair Allocation Problems

A **fair allocation problem** is a quadruple $\langle I, A, V, m \rangle$ where $I = \{i_1, \dots, i_n\}$ is a set of agents, $A = \{a_1, \dots, a_n\}$ is a set of objects, $V = [v_a^i]_{i \in I, a \in A}$ is a value matrix where $v_a^i \in \mathbb{R}$ denotes the value of object $a \in A$ for agent $i \in I$, and $m \in \mathbb{R}$ is the money endowment. Each agent $i \in I$ can use one and only one object.[2] Let $\mathbb{F}$ be the set of all fair allocation problems. We will fix a problem $\langle I, A, V, m \rangle$ $\in \mathbb{F}$ for the rest of our analysis.

A **money distribution** $t = (t_{a_1}, ..., t_{a_n})$ is a list such that $\sum_{a \in A} t_a = m$ where **money share** $t_a \in \mathbb{R}$ shows the money share attached to object $a \in A$ under money distribution t. Let $\mathcal{T}$ be the set of money distributions.

The **utility** of each agent $i \in I$ is a function $u_i : A \times \mathbb{R} \to \mathbb{R}$ which is defined as

$$u_i(a, t_a) = v_a^i + t_a$$

for all objects $a \in A$ at all money shares $t_a \in \mathbb{R}$. Each agent has a reservation utility, which is equal to zero, denoting her outside options.

We shall assume that

$$\sum_{a \in A} v_a^i + m \geq 0$$

for each agent $i \in I$. This assumption is fairly standard in the literature.[3] It will ensure individual rationality of the mechanisms that we will introduce. There are two interpretations of this assumption: (i) Each agent thinks that the total disutility of the objects can be compensated by the money endowment. It is natural to assume that if an agent thinks it is not worth to obtain the objects then she will not be in this coalition of agents. (ii) If an agent would like to form a coalition with agents who are exact copies of her, then this coalition of agents will be willing to obtain these objects.

[1] Alkan, Demange and Gale (1991) prove the existence of extreme allocations of this sort. In different domains of problems, there are extreme allocations and mechanisms of this sort as well. For example, in "two-sided matching markets" Gale and Shapley (1962) propose two mechanisms which find optimal stable matchings for each side of the market.

[2] In reality, there may be more agents than objects. In this case, we create $|I| - |A|$ new **dummy** objects, add these to set A, and make the number of objects equal to number of agents. We modify the value vector of each agent by adding value $v_a^i = 0$ (the reservation utility) for any dummy object a. In a solution (that is described below) of the problem, some agents will be assigned dummy objects, or no real objects, and they will be only compensated/taxed by money shares.

[3] See Su (1999), Brams and Kilgour (2001), Haake, Raith and Su (2002), and Abdulkadiroğlu, Sönmez and Ünver (2004).

A **matching** $\mu = \{\{i_1, \mu_{i_1}\}, ..., \{i_n, \mu_{i_n}\}\}$ is a list of the assignments of objects to agents such that each object is assigned to one agent and each agent is assigned one object and component $\mu_i \in A$ denotes the assignment of agent $i \in I$ under matching μ. Let $\mathcal{M}$ denote the set of matchings. That is, for any $i \in I$, $\mu_i \in A$, and for any $\{i, j\} \subseteq I$, we have $\mu_i \neq \mu_j$.

An **allocation** is a matching - money distribution pair $(\mu, t) \in \mathcal{M} \times \mathcal{T}$. At an allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ each agent $i \in I$ obtains object μ_i and money share t_{μ_i}.

An **allocation mechanism** is a systematic procedure which finds an allocation for each problem.

3 A Market Approach

We adopt a market approach in our analysis. The **demand** of each agent $i \in I$ is a correspondence $D_i : \mathcal{T} \twoheadrightarrow A$ that is defined as

$$D_i(t) = \left\{a \in A : u_i(a, t_a) \geq u_i(a', t_{a'}) \;\forall\; a' \in A\right\}$$

for all $t \in \mathcal{T}$. By definition, $D_i(t) \neq \emptyset$ for any $i \in I$ and $t \in \mathcal{T}$.

The **demand profile** at money distribution $t \in \mathcal{T}$ is defined as $D(t) = (D_i(t))_{i \in I}$.

The **supply** of each object $a \in A$ is a correspondence $S_a : \mathcal{T} \twoheadrightarrow I$ which is defined as

$$S_a(t) = \{i \in I : a \in D_i(t)\}$$

for all $t \in \mathcal{T}$.

The **supply profile** at money distribution $t \in \mathcal{T}$ is defined as $S(t) = (S_a(t))_{a \in A}$.

The **indirect utility** function of each agent $i \in I$, $\widetilde{u}_i : \mathcal{T} \to \mathbb{R}$, is defined as

$$\widetilde{u}_i(t) = \max_{a \in A} \; u_i(a, t_a)$$

for all $t \in \mathcal{T}$.

For any $t \in \mathcal{T}$, a matching $\mu \in \mathcal{M}$ **clears the market** if $\mu_i \in D_i(t)$ for any $i \in I$.

In order to formulate our market approach, we need to define the objects in excess supply, the objects in excess demand, and the objects in perfect demand. This will be crucial for the definition of the "tatônnement" process.

3.1 Gallai-Edmonds Decomposition

We use a graph theoretic approach to define the notions of overdemand, underdemand and perfect demand.

We say that a set $\{i, a\}$ consisting of an agent $i \in I$ and an object $a \in A$ is a **link** at money distribution $t \in \mathcal{T}$ if and only if $a \in D_i(t)$. The **set of links** at a money distribution $t \in \mathcal{T}$ is defined as follows:

$$L(t) = \{\{i, a\} : i \in I,\; a \in A, \text{ and } a \in D_i(t)\}$$

for each $t \in \mathcal{T}$.

The **demand-supply graph** at money distribution $t \in \mathcal{T}$ is a pair $\langle I \cup A, L(t)\rangle$ where agents in I and objects in A are the nodes of the graph and links in $L(t)$ are the arcs of the graph. We denote the demand-supply graph at money distribution $t \in \mathcal{T}$ by $\mathcal{G}(t)$.

Fix $t \in \mathcal{T}$. A **market assignment at** t is a set of links $Q \subseteq L(t)$ such that any agent receives either no object or one object in her demand and any object is assigned to either nobody or one agent in its supply. That is, $\{i, a\} \in Q$ and $\{j, b\} \in Q \setminus \{\{i, a\}\}$ implies $i \neq j$ and $a \neq b$. Component Q_i is the assigned object of agent $i \in I$ under market assignment Q : if agent i is assigned an object under

Q then $Q_i \in A$, and if agent i is not assigned any object under Q then $Q_i = \emptyset$. Let $\mathcal{Q}(t)$ be the set of market assignments at t. For any $x \in A \cup I$ and $Q \in \mathcal{Q}(t)$ we say that x is **unmatched under** Q if there is no $\{i, a\} \in Q$ such that $x \in \{i, a\}$. A market assignment $Q \in \mathcal{Q}(t)$ is **maximal** if for all $R \in \mathcal{Q}(t)$ we have $|Q| \geq |R|$. Let $\mathcal{M}(t)$ be the set of maximal market assignments at t. The following observations are trivial, and yet crucial for our market approach:

Observation 1: A market assignment $Q \in \mathcal{Q}(t)$ is a matching if and only if $|Q| = n$.

Observation 2: A matching $\mu \in \mathcal{M}$ clears the market at t if and only if it is a market assignment at t.

Since $|Q| \leq n$ for any $Q \in \mathcal{Q}(t)$, if a market assignment $Q \in \mathcal{Q}(t)$ is a matching then it is maximal.

Consider the following partitions $\{UD(t), OD(t), PD(t)\}$ of A and $\{US(t), OS(t), PS(t)\}$ of I defined at t:

$$
\begin{aligned}
UD(t) &= \{a \in A : \exists Q \in \mathcal{M}(t) \text{ such that } \forall i \in I, (i,a) \notin Q\}. \\
US(t) &= \{i \in I : \exists Q \in \mathcal{M}(t) \text{ such that } \forall a \in A, (i,a) \notin Q\}. \\
OD(t) &= \{a \in A \backslash UD(t) : \exists i \in US(t) \text{ such that } (i,a) \in L(t)\}. \\
OS(t) &= \{i \in I \backslash US(t) : \exists a \in UD(t) \text{ such that } (i,a) \in L(t)\}. \\
PD(t) &= \{a \in A \backslash UD(t) : \forall i \in US(t),\ (i,a) \notin L(t)\}. \\
PS(t) &= \{i \in I \backslash US(t) : \forall a \in UD(t),\ (i,a) \notin L(t)\}.
\end{aligned}
$$

Set $UD(t)$ ($US(t)$) is the set of objects (agents) for each of which (whom) there is a maximal market assignment leaving it (her) unmatched at t. Set $OD(t)$ ($OS(t)$) is the set of objects (agents) each of which (whom) is matched under any maximal market assignment at t and has a link to an agent in $US(t)$ (an object in $UD(t)$). Set $PD(t)$ ($PS(t)$) is the set of objects (agents) each of which (whom) is matched under any maximal market assignment at t and does not have any link to agents in $US(t)$ (objects in $UD(t)$). These sets are crucial for the structure of maximal market assignments. This structure is well-studied in the combinatorial optimization literature. Gallai (1963, 1964) and Edmonds (1965)) Decomposition (GED) Lemma for bipartite graphs is about the structure of maximal market assignments, we derive this lemma from the GED Lemma for general graphs in Appendix E.[4]

Gallai-Edmonds Decomposition (GED) Lemma for Bipartite Graphs: Let $t \in \mathcal{T}$. We have $D_i(t) \subseteq OD(t)$ for any $i \in US(t)$ and $S_a(t) \subseteq OS(t)$ for any $a \in UD(t)$. Let $Q \in \mathcal{M}(t)$ be a maximal market assignment at t.

1. Every object in $OD(t)$ is assigned to an agent in $US(t)$ under Q.

2. Every agent in $OS(t)$ is assigned an object in $UD(t)$ under Q.

3. Every object in $PD(t)$ is assigned to an agent in $PS(t)$ and every agent in $PS(t)$ is assigned an object in $PD(t)$ under Q.[5]

We will refer to this lemma as the GED Lemma. Based on the GED Lemma, we refer to $OD(t)$ as the **set of overdemanded objects**, $UD(t)$ as the **set of underdemanded objects**, and $PD(t)$ as the **set of perfectly demanded objects** at money distribution t. Similarly, we refer to $OS(t)$ as the **set of oversupplied agents**, $US(t)$ as the **set of undersupplied agents**, and $PS(t)$

[4] See Bogomolnaia and Moulin (2004) and Roth, Sönmez and Ünver (2005) for use of the Gallai-Edmonds Decomposition Lemma in other allocation problems.

[5] See Bogomolnaia and Moulin (2004) for an alternative formulation of the GED Lemma for bipartite graphs.

as the **set of perfectly supplied agents** at money distribution t.[6] We refer to the partitions $\{UD(t), OD(t), PD(t)\}$ of A and $\{US(t), OS(t), PS(t)\}$ of I as the **Gallai-Edmonds Decomposition (GED)** of the problem at t. The next result is a direct corollary of the GED Lemma and the definitions:

Corollary 1: Let $t \in \mathcal{T}$. We have

1. If $UD(t) \neq \emptyset$ then $|UD(t)| > |OS(t)|$, otherwise $OS(t) = \emptyset$.
2. If $US(t) \neq \emptyset$ then $|US(t)| > |OD(t)|$, otherwise $OD(t) = \emptyset$.
3. $|PD(t)| = |PS(t)|$.

The next lemma is crucial for our analysis and our understanding of the conditions of a market clearing money distribution:

Lemma 1: Let $t \in \mathcal{T}$ and $Q \in \mathcal{M}(t)$.

1. $|Q| = n$ if and only if $UD(t) = \emptyset$.
2. $|Q| = n$ if and only if $US(t) = \emptyset$.

Proofs of all results are given in Appendix A.

Sets $UD(t)$ and $US(t)$ can be constructed in polynomial time complexity in number of agents n using several well-known algorithms in combinatorial optimization literature. For example, **Edmonds'** (1965) **algorithm** finds a maximal market assignment Q in $O(n^3)$ complexity, and then construction of sets $UD(t), OD(t), PD(t)$ and $US(t), OS(t), PS(t)$ is straightforward. These are explained in Appendix B.

4 Market Mechanisms

A market mechanism incrementally increases the money shares of the underdemanded objects. It incrementally decreases the money shares of the overdemanded objects. It changes the money shares of the perfectly demanded objects by no greater than the increment for the underdemanded objects and no smaller than the decrement for the overdemanded objects such that the sum of the money shares of all objects is equal to m. We are ready to state a market mechanism "informally" as follows:

Step 0: Initially set the money share of each object $a \in A$ to $t_a^0 = \frac{m}{n}$ such that $t^0 = (\frac{m}{n}, \frac{m}{n}, ..., \frac{m}{n})$. Find a maximal market assignment Q^0.

(a) If $|Q^0| = n$ then Q^0 is a matching such that $Q_i^0 \in D_i(t^0)$ for each agent i by Observation 1. We terminate the procedure and (Q^0, t^0) is the outcome.

(b) If $|Q^0| < n$ then $UD(t^0) \neq \emptyset$ by Lemma 1 and we proceed to the next step.

In a general step s,

Step s: Construct GED at t^{s-1} using Q^{s-1}. Let $t^s \in \mathcal{T}$ be defined as $t_a^s = t_a^{s-1} + \alpha$ for any $a \in UD(t^{s-1})$, $t_a^s = t_a^{s-1} + \beta$ for any $a \in PD(t^{s-1})$, and $t_a^s = t_a^{s-1} + \gamma$ for any $a \in OD(t^{s-1})$ such that $\alpha \to 0$, $\beta \to 0$, $\gamma \to 0$ with

(i) $\alpha \geq \beta \geq \gamma$ and

(ii) $|UD(t^{s-1})|\alpha + |PD(t^{s-1})|\beta + |OD(t^{s-1})|\gamma = 0$.

Construct a maximal market assignment Q^s at t^s.

[6] For alternative approaches in constructing overdemanded and oversupplied sets using Hall's (1935) Theorem, see Demange, Gale and Sotomayor (1986) and de Vries, Schummer and Vohra (2005).

(a) If $|Q^s| = n$ then Q^s is a matching such that $Q^s_i \in D_i(t^s)$ for each agent i by Observation 1. We terminate the procedure and (Q^s, t^s) is the outcome.
(b) If $|Q^s| < n$ then $UD(t^s) \neq \emptyset$ by Lemma 1 and we proceed to the next step.

The above money adjustment rule definition is in the flavor of an informal "tâtonnement" process in exchange economies. In the next subsection, we will make a formal definition of the money adjustment rule and be precise about the properties of increments (or decrements) α, β, γ.

4.1 Formalization of a Market Mechanism

We claim that the crucial money distribution levels in an informal tâtonnement process are those at which a new object joins the demand of an agent. Because that is only when the set of overdemanded objects, the set of underdemanded objects and the set of perfectly demanded objects can change. We prove this with a lemma.

Lemma 2: Let $\{t^s\}$ be the money distribution sequence of a market mechanism. For any step s, if we have $D_i(t^{s+1}) \subseteq D_i(t^s)$ for all $i \in I$, then $UD(t^{s+1}) = UD(t^s)$, $US(t^{s+1}) = US(t^s)$, $PD(t^{s+1}) = PD(t^s)$, $PS(t^{s+1}) = PS(t^s)$, $OD(t^{s+1}) = OD(t^s)$, and $OS(t^{s+1}) = OS(t^s)$.

This result shows that as long as new objects do not join the demands of agents, underdemand, undersupply, perfect demand, perfect supply, overdemand, and oversupply will not change in a market mechanism. Note that even if some objects are dropped from demands of agents during this process, these crucial sets will stay put.

We can directly find the next money distribution when a new object joins the demand of an agent. This new money distribution may possibly induce a change in overdemand, underdemand and perfect demand.

Let $t \in \mathcal{T}$ be a money distribution reached in the market mechanism. We continuously increase or decrease money shares. Let $t' \in \mathcal{T}$ be the first money distribution level reached after t where the demand of an agent includes a new object.

We will distinguish momentary rates α,β and γ and the discrete changes achieved in money distribution after a certain amount of money adjustments. Let $\triangle t_a$ be a discrete increment for the money share of object a such that

$$\begin{aligned} t'_a &= t_a + \alpha(t) \qquad \forall\, a \in UD(t) \\ t'_a &= t_a + \beta(t) \qquad \forall\, a \in PD(t) \\ t'_a &= t_a + \gamma(t) \qquad \forall\, a \in OD(t) \end{aligned}$$

The agent who demands a new object at t' is necessarily a member of $US(t)$ or $PS(t)$. That is, because (i) each agent in $OS(t)$ demands an object in $UD(t)$ at money distribution t and the money shares of objects in $UD(t)$ are increasing uniformly at the highest rate, and (ii) utilities are quasi-linear in money. Three observations hold for the discrete changes $\alpha(t)$, $\beta(t)$, and $\gamma(t)$:

A new object can join the demand of an agent in $US(t)$ in two ways. Either this object is a perfectly demanded object at money distribution t and its money share is rising faster than money shares of the overdemanded objects or this object is an underdemanded object.

We define

$$x(t) = \begin{cases} \min_{i \in US(t)} \left(\tilde{u}_i(t) - \max_{a \in UD(t)} u_i(a, t_a)\right) & \text{if } UD(t) \neq \emptyset \text{ and } US(t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$y(t) = \begin{cases} \min_{i \in US(t)} \left(\tilde{u}_i(t) - \max_{a \in PD(t)} u_i(a, t_a)\right) & \text{if } PD(t) \neq \emptyset \text{ and } US(t) \neq \emptyset \\ x(t) & \text{otherwise} \end{cases}.$$

Consider objects $a, b, c \in A$ such that $a \in UD(t)$, $b \in PD(t)$, and $c \in OD\,(t)$. The money share of a increases at the rate α, the money share of b increases at the rate β, and the money share of c increases at the rate γ initially at t until t' is reached.

- The money differential $t_a - t_c$ increases at the same momentary rate, $\alpha - \gamma$, for any pair of such objects until a new object joins the demand of an agent. This may happen when an agent in $US\,(t)$ demands an underdemanded object and $x(t)$ is the minimum differential for that to occur. Therefore, we need $\alpha\,(t) - \gamma\,(t) \leq x\,(t)$ to have the same momentary rates, α and γ, to use in our money adjustments between t and t' by Lemma 2.

- The money differential $t_b - t_c$ increases at the same momentary rate, $\beta - \gamma$, for any pair of such objects until a new object joins the demand of an agent. This may happen when an agent in $US\,(t)$ demands a perfectly demanded object and $y(t)$ is the minimum differential for that to occur. Therefore, we need $\beta\,(t) - \gamma\,(t) \leq y\,(t)$ to have the same momentary rates, β and γ, to use in our money adjustments between t and t' by Lemma 2.

A new object can join the demand of a member of $PS(t)$, if it is an underdemanded object and money shares of the underdemanded objects are rising faster than money shares of the perfectly demanded objects. We define

$$z(t) = \begin{cases} \min_{i \in PS(t)} \left(\tilde{u}_i\,(t) - \max_{a \in UD(t)} u_i(a, t_a) \right) & \text{if } UD(t) \neq \emptyset \text{ and } PS\,(t) \neq \emptyset \\ x(t) & \text{otherwise} \end{cases} .$$

Consider objects $a, b \in A$ such that $a \in UD(t)$, $b \in PD(t)$. The money share of a increases at the rate α, the money share of b increases at the rate β initially at t until t' is reached.

- The money differential $t_a - t_b$ increases at the same momentary rate, $\alpha - \beta$, for any pair of such objects until a new object joins the demand of an agent. This may happen when an agent in $PS\,(t)$ demands an underdemanded object and $z(t)$ is the minimum differential for that to occur. Therefore, we need $\alpha\,(t) - \beta\,(t) \leq z\,(t)$ to have the same momentary rates, α and β, to use in our money adjustments between t and t' by Lemma 2.

One of these three situations will occur before the others causing a new object to join the demand of an agent. Using this information we can construct a "discrete algorithm" to find the outcome of a market mechanism.

To summarize, we have

$$\begin{aligned} &\alpha(t) - \gamma(t) \leq x(t), \\ &\beta(t) - \gamma(t) \leq y(t), \text{ and} \\ &\alpha(t) - \beta(t) \leq z(t) \end{aligned} \tag{1}$$

and, since at money distribution t' a new object joins the demand of an agent, **one of the above inequalities is binding.**

Since t' is a money distribution level reached in a market mechanism, we have

$$\gamma(t) \leq \beta(t) \leq \alpha(t) \tag{2}$$

and

$$|UD(t)|\,\alpha(t) + |PD(t)|\,\beta(t) + |OD(t)|\,\gamma(t) = 0\,. \tag{3}$$

If $UD(t) \neq \emptyset$, Equation 1 implies that $\alpha(t) - \gamma(t) > 0$, or $\beta(t) - \gamma(t) > 0$, or $\alpha(t) - \beta(t) > 0$. In either case, Equation 2 implies that $\alpha(t) > \gamma(t)$, which in turn implies with Equation 3 that $\alpha(t) > 0$ and $\gamma(t) < 0$.

We are ready to introduce an iterative discrete algorithm that is used to compute the outcome of a market mechanism.

Step 0: Initially set the money share of each object to $\frac{m}{n}$. Let $t^0 = (\frac{m}{n}, \frac{m}{n}, ..., \frac{m}{n})$. Find a maximal market assignment $Q^0 \in \mathcal{M}(t^0)$ at t^0.
(a) If $|Q^0| = n$ then Q^0 is a matching such that $Q_i^0 \in D_i(t^0)$ for each agent i. We terminate the procedure and (Q^0, t^0) is the outcome.
(b) If $|Q^0| < n$ then $UD(t^0) \neq \emptyset$ by Lemma 1 and we proceed to the next step.

In a general step s,

Step s: Construct GE Decomposition at t^{s-1} using Q^{s-1}. Let $t^s \in \mathcal{T}$ be defined as $t_a^s = t_a^{s-1} + \alpha(t^{s-1})$ for all $a \in UD(t^{s-1})$, $t_a^s = t_a^{s-1} + \beta(t^{s-1})$ for all $a \in PD(t^{s-1})$, and $t_a^s = t_a^{s-1} + \gamma(t^{s-1})$ for all $a \in OD(t^{s-1})$. Find a maximal market assignment $Q^s \in \mathcal{M}(t^s)$ at t^s.
(a) If $|Q^s| = n$ then Q^s is a matching such that $Q_i^s \in D_i(t^s)$ for each agent i. We terminate the procedure and (Q^s, t^s) is the outcome.
(b) If $|Q^s| < n$ then $UD(t^s) \neq \emptyset$ by Lemma 1 and we proceed to the next step.

We use the discrete algorithm outlined above for showing that a market mechanism converges to a market outcome and for computing the outcome of a market mechanism.

Note that one may uniquely define a market mechanism by choosing one of the functions α, β, and γ as a function of the remaining two. For instance, different choices of function β induce algorithms for different market mechanisms. Three interesting mechanisms are (i) the market mechanism with $\beta = \alpha$, (ii) the market mechanism with $\beta = 0$, and (iii) the market mechanism with $\beta = \gamma$. In the first mechanism, the money shares of the perfectly demanded objects increase at the same rate as the underdemanded objects. In the second mechanism, the money shares of the perfectly demanded objects are kept constant. In the third mechanism, the money shares of the perfectly demanded objects decrease at the same rate as the overdemanded objects. In the below example, we find functions α, β, and γ used in the algorithms for these market mechanisms.

Example 1: For each of these mechanisms, we find functions α, β, and γ using Equation System 1, Equation 2, and Equation 3:

1. **Market mechanism with $\beta = \alpha$:**

$$\begin{aligned} \alpha(t) &= \frac{|OD(t)|}{n} \min\{x(t), y(t)\}, \\ \gamma(t) &= -\frac{|UD(t)| + |PD(t)|}{n} \min\{x(t), y(t)\}, \\ \beta(t) &= \frac{|OD(t)|}{n} \min\{x(t), y(t)\} \end{aligned}$$

   for all $t \in \mathcal{T}$.

2. **Market mechanism with $\beta = 0$:**

$$\begin{aligned} \alpha(t) &= \min\{\frac{|OD(t)|}{|OD(t)| + |UD(t)|} x(t), \frac{|OD(t)|}{|UD(t)|} y(t), z(t)\}, \\ \gamma(t) &= \max\{-\frac{|UD(t)|}{|OD(t)| + |UD(t)|} x(t), -y(t), -\frac{|UD(t)|}{|OD(t)|} z(t)\}, \\ \beta(t) &= 0 \end{aligned}$$

   for all $t \in \mathcal{T}$.

3. **Market mechanism with** $\beta = \gamma$:

$$\alpha(t) = \frac{|OD(t)| + |PD(t)|}{n} \min\{x(t), z(t)\},$$
$$\gamma(t) = -\frac{|UD(t)|}{n} \min\{x(t), z(t)\},$$
$$\beta(t) = -\frac{|UD(t)|}{n} \min\{x(t), z(t)\}$$

for all $t \in \mathcal{T}$. ♦

In Appendix C, we show how the discrete algorithm can be used to find the outcome of a market mechanism for a fair allocation problem with an example.

4.2 Convergence of a Market Mechanism

We will prove that a market mechanism converges to a market outcome. We will use the discrete algorithm to prove this result. The following proposition shows that summation of indirect utilities monotonically decreases in the discrete algorithm of a market mechanism.

Proposition 1: Let $\{t^s\}$ be the money distribution sequence of the discrete algorithm of a market mechanism. For every step $s \geq 0$ with $UD(t^s) \neq \emptyset$ we have $\sum_{i \in I} \tilde{u}_i(t^s) \geq \sum_{i \in I} \tilde{u}_i(t^{s+1}) + \alpha(t^s) - \gamma(t^s)$.

We can state our main convergence result.

Theorem 1: Let $\{t^s\}$ be the money distribution sequence in a market mechanism. Then, there exists some equivalent discrete algorithm which converges to a money distribution in finite number of steps.

4.3 Characteristics of a Market Outcome

Envy-freeness and efficiency are central notions in fair allocation problems. An allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ is **envy-free** if and only if

$$u_i(\mu_i, t_{\mu_i}) \geq u_i(a, t_a) \qquad \forall\, a \in A.$$

Note that an allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ is envy-free if and only if $\mu_i \in D_i(t)$ for each agent $i \in I$. An allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ is **efficient** if and only if

$$\sum_{i \in I} u_i(\mu_i, t_{\mu_i}) \geq \sum_{i \in I} u_i(\lambda_i, x_{\lambda_i}) \qquad \forall\ (\lambda, x) \in \mathcal{M} \times \mathcal{T}.$$

Svensson (1983) showed that if an allocation is envy-free then it is efficient in this class of fair allocation problems.

Another central notion is individual rationality. An allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ is **individually rational** if and only if

$$u_i(\mu_i, t_{\mu_i}) \geq 0 \qquad \forall\, a \in A.$$

An individually rational allocation guarantees at least the reservation utility for each agent. Our market mechanisms find allocations, which satisfy the above properties.

Proposition 2: An outcome of a market mechanism is envy-free, efficient, and individually rational.

5 The Family of Market Mechanisms and the Compromised Egalitarian Mechanism

Different choices of function β in the discrete algorithm find outcomes of different market mechanisms. Three examples are as follows:

- Consider a fair allocation problem with a negative money endowment. An example is room assignment-rent division problem. Hence, a natural allocation to this problem involves an allocation with non-positive money shares. Envy-free allocations with non-positive money shares may not exist (Maskin, 1987). For example, let $I = \{i_1, i_2\}$, $A = \{a_1, a_2\}$, $v^{i_1} = v^{i_2} = (2, 16)$ and $m = -10$. The unique envy-free money distribution is $t = (2, -12)$.

  The market mechanism with $\beta(t) = \alpha(t)$ for all $t \in \mathcal{T}$ finds an envy-free allocation with non-positive money shares whenever such an allocation exists. This mechanism is suggested by ASÜ for a room assignment-rent division problem using a different definition for $OD(t)$. In Appendix D, we prove that the ASÜ mechanism is equivalent to market mechanism with $\beta = \alpha$. They also prove the mentioned property of this mechanism in their Theorem 2. The money distribution that this mechanism converges to is the money distribution that solves[7]

  $$\min_{t \in \mathcal{T}} \left(\max_{a \in A} t_a \right).$$

- Consider a fair allocation problem with a positive money endowment. An example is the division of a bequest consisting of houses and money among inheritors. A natural allocation has "non-negative" money shares for each inheritor. Envy-free allocations with non-negative money shares may not exist. An allocation with a negative money share involves taxation on an agent by the others. It directly follows from the dual statement of ASÜ Theorem 2 that the market mechanism with $\beta(t) = \gamma(t)$ for all $t \in \mathcal{T}$ finds an envy-free allocation with non-negative money shares whenever such an allocation exists. The money distribution that this mechanism converges to is the money distribution that solves

  $$\max_{t \in \mathcal{T}} \left(\min_{a \in A} t_a \right).$$

Alkan, Demange and Gale (1991) refer to the above two allocations as money-Rawlsian envy-free allocations. Aragones (1995) also presents a method which finds the last money-Rawlsian allocation described above in a fair allocation problem. We will refer to these mechanisms as **maximal-share** and **minimal-share egalitarian mechanisms**, respectively. Note that the outcome of these mechanisms can also be found using Demange, Gale and Sotomayor (1986) auction and its dual: apply a Vickrey auction and find the buyer- or seller-optimal competitive price/money distribution and balance the budget by equally subsidizing/taxing each agent. However, this is not true for other market mechanisms, including the following interesting mechanism:

---

[7] This result directly follows from Theorem 2 of ASÜ. This theorem proves that the ASÜ mechanism will converge to a non-positive money distribution (non-negative *price vector* in the terminology of ASÜ) as long as such an envy-free money distribution exists. This property is equivalent to the money distribution found by the ASÜ mechanism is a solution to

$$\min_{t \in \mathcal{T}} \left(\max_{a \in A} t_a \right)$$

- The market mechanism with $\beta = 0$ neither favors nor disfavors perfectly demanded goods in money adjustments. Hence, overdemanded and underdemanded objects are treated symmetrically in this mechanism. Therefore, this mechanism is a compromise between the two egalitarian mechanisms introduced above. We refer to this mechanism as the **compromised egalitarian mechanism**.

We can also start from arbitrary initial money distributions $t \in \mathcal{T}$ instead of equal shares as t^0. All our convergence results translate to this case without loss of generality. These can reflect object specific weights and these will induce new market allocations.

6 Appendix A: Proofs of Results

Proof of Lemma 1: Let $t \in \mathcal{T}$ and $Q \in \mathcal{M}(t)$. First suppose $|Q| = n$, that is each agent is matched with an object in her demand set at t under Q. Then there is no agent and no object unmatched. Since each maximal market assignment has cardinality n, none of them leaves any agent or any object unmatched. Therefore, $UD(t) = US(t) = \emptyset$.

Next, suppose $UD(t) = \emptyset$. Each object is matched with an agent in its supply at a maximal market assignment. Since $|I| = |A| = n$, we have $|Q| = n$ for any $Q \in \mathcal{M}(t)$.

The proof of $US(t) = \emptyset$ implies $|Q| = n$ is the dual of the above proof. ♦

Proof of Lemma 2: Let $\{t^s\}$ be the money distribution sequence of a market mechanism. Let step s be such that $D_i(t^{s+1}) \subseteq D_i(t^s)$ for all $i \in I$ and $UD(t^s) \neq \emptyset$. Let α, β, $\gamma \in \mathbb{R}$ satisfy

(a) $\alpha \geq \beta \geq \gamma$,

(b) $\alpha \to 0$, $\beta \to 0$, $\gamma \to 0$ such that $\alpha - \gamma \leq \tilde{u}_i(t^s) - u_i(a, t^s_a)$ for all $i \in I$ and $a \in A \backslash D_i(t^s)$, and

(c) $t^{s+1}_a = t^s_a + \alpha$ for all $a \in UD(t^s)$, $t^{s+1}_a = t^s_a + \beta$ for all $a \in PD(t^s)$, and $t^{s+1}_a = t^s_a + \gamma$ for all $a \in OD(t^s)$.

We will prove the lemma by showing that $\mathcal{M}(t^{s+1}) = \mathcal{M}(t^s)$, that is the set of maximal market assignments are identical for t^{s+1} and t^s. The following observations will be useful in establishing this result:

1. We have $UD(t^s) \neq \emptyset$ and $OS(t^s) \neq \emptyset$. If $S_a(t^s) = \emptyset$ then $S_a(t^{s+1}) \neq \emptyset$ for any $a \in UD(t^s)$. Let $i \in OS(t^s)$ and $a \in US(t^s)$ be such that $i \in S_a(t^s)$. Note that $S_a(t^s) \subseteq OS(t^s)$ by the GED Lemma. For any $b \in A$

$$\tilde{u}_i(t^s) = u_i(a, t^s_a) = v^i_a + t^s_a \geq u_i(b, t^s_b) = v^i_b + t^s_b,$$

and money share of b increases at most by α, implying

$$\begin{aligned} u_i\left(a, t^{s+1}_a\right) &= v^i_a + t^{s+1}_a = v^i_a + t^s_a + \alpha \\ &\geq v^i_b + t^s_b + \alpha \\ &\geq v^i_b + t^{s+1}_b = u_i\left(b, t^{s+1}_b\right). \end{aligned}$$

Hence $a \in D_i(t^{s+1})$ and $i \in S_a(t^{s+1})$. We showed that for any link $\{i, a\} \in \mathcal{L}(t^s)$ such that $a \in UD(t^s)$ and $i \in OS(t^s)$ we have $\{i, a\} \in \mathcal{L}(t^{s+1})$.

2. If $PD(t^s) = \emptyset$ (or equivalently $PS(t^s) = \emptyset$ by the GED Lemma) skip to the next observation. For any $a \in PD(t^s)$ we have $S_a(t^s) \subseteq PS(t^s) \cup OS(t^s)$ and $S_a(t^s) \cap PS(t^s) \neq \emptyset$ by the GED Lemma, and similarly for any $i \in PS(t^s)$, $D_i(t^s) \subseteq PD(t^s) \cup OD(t^s)$ and $D_i(t^s) \cap PD(t^s) \neq \emptyset$ by the GED Lemma. Let $a \in PD(t^s)$ and $i \in S_a(t^s) \cap PS(t^s)$. We have

$$\tilde{u}_i(t^s) = u_i(a, t^s_a) = v^i_a + t^s_a \geq u_i(b, t^s_b) = v^i_b + t^s_b$$

for any $b \in A$. For any $b \in A \backslash UD(t^s)$, money share of b increases by at most β and we have,

$$\begin{aligned} u_i\left(a, t_a^{s+1}\right) &= v_a^i + t_a^{s+1} = v_a^i + t_a^s + \beta \\ &\geq v_b^i + t_b^s + \beta \\ &\geq v_b^i + t_b^{s+1} = u_i\left(b, t_b^{s+1}\right). \end{aligned}$$

Since $D_i\left(t^{s+1}\right) \subseteq D_i\left(t^s\right) \subseteq PD(t^s) \cup OD(t^s)$, $a \in D_i\left(t^{s+1}\right)$ and $i \in S_a\left(t^{s+1}\right)$. We showed that for any link $\{i, a\} \in \mathcal{L}\left(t^s\right)$ such that $a \in PD(t^s)$ and $i \in PS(t^s)$, we have $\{i, a\} \in \mathcal{L}\left(t^{s+1}\right)$.

3. We have $OD(t^s) \neq \emptyset$ and $US(t^s) \neq \emptyset$. Note that for any $i \in US(t^s)$, $D_i(t^s) \neq \emptyset$. Let $a \in OD(t^s)$ and $i \in US(t^s)$ be such that $i \in S_a\left(t^s\right)$. We have $D_i\left(t^s\right) \subseteq OD\left(t^s\right)$ by the GED Lemma. We have

$$\tilde{u}_i\left(t^s\right) = u_i(a, t_a^s) = v_a^i + t_a^s \geq u_i(b, t_b^s) = v_b^i + t_b^s$$

for any $b \in A$. For any $b \in OD(t^s)$, its money share changes by γ and we have,

$$\begin{aligned} u_i\left(a, t_a^{s+1}\right) &= v_a^i + t_a^{s+1} = v_a^i + t_a^s + \gamma \\ &\geq v_b^i + t_b^s + \gamma \\ &= v_b^i + t_b^{s+1} = u_i\left(b, t_b^{s+1}\right). \end{aligned}$$

Since $D_i(t^{s+1}) \subseteq D_i\left(t^s\right)$, we have $D_i(t^{s+1}) \subseteq OD\left(t^s\right)$, in turn implying with the above finding that $a \in D_i\left(t^{s+1}\right)$ and $i \in S_a\left(t^{s+1}\right)$. We showed that for any link $\{i, a\} \in \mathcal{L}\left(t^s\right)$ such that $a \in OD(t^s)$ and $i \in US(t^s)$, we have $\{i, a\} \in \mathcal{L}\left(t^{s+1}\right)$.

Next, we finish the proof of the lemma.

First we prove that $\mathcal{M}\left(t^s\right) \subseteq \mathcal{M}\left(t^{s+1}\right)$. Let $Q \in \mathcal{M}\left(t^s\right)$. By the GED Lemma, under Q oversupplied agents are matched with underdemanded objects, overdemanded objects are matched with undersupplied agents, and perfectly demanded objects are perfectly matched with perfectly supplied agents. Therefore by the above three observations, we have $Q \in \mathcal{Q}\left(t^{s+1}\right)$. Since $D_i\left(t^{s+1}\right) \subseteq D_i\left(t^s\right)$ for any $i \in I$, $S_a\left(t^{s+1}\right) \subseteq S_a\left(t^s\right)$ for any $a \in A$, we have $\mathcal{L}\left(t^{s+1}\right) \subseteq \mathcal{L}\left(t^s\right)$. This together with the fact that Q is maximal at t^s imply that Q is maximal at t^{s+1}, that is $Q \in \mathcal{M}\left(t^{s+1}\right)$.

Next we prove that $\mathcal{M}\left(t^{s+1}\right) \subseteq \mathcal{M}\left(t^s\right)$. Let $Q^* \in \mathcal{M}\left(t^{s+1}\right)$. Since $\mathcal{L}\left(t^{s+1}\right) \subseteq \mathcal{L}\left(t^s\right)$, $Q^* \in \mathcal{Q}(t^s)$. We already proved above that for any $Q \in \mathcal{M}(t^s)$, we have $|Q| = |Q^*|$, therefore $Q^* \in \mathcal{M}\left(t^s\right)$.

We established $\mathcal{M}\left(t^{s+1}\right) = \mathcal{M}\left(t^s\right)$. This implies that $UD(t^{s+1}) = UD(t^s)$, $US(t^{s+1}) = US(t^s)$, $PD(t^{s+1}) = PD(t^s)$, $PS(t^{s+1}) = PS(t^s)$, $OD(t^{s+1}) = OD(t^s)$, and $OS(t^{s+1}) = OS(t^s)$. ♦

Proof of Proposition 1: Let $\{t^s\}$ be the money distribution sequence of the discrete algorithm of a market mechanism. Let s be a step with $UD(t^s) \neq \emptyset$. We determine indirect utility $\tilde{u}_i(t^{s+1})$ for each agent $i \in I$. We consider agents in $US(t)$, $OS(t)$, and $PS(t)$ separately.

1. Let $i \in US(t^s)$ and $a \in D_i(t^s)$. By the GED Lemma, we have $a \in OD(t^s)$. By the construction of t_a^{s+1} we have $t_a^{s+1} = t_a^s + \gamma(t^s)$. We obtain

$$u_i(a, t_a^{s+1}) = v_a^i + t_a^{s+1} = v_a^i + t_a^s + \gamma(t^s) = u_i(a, t_a^s) + \gamma(t^s). \tag{4}$$

We will show that $u_i(a, t_a^{s+1}) \geq u_i(b, t_b^{s+1})$ for all $b \in A$. Let $b \in A \backslash \{a\}$. Three cases are possible:

(a) $b \in UD(t^s)$. By the construction of $\alpha\left(t^s\right)$ and $\gamma(t^s)$, we have

$$\begin{aligned} \alpha\left(t^s\right) - \gamma(t^s) &\leq \min_{j \in US(t^s)} \left(\tilde{u}_j(t^s) - \max_{c \in UD(t^s)} u_j(c, t_c^s)\right) \leq \tilde{u}_i(t^s) - \max_{c \in UD(t^s)} u_i(c, t_c^s) \\ &\leq \tilde{u}_i(t^s) - u_i(b, t_b^s) = u_i(a, t_a^s) - u_i(b, t_b^s). \end{aligned}$$

This implies

$$u_i(b, t_b^s) + \alpha(t^s) \leq u_i(a, t_a^s) + \gamma(t^s). \tag{5}$$

By the construction of t_b^{s+1}, the money share of object b increases by $\alpha(t^s)$. We have $t_b^{s+1} = t_b^s + \alpha(t^s)$. This together with Equation 4 and Equation 5 implies

$$u_i(b, t_b^{s+1}) = v_b^i + t_b^{s+1} = v_b^i + t_b^s + \alpha(t^s) = u_i(b, t_b^s) + \alpha(t^s) \leq u_i(a, t_a^s) + \gamma(t^s) = u_i(a, t_a^{s+1}).$$

(b) $b \in PD(t^s)$. By the construction of $\gamma(t^s)$ and $\beta(t^s)$, we have

$$\begin{aligned}\beta(t^s) - \gamma(t^s) &\leq \min_{j \in US(t^s)} \left(\tilde{u}_j(t^s) - \max_{c \in PD(t^s)} u_j(c, t_c^s) \right) \leq \tilde{u}_i(t^s) - \max_{c \in PD(t^s)} u_i(c, t_c^s) \\ &\leq \tilde{u}_i(t^s) - u_i(b, t_b^s) = u_i(a, t_a^s) - u_i(b, t_b^s).\end{aligned}$$

This implies

$$u_i(b, t_b^s) + \beta(t^s) \leq u_i(a, t_a^s) + \gamma(t^s). \tag{6}$$

By the construction of t_b^{s+1}, the money share of object b increases by $\beta(t^s)$. We have $t_b^{s+1} = t_b^s + \beta(t^s)$. This together with Equation 4 and Equation 6 implies

$$u_i(b, t_b^{s+1}) = v_b^i + t_b^{s+1} = v_b^i + t_b^s + \beta(t^s) = u_i(b, t_b^s) + \beta(t^s) \leq u_i(a, t_a^s) + \gamma(t^s) = u_i(a, t_a^{s+1}).$$

(c) $b \in OD(t^s) \backslash \{a\}$. By the construction of t_b^{s+1}, the money share of object b increases by $\gamma(t^s)$. We have $t_b^{s+1} = t_b^s + \gamma(t^s)$. Since $a \in D_i(t^s)$, we have $u_i(b, t_b^s) \leq u_i(a, t_a^s)$. These together with Equation 4 imply

$$u_i(b, t_b^{s+1}) = v_b^i + t_b^{s+1} = v_b^i + t_b^s + \gamma(t^s) = u_i(b, t_b^s) + \gamma(t^s) \leq u_i(a, t_a^s) + \gamma(t^s) = u_i(a, t_a^{s+1}).$$

We showed that $a \in D_i(t^{s+1})$ and

$$\tilde{u}_i(t^{s+1}) = u_i(a, t_a^{s+1}) = u_i(a, t_a^s) + \gamma(t^s) = \tilde{u}_i(t^s) + \gamma(t^s). \tag{7}$$

2. Let $i \in OS(t^s)$ and $a \in D_i(t^s) \cap UD(t^s)$. By the construction of t_a^{s+1}, we have $t_a^{s+1} = t_a^s + \alpha(t^s)$. We have

$$u_i(a, t_a^{s+1}) = v_a^i + t_a^{s+1} = v_a^i + t_a^s + \alpha(t^s) = u_i(a, t_a^s) + \alpha(t^s). \tag{8}$$

We will show that $u_i(a, t_a^{s+1}) \geq u_i(b, t_b^{s+1})$ for all $b \in A$.

Let $b \in A \backslash \{a\}$. We have $u_i(a, t_a^s) \geq u_i(b, t_b^s)$, since $a \in D_i(t^s)$. By the construction of t_b^{s+1}, the money share of object b increases at most by $\alpha(t^s)$. We have $t_b^{s+1} \leq t_b^s + \alpha(t^s)$. These and Equation 8 imply

$$u_i(b, t_b^{s+1}) = v_b^i + t_b^{s+1} \leq v_b^i + t_b^s + \alpha(t^s) = u_i(b, t_b^s) + \alpha(t^s) \leq u_i(a, t_a^s) + \alpha(t^s) = u_i(a, t_a^{s+1}).$$

We showed that $a \in D_i(t^{s+1})$ and

$$\tilde{u}_i(t^{s+1}) = u_i(a, t_a^{s+1}) = u_i(a, t_a^s) + \alpha(t^s) = \tilde{u}_i(t^s) + \alpha(t^s). \tag{9}$$

3. Let $i \in PS(t^s)$ and $a \in D_i(t^s) \backslash OD(t^s)$. We have $a \in PD(t^s)$. By the construction of t_a^{s+1} we have $t_a^{s+1} = t_a^s + \beta(t^s)$. We obtain

$$u_i(a, t_a^{s+1}) = v_a^i + t_a^{s+1} = v_a^i + t_a^s + \beta(t^s) = u_i(a, t_a^s) + \beta(t^s). \tag{10}$$

We will show that $u_i(a, t_a^{s+1}) \geq u_i(b, t_b^{s+1})$ for all $b \in A$. Let $b \in A \backslash \{a\}$. Two cases are possible:

(a) $b \in UD(t^s)$. By the construction of $\alpha(t^s)$ and $\beta(t^s)$ we have

$$\begin{aligned}\alpha(t^s) - \beta(t^s) &\leq \min_{j \in PS(t^s)} \left(\tilde{u}_j(t^s) - \max_{c \in UD(t^s)} u_j(c, t^s_c) \right) \leq \tilde{u}_i(t^s) - \max_{c \in UD(t^s)} u_i(c, t^s_c) \\ &\leq \tilde{u}_i(t^s) - u_i(b, t^s_b) = u_i(a, t^s_a) - u_i(b, t^s_b).\end{aligned}$$

We have

$$u_i(b, t^s_b) + \alpha(t^s) \leq u_i(a, t^s_a) + \beta(t^s). \tag{11}$$

By the construction of t^{s+1}_b, the money share of object b increases by $\alpha(t^s)$. We have $t^{s+1}_b = t^s_b + \alpha(t^s)$. This together with Equation 10 and Equation 11 implies

$$u_i(b, t^{s+1}_b) = v^i_b + t^{s+1}_b = v^i_b + t^s_b + \alpha(t^s) = u_i(b, t^s_b) + \alpha(t^s) \leq u_i(a, t^s_a) + \beta(t^s) = u_i(a, t^{s+1}_a).$$

(b) $b \in OD(t^s) \cup PD(t^s) \backslash \{a\}$. We have $b \in OD(t^s)$ or $b \in PD(t^s)$. By the construction of t^{s+1}_b, the money share of object b increases at most by $\beta(t^s)$. Hence, $t^{s+1}_b \leq t^s_b + \beta(t^s)$. Since $a \in D_i(t^s)$, we have $u_i(b, t^s_b) \leq u_i(a, t^s_a)$. These together with Equation 10 imply

$$u_i(b, t^{s+1}_b) = v^i_b + t^{s+1}_b \leq v^i_b + t^s_b + \beta(t^s) = u_i(b, t^s_b) + \beta(t^s) \leq u_i(a, t^s_a) + \beta(t^s) = u_i(a, t^{s+1}_a).$$

We showed that $a \in D_i(t^{s+1})$ and

$$\tilde{u}_i(t^{s+1}) = u_i(a, t^{s+1}_a) = u_i(a, t^s_a) + \beta(t^s) = \tilde{u}_i(t^s) + \beta(t^s). \tag{12}$$

Next, we will inspect the sum of indirect utilities at money distribution t^{s+1}. We have

$$\begin{aligned}\sum_{i \in I} \tilde{u}_i(t^{s+1}) &= \sum_{i \in OS(t^s)} (\tilde{u}_i(t^s) + \alpha(t^s)) + \sum_{i \in PS(t^s)} (\tilde{u}_i(t^s) + \beta(t^s)) + \sum_{i \in US(t^s)} (\tilde{u}_i(t^s) + \gamma(t^s)) \\ &= \sum_{i \in I} \tilde{u}_i(t^s) + \underbrace{|OS(t^s)|}_{\leq |UD(t^s)|-1} \alpha(t^s) + \underbrace{|PS(t^s)|}_{=PD(t^s)} \beta(t^s) + \underbrace{|US(t^s)|}_{\geq |OD(t^s)|+1} \gamma(t^s) \\ &\leq \sum_{i \in I} \tilde{u}_i(t^s) + (|UD(t^s)| - 1)\, \alpha(t^s) + |PD(t^s)|\, \beta(t^s) + (|OD(t^s)| + 1)\, \gamma(t^s) \text{ since } \alpha > 0 \text{ and } \gamma < 0 \\ &= \sum_{i \in I} \tilde{u}_i(t^s) + \gamma(t^s) - \alpha(t^s) + \underbrace{|UD(t^s)|\, \alpha(t^s) + |PD(t^s)|\, \beta(t^s) + |OD(t^s)|\, \gamma(t^s)}_{=0 \text{ by Equation 3}}\end{aligned}$$

This completes the proof of Proposition 1. ♦

Proof of Theorem 1: Let $\{t^s\}$ be the money distribution sequence in the discrete algorithm of a market mechanism. Consider any step s. For any $i \in I$ we have $\tilde{u}_i(t^s) \geq u_i(a, t^s_a)$ for all $a \in A$. We have

$$n\tilde{u}_i(t^s) \geq \sum_{a \in A} u_i(a, t^s_a) = \sum_{a \in A} v^i_a + \sum_{a \in A} t^s_a = \sum_{a \in A} v^i_a + m$$

for all $i \in I$. Hence,

$$\sum_{i \in I} \tilde{u}_i(t^s) \geq \frac{1}{n} \left(\sum_{i \in I} \sum_{a \in A} v^i_a \right) + m.$$

Therefore the sum of the indirect utilities of agents is bounded below in the market mechanism. We claim that $D(t^s) \neq D(t^u)$ for any steps s, u with $UD(t^s) \neq \emptyset$ and $UD(t^u) \neq \emptyset$ by contradiction.

Suppose there exists two steps s and u with $u > s$ such that $D(t^s) = D(t^u)$, $UD(t^s) \neq \emptyset$, and $UD(t^u) \neq \emptyset$. We have $\sum_{i\in I} \tilde{u}_i(t^s) > \sum_{i\in I} \tilde{u}_i(t^u)$ by Proposition 1. Moreover $s+1 \neq u$, since at step $s+1$ a new object joins the demand of at least one agent. Hence $\sum_{i\in I} \tilde{u}_i(t^{s+1}) > \sum_{i\in I} \tilde{u}_i(t^u)$ by Proposition 1.

Claim 1: The set of money distributions that attain the same demand is convex.

Proof of Claim 1: To see this, let $t, t' \in \mathcal{T}$ be such that $D(t) = D(t')$. Let $\sigma \in [0,1]$ and $t'' = \sigma t' + (1-\sigma)t$. We have $\sum_{a\in A} t''_a = \sum_{a\in A} \sigma t'_a + (1-\sigma)t_a = m$ hence $t'' \in \mathcal{T}$. Moreover for any $i \in I$, $a \in D_i(t)$, and $b \in A \backslash D_i(t)$, we have

$$\begin{aligned} u_i\left(a, t''_a\right) &= v^i_a + \sigma t'_a + (1-\sigma)t_a = \sigma\left(v^i_a + t'_a\right) + (1-\sigma)\left(v^i_a + t_a\right) \\ &> \sigma\left(v^i_b + t'_b\right) + (1-\sigma)\left(v^i_b + t_b\right) = u_i\left(b, t''_b\right). \end{aligned}$$

Hence, $D(t'') = D(t) = D(t')$. QED.

Claim 2: Let $C(t)$ be the set of money distributions that achieve the same demand as t. Then $U(t) = \left\{\sum_{i\in I} \tilde{u}_i(t') : t' \in C(t)\right\}$ is convex.

Proof of Claim 2: let u^1 and $u^2 \in U(t)$. Let $u^* = \sigma u^1 + (1-\sigma)u^2$ for $\sigma \in (0,1)$. Let $t^h \in C(t)$ such that $\sum_{i\in I} \tilde{u}_i(t^h) = u^h$ for each $h = 1, 2$. Then for $a_i \in D(t)$ for all i,

$$\begin{aligned} u^* = \sigma u^1 + (1-\sigma)u^2 &= \sigma \sum_i u_i\left(a_i, t^1_{a_i}\right) + (1-\sigma)\sum_i u_i\left(a_i, t^2_{a_i}\right) \\ &= \sum_i \sigma v^i_{a_i} + \sigma t^1_{a_i} + (1-\sigma)v^i_{a_i} + (1-\sigma)t^2_{a_i} \\ &= \sum_i v^i_{a_i} + \sigma t^1_{a_i} + (1-\sigma)t^2_{a_i} = \sum_i \tilde{u}_i(\sigma t^1 + (1-\sigma)t^2) \end{aligned}$$

Since $\sigma t^1 + (1-\sigma)t^2 \in C(t)$, $u^* \in U(t)$. QED

Let $\underline{u}(t^s) = \inf_{t^{s'} \in U(t^s)} U\left(t^{s'}\right)$.

Claim 3: There exists some market mechanism such that some t^* with $\sum_{i\in I} \tilde{u}_i(t^*) = \underline{u}(t^s)$ can be reached from t^s in the same step.

Proof of Claim 4: Observe that by Claims 1 and 2, we can choose t^* by for any $\sigma \in (0,1)$, $t = \sigma t^s + (1-\sigma)t^*$ satisfies $D(t) = D(t^s)$. Fix the rate of change of transfers for overdemanded, perfectly demanded, and underdemanded objects as $\alpha^*, \beta^*, \gamma^*$ such that $\alpha^* = \max_{a\in UD(t^s)}(t^*_a - t^s_a)$, $\beta^* = \max_{a\in PD(t^s)}(t^*_a - t^s_a)$, and $\gamma^* = \max_{a\in OD(t^s)}(t^*_a - t^s_a)$. QED

Therefore, we can make sure that $D(t^s) \neq D(t^u)$ for any two different steps s, u with $UD(t^s) \neq \emptyset$ and $UD(t^u) \neq \emptyset$ under an equivalent discrete algorithm. Since in every step a new demand configuration is reached and there are finite number of such configurations, $\{t^s\}$ converges to t^S for some finite step S. ♦

Proof of Proposition 2: Identical to the proofs of ASÜ Proposition 1, Proposition 2 and Proposition 3. ♦

7 Appendix B: Construction of a Maximal Market Assignment, Sets of Underdemanded Objects and Undersupplied Agents

Fix $t \in \mathcal{T}$. Take any market assignment $Q \in \mathcal{Q}(t)$. Let $a \in A$. An **odd-length alternating path for** Q **from** a is a path $(a, i_1, a_1, ..., i_k, a_k, i)$ of distinct agents and objects such that $\{i_\ell, a_\ell\} \in Q$ for any $\ell \in \{1, 2, ..., k\}$; $\{a, i_1\}, \{a_k, i\} \in L(t) \backslash Q$, and $\{a_{\ell-1}, i_\ell\} \in L(t) \backslash Q$ for any $\ell \in \{2, 3, ..., k\}$.[8] That is, this path has $2k + 1$ links in it (odd-length) and the first link *is not* in Q, the second link *is* in Q, the third link *is not* in Q,..., the last link *is* in Q (alternating). An odd-length alternating path from an agent is symmetrically defined. If Q has an odd-length alternating path, then it cannot be maximal: since we can have the assignments $\{a, i_1\}$,$\{a_1, i_2\}$,...,$\{a_k, i\}$ instead of the assignments $\{i_1, a_1\}$,$\{i_2, a_2\}$,..., $\{i_k, a_k\}$ and obtain $k + 1$ agents matched instead of k without affecting the rest of the assignments under Q at t. Edmonds' algorithm starts from any market assignment, finds an odd-length alternating path in it, and then creates a new market assignment from the old one as explained above, and repeats the process until there are no odd-length alternating paths left. Berge (1957) showed that a market assignment is maximal if and only if it does not have any odd-length alternating path. Therefore, Edmonds' algorithm converges to a maximal market assignment. It has $O(n^3)$ time complexity, and faster algorithms than Edmonds' algorithm have been introduced.[9]

Once a maximal market assignment $Q \in \mathcal{M}(t)$ is determined, we can construct $US(t)$ and $UD(t)$ also in polynomial time complexity. Take an object a that is left unmatched under Q. Define an **even-length alternating path for** Q **from** a as a path $(a, i_1, a_1, ..., i_k, a_k)$ of distinct agents and objects such that $\{i_\ell, a_\ell\} \in Q$ for any $\ell \in \{1, 2, ..., k\}$; $\{a, i_1\} \in L(t) \backslash Q$ and $\{a_{\ell-1}, i_\ell\} \in L(t) \backslash Q$ for any $\ell \in \{2, 3, ..., k\}$. Note that (a) is a degenerate even-length alternating path with zero links. We define an even-length alternating path starting from an unmatched agent under Q in a symmetric manner. That is, this path has $2k$ links in it (even-length) and the first link *is not* in Q, the second link *is* in Q, the third link *is not* in Q..., the last link *is* in Q (alternating). Suppose that path $(a, i_1, a_1, ..., i_k, a_k)$ is an even-length alternating path from a. We can leave a_1 unmatched instead of a under a maximal market assignment: modify Q to include $\{a, i_1\}$ instead of $\{i_1, a_1\}$. Similarly, for any $\ell \in \{2, 3, ..., k\}$, we can leave a_ℓ unmatched instead of a under a maximal market assignment: modify Q to include links $\{a, i_1\}, \{a_1, i_2\}, ..., \{a_{\ell-1}, i_\ell\}$ instead of links $\{i_1, a_1\}, \{i_2, a_2\}, ..., \{i_\ell, a_\ell\}$. As explained in Goemans (2004), (i) the set of underdemanded objects is the union of all objects that can be reached by an even-length alternating path from at least one unmatched object under Q; and (ii) the set of undersupplied agents is the union of all agents that can be reached by an even-length alternating path from at least one unmatched agent under Q.

8 Appendix C: Example

Example 2: Consider the market mechanism in which the money shares of perfectly demanded objects are increased half as fast as the money shares of underdemanded objects, i.e. $\beta = \frac{1}{2}\alpha$.

[8] It is also known as an **augmenting path**.

[9] See Korte and Vygen (2000) for an excellent reference on combinatorial optimization theory.

We find functions α, β, and γ for the discrete algorithm as

$$\alpha(t) = \min\left\{\frac{|OD(t)|}{|OD(t)| + |UD(t)| + \frac{1}{2}|PD(t)|}x(t), \frac{|OD(t)|}{\frac{1}{2}|OD(t)| + |UD(t)| + \frac{1}{2}|PD(t)|}y(t), 2z(t)\right\},$$

$$\gamma(t) = -\frac{|UD(t)| + \frac{1}{2}|PD(t)|}{|OD(t)|}\alpha(t), \text{ and}$$

$$\beta(t) = \frac{1}{2}\alpha(t) \qquad \forall\, t \in \mathcal{T}.$$

Consider fair allocation problem $\langle I, A, V, m\rangle$ with agent set $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, object set $A = \{a_1, a_2, a_3, a_4, a_5\}$, the value matrix

$$V = \left[v^i_a\right]_{i \in I, a \in A} =$$

	a_1	a_2	a_3	a_4	a_5
i_1	37	62	13	14	12
i_2	-34	-47	1	-10	-24
i_3	58	-26	34	47	58
i_4	0	47	24	56	72
i_5	-36	47	-50	12	47
i_6	2	16	-81	-104	-69

,

and money endowment $m = 600$. Since $|A| < |I|$, we introduce a dummy object a_6 and set the value of each agent for this object to 0.

We will find the outcome of the market mechanism introduced above for this fair allocation problem using the discrete algorithm.

Step 0: We set the initial money distribution as

$$t^0 = (100, 100, 100, 100, 100, 100).$$

Below, we give the utility profile of agents at t^0. The indirect utilities of agents are highlighted in bold.

$$\left[u_i(a, t^0_a)\right]_{i \in I, a \in A} =$$

	a_1	a_2	a_3	a_4	a_5	a_6
i_1	137	**162**	113	114	112	100
i_2	66	53	**101**	90	76	100
i_3	**158**	74	134	147	**158**	100
i_4	100	147	124	156	**172**	100
i_5	64	**147**	50	112	**147**	100
i_6	102	**116**	19	-4	31	100

.

The set of links at t^0 is given as

$$L\left(t^0\right) = \{\{i_1, a_2\}, \{i_2, a_3\}, \{i_3, a_1\}, \{i_3, a_5\}, \{i_4, a_5\}, \{i_5, a_2\}, \{i_5, a_5\}, \{i_6, a_2\}\}.$$

We find

$$Q^0 = \{\{i_1, a_2\}, \{i_2, a_3\}, \{i_3, a_1\}, \{i_5, a_5\}\}$$

as a maximal market assignment.[10] Since $\left|Q^0\right| < 6 = n$, $US(t^0) \neq \emptyset$ and $UD\left(t^0\right) \neq \emptyset$ by Lemma 1 and hence, we proceed to the next step.

[10] It is easy to find a maximal market assignment for this example, for more complicated problems we can use Edmonds' (1965) algorithm.

Step 1: We first find the GED of the problem at t^0. We find $UD\left(t^0\right)$ as follows: Q^0 leaves a_4 and a_6 unmatched, implying that a_4 and a_6 are underdemanded. There are no non-degenerate even-length alternating paths starting from either a_4 or a_6, implying $UD\left(t^0\right) = \{a_4, a_6\}$. We find $US\left(t^0\right)$ as follows: Q^0 leaves i_4 and i_6 unmatched implying i_4 and i_6 are undersupplied. Path $(i_4, a_5, i_5, a_2, i_1)$ is an even-length alternating path starting from i_4. Therefore, i_5 and i_1 are undersupplied. No other even-length alternating paths starting from i_4 or i_6 contain other agents. Hence, $US\left(t^0\right) = \{i_1, i_4, i_5, i_6\}$. The agents who have links with objects in $UD\left(t^0\right)$ are oversupplied. There are no links including either object a_4 or a_6 in $L\left(t^0\right)$, therefore, $OS\left(t^0\right) = \emptyset$. The objects which have links with agents in $US\left(t^0\right)$ are overdemanded. We have $OD\left(t^0\right) = \{a_2, a_5\}$. The rest of the objects are perfectly demanded, implying $PD\left(t^0\right) = I\backslash\left(UD\left(t^0\right) \cup OD\left(t^0\right)\right) = \{a_1, a_3\}$. The rest of the agents are perfectly supplied, implying $PS\left(t^0\right) = I\backslash\left(US\left(t^0\right) \cup OS\left(t^0\right)\right) = \{i_2, i_3\}$. In summary, we have

$$\begin{aligned} UD\left(t^0\right) &= \{a_4, a_6\}, \ OD\left(t^0\right) = \{a_2, a_5\}, \text{ and } PD\left(t^0\right) = \{a_1, a_3\}; \\ US\left(t^0\right) &= \{i_1, i_4, i_5, i_6\}, \ OS\left(t^0\right) = \emptyset, \text{ and } PS\left(t^0\right) = \{i_2, i_3\}. \end{aligned}$$

We determine $x(t^0)$, $y(t^0)$, and $z(t^0)$ in order to calculate money distribution t^1.

$$x(t^0) = \min_{i \in US(t^0)} \left(\tilde{u}_i\left(t^0\right) - \max_{a \in UD(t^0)} u_i(a, t^0_a) \right) = \tilde{u}_{i_6}\left(t^0\right) - u_{i_6}\left(a_6, t^0_{a_6}\right) = 116 - 100 = 16.$$

$$y(t^0) = \min_{i \in US(t^0)} \left(\tilde{u}_i\left(t^0\right) - \max_{a \in PD(t^0)} u_i(a, t^0_a) \right) = \tilde{u}_{i_6}\left(t^0\right) - u_{i_6}\left(a_1, t^0_{a_1}\right) = 116 - 102 = 14.$$

$$z(t^0) = \min_{i \in PS(t^0)} \left(\tilde{u}_i\left(t^0\right) - \max_{a \in UD(t^0)} u_i(a, t^0_a) \right) = \tilde{u}_{i_2}\left(t^0\right) - u_{i_2}\left(a_6, t^0_{a_6}\right) = 101 - 100 = 1.$$

We determine $\alpha(t^0)$, $\gamma(t^0)$, and $\beta(t^0)$ as

$$\begin{aligned} \alpha(t^0) &= \min\{\frac{2}{5}x\left(t^0\right), \frac{1}{2}y\left(t^0\right), 2z\left(t^0\right)\} = \min\{\frac{32}{5}, 7, 2\} = 2, \\ \gamma(t^0) &= -\frac{3}{2}\alpha(t^0) = -3, \text{ and} \\ \beta(t^0) &= \frac{1}{2}\alpha(t^0) = 1. \end{aligned}$$

We determine the new money distribution as

$$t^1 = (101, 97, 101, 102, 97, 102).$$

The utility matrix at t^1 is given below:

$\left[u_i(a, t^1_a)\right]_{i \in I, a \in A} =$

	a_1	a_2	a_3	a_4	a_5	a_6
i_1	138	**159**	114	116	109	102
i_2	67	50	**102**	92	73	**102**
i_3	**159**	71	135	149	155	102
i_4	101	144	125	158	**169**	102
i_5	65	**144**	51	114	**144**	102
i_6	103	**113**	20	-2	28	102

The set of links at t^1 is given as

$$L\left(t^{1}\right)=\left\{\left\{i_{1},a_{2}\right\},\left\{i_{2},a_{3}\right\},\left\{i_{2},a_{6}\right\},\left\{i_{3},a_{1}\right\},\left\{i_{4},a_{5}\right\},\left\{i_{5},a_{2}\right\},\left\{i_{5},a_{5}\right\},\left\{i_{6},a_{2}\right\}\right\}.$$

We find

$$Q^{1}=\left\{\left\{i_{1},a_{2}\right\},\left\{i_{2},a_{3}\right\},\left\{i_{3},a_{1}\right\},\left\{i_{4},a_{5}\right\}\right\}$$

as a maximal market assignment at t^1. Since $\left|Q^1\right| < 6 = n$, $UD\left(t^1\right) \neq \emptyset$ and $US\left(t^1\right) \neq \emptyset$ by Lemma 1 and hence, we proceed to the next step.

Step 2: First, we find the GED of the problem at t^1 using Q^1. We determine $UD\left(t^1\right)$ as follows: Q^1 leaves a_4 and a_6 unmatched, implying a_4 and a_6 are underdemanded. Path (a_6, i_2, a_3) is an even-length alternating path starting from a_6. Therefore, a_3 is underdemanded, as well. There are no other non-degenerate even-length alternating paths starting from a_4 or a_6. Hence, $UD\left(t^1\right) = \{a_3, a_4, a_6\}$. We determine $US\left(t^1\right)$ as follows: Q^1 leaves i_5 and i_6 unmatched, implying i_5 and i_6 are undersupplied. Path (i_5, a_2, i_1) is an even-length alternating path starting from i_5, implying i_1 is undersupplied. Path (i_5, a_5, i_4) is an even-length alternating path starting from i_5, implying i_4 is undersupplied. Other even-length alternating paths starting from i_5 or i_6 do not include any other new agents. Therefore, $US\left(t^1\right) = \{i_1, i_4, i_5, i_6\}$. Set $OD\left(t^1\right)$ is the set of objects which have links to agents in $US\left(t^1\right)$. We have $OD\left(t^1\right) = \{a_2, a_5\}$. Set $OS(t^1)$ is the set of agents who have links to objects in $UD\left(t^1\right)$. We have $OS\left(t^1\right) = \{i_2\}$. The remaining objects are perfectly demanded: $PD\left(t^1\right) = A\backslash\left(UD\left(t^1\right) \cup OD\left(t^1\right)\right) = \{a_1\}$. The remaining agents are perfectly supplied: $PS\left(t^1\right) = I\backslash\left(US\left(t^1\right) \cup OS\left(t^1\right)\right) = \{i_3\}$. In summary, we have

$$\begin{gathered}UD\left(t^{1}\right)=\left\{a_{3},a_{4},a_{6}\right\},\ OD\left(t^{1}\right)=\left\{a_{2},a_{5}\right\},\text{ and }PD\left(t^{1}\right)=\left\{a_{1}\right\};\\ US\left(t^{1}\right)=\left\{i_{1},i_{4},i_{5},i_{6}\right\},\ OS\left(t^{1}\right)=\left\{i_{2}\right\},\text{ and }PS\left(t^{1}\right)=\left\{i_{3}\right\}.\end{gathered}$$

We find,

$$x(t^{1})=\min_{i\in US(t^{1})}\left(\tilde{u}_{i}\left(t^{1}\right)-\max_{a\in UD(t^{1})}u_{i}(a,t_{a}^{1})\right)=\tilde{u}_{i_{6}}\left(t^{1}\right)-u_{i_{6}}\left(a_{6},t_{a_{6}}^{1}\right)=113-102=11.$$

$$y(t^{1})=\min_{i\in US(t^{1})}\left(\tilde{u}_{i}\left(t^{1}\right)-\max_{a\in PD(t^{1})}u_{i}(a,t_{a}^{1})\right)=\tilde{u}_{i_{6}}\left(t^{1}\right)-u_{i_{6}}\left(a_{1},t_{a_{1}}^{1}\right)=113-103=10.$$

$$z(t^{1})=\min_{i\in PS(t^{1})}\left(\tilde{u}_{i}\left(t^{1}\right)-\max_{a\in UD(t^{1})}u_{i}(a,t_{a}^{1})\right)=\tilde{u}_{i_{3}}\left(t^{1}\right)-u_{i_{3}}\left(a_{4},t_{a_{4}}^{1}\right)=159-149=10.$$

We determine $\alpha(t^1)$, $\gamma(t^1)$, and $\beta(t^1)$ as

$$\begin{aligned}&\alpha(t^{1})=\min\{\frac{4}{11}x\left(t^{1}\right),\frac{4}{9}y\left(t^{1}\right),2z\left(t^{1}\right)\}=\min\{4,\frac{40}{9},20\}=4,\\ &\gamma(t^{1})=-\frac{7}{4}\alpha(t^{1})=-7,\text{ and}\\ &\beta(t^{1})=\frac{1}{2}\alpha(t^{1})=2.\end{aligned}$$

We determine the new money distribution as

$$t^{2}=(103,90,105,106,90,106).$$

The utility matrix at t^2 is given below:

$$\left[u_i(a, t^2_a)\right]_{i\in I, a\in A} = $$

	a_1	a_2	a_3	a_4	a_5	a_6
i_1	140	**152**	118	120	102	106
i_2	69	43	**106**	96	66	**106**
i_3	**161**	64	139	153	148	106
i_4	103	137	129	**162**	**162**	106
i_5	67	**137**	55	118	**137**	106
i_6	105	**106**	24	2	21	**106**

The set of links at t^2 is given by

$$L\left(t^2\right) = \{\{i_1, a_2\}, \{i_2, a_3\}, \{i_2, a_6\}, \{i_3, a_1\}, \{i_4, a_4\}, \{i_4, a_5\}, \{i_5, a_2\}, \{i_5, a_5\}, \{i_6, a_2\}, \{i_6, a_6\}\}.$$

A maximal market assignment is given by

$$Q^2 = \{\{i_1, a_2\}, \{i_2, a_3\}, \{i_3, a_1\}, \{i_4, a_4\}, \{i_5, a_5\}, \{i_6, a_6\}\}.$$

We have $\left|Q^2\right| = 6 = n$, implying Q^2 is a market matching by Observation 1. We terminate the procedure and Q^2 is a matching that clears the market and $\left(Q^2, t^2\right)$ is an outcome of the market mechanism with $\beta = \frac{\alpha}{2}$ for fair allocation problem $\langle I, A, V, m\rangle$. ♦

9 Appendix D: Equivalence between Market Mechanism with $\beta = \alpha$ and ASÜ Mechanism

Fix $t \in \mathcal{T}$. Let $OD^*(t)$ be the *full set of overdemanded objects* at t as defined by ASÜ. It is constructed as follows: $B \subset A$ is **overdemanded** if $|\{i \in I : D_i(t) \subseteq B\}| > |B|$. A set of objects is a **minimal overdemanded set** if it is overdemanded and none of its proper subsets is overdemanded. ASÜ iteratively define the **full set of overdemanded objects** at t as follows: Given t find all minimal overdemanded sets. Remove these objects from the demand of each agent and find the minimal overdemanded sets for the modified demand profiles. Proceed in a similar way until there is no minimal overdemanded set for the modified demand profiles. The full set of overdemanded objects, $OD^*(t)$, is the union of each of the sets encountered in the procedure.

We prove that $OD^*(t) = OD(t)$:

- $OD^*(t) \subseteq OD(t)$: Let $a \in OD^*(t)$.
  - Suppose a is removed in the above construction in round 1 in minimal overdemanded set B_1. Let $J_1 = \{i \in I : D_i(t) \subseteq B\}$. Let $i \in S_a(t)$. We have $|J_1| > |B_1|$, since B_1 is overdemanded. Let $J \subseteq J_1 \setminus \{i\}$ be such that $|J| = |B_1|$. We have for all $B \subseteq B_1$, $|\{j \in J : D_j(t) \subseteq B\}| \leq |B|$, since B_1 is a *minimal* overdemanded set and none of its proper subsets are overdemanded. This implies by Hall's (1935) Theorem that all objects in B_1 can be distributed to agents in J. Since i only demands objects in B_1, i can remain unmatched in a maximal market assignment, implying $i \in US(t)$. Since $i \in S_a(t)$, we have $a \in OD(t)$ by the GED Lemma.
  - Suppose a is removed in the above construction in round 2 in the remaining minimal overdemanded set B_2. We showed above that in a maximal market assignment all overdemanded objects removed in the first round can be matched to agents removed in the first round.

Let J_1 be the agents removed in first round, let B_1 be the objects removed in the first round. Let $J_2 = \{i \in I : D_i(t) \backslash B_1 \subseteq B_2\}$. Let $i \in S_a(t)$. Let $J \subseteq J_2 \backslash \{i\}$ be such that $|J| = |B_2|$. We have for any $B \subseteq B_2$, $|\{j \in J : D_j(t) \backslash B_1 \subseteq B\}| \leq |B|$, since B_2 is a *minimal* overdemanded set and none of its subsets are overdemanded after B_1 is removed, implying by Hall's Theorem that all objects in B_2 can be distributed to agents in J_2. Since all objects in B_1 and B_2 can be committed to agents in J_1 and J respectively, and since i does not demand any other objects (i.e. $D_i(t) \subseteq B_1 \cup B_2$), there is $Q \in \mathcal{M}(t)$ such that $Q_i = \emptyset$, implying $i \in US(t)$. Since $i \in S_a(t)$, we have $a \in OD(t)$ by the GED Lemma.

Continuing in a similar iterative manner, we obtain $OD^*(t) \subseteq OD(t)$.

- $OD(t) \subseteq OD^*(t)$: Let $J^* = \{i \in I : D_i(t) \subseteq OD^*(t)\}$. There are no overdemanded sets in $A \backslash OD^*(t)$: for any $B \subseteq A \backslash OD^*(t)$ we have $|\{i \in I : D_i(t) \subseteq I\}| \leq |B|$. Since $|J^*| > |OD^*(t)|$, we have $|I \backslash J^*| < |A \backslash OD^*(t)|$. Therefore, by Hall's Theorem all agents in $I \backslash J^*$ can be matched with objects in $A \backslash OD^*(t)$. Since all agents $I \backslash J^*$ are matched under all maximal market assignments, $I \backslash J^* \subseteq I \backslash US(t)$, this in turn implies that $US(t) \subseteq J^*$. $OD(t)$ is the set of objects that agents in $US(t)$ demand, and $OD^*(t)$ is the set of objects that agents in J^* demand, implying $OD(t) \subseteq OD^*(t)$.

10 Appendix E: Derivation of GED Lemma for Bipartite Graphs

A **connected component** of a graph is a maximal connected subgraph. An **odd component** is a connected component with odd number of nodes and an **even component** is a connected component with an even number of nodes. The GED Lemma for general graphs can be stated as follows (it is part of Theorem 10.32 on page 227 in Korte and Vygen (2000) due to Gallai (1964)) in the context of the demand-supply graph:

Gallai-Edmonds Decomposition Lemma for General Graphs: Let $t \in \mathcal{T}$.

1. Every object in $OD(t)$ is assigned to an agent in $US(t)$, every agent in $OS(t)$ is assigned an object in $UD(t)$, every object in $PD(t)$ is assigned to an agent in $PS(t)$ and every agent in $PS(t)$ is assigned an object in $PD(t)$ under any maximal market assignment.

2. Construct demand-supply graph $\mathcal{G}(t)$ at t. Consider the demand-supply subgraph obtained by deleting the objects in $OD(t)$ and their links with all agents, and also agents in $OS(t)$ and their links with all objects. Let $\mathcal{R}(t)$ be this remaining graph.

   - Every even component of $\mathcal{R}(t)$ only contains nodes in $PD(t) \cup PS(t)$ and it is possible to match all nodes in every even component in $\mathcal{R}(t)$ with each other under a market assignment.
   - Every odd component of $\mathcal{R}(t)$ only contains nodes in $UD(t) \cup US(t)$. Let U be the set of nodes in an odd component of $\mathcal{R}(t)$. Then it is always possible to match any $|U| - 1$ nodes of set U under a market assignment with each other. Moreover under any maximal market assignment, at most one node of U is matched with a node in $OS(t) \cup OD(t)$ and $|U| - 1$ nodes of U are matched with each other.

The following lemma is the main tool in deriving the GED Lemma for bipartite graphs like the demand-supply graph.

Lemma 3: For any $t \in \mathcal{T}$, every odd component of $\mathcal{R}(t)$ is a singleton, where $\mathcal{R}(t)$ is defined as in the statement of the GED Lemma for general graphs.

Proof of Lemma 3: Fix a money distribution $t \in \mathcal{T}$. Find demand-supply graph $\mathcal{G}(t)$. Find the remainder demand-supply graph $\mathcal{R}(t)$. Consider an odd component of graph $\mathcal{R}(t)$. Let U be the set of nodes in it. We have $U \subseteq UD(t) \cup US(t)$ by the GED Lemma for general graphs. Since $|U|$ is odd, and since within U it is possible to match $|U|-1$ nodes of U (by the GED Lemma for general graphs), it should be the case that either there are $\frac{|U|-1}{2}$ agents and $\frac{|U|+1}{2}$ objects, or $\frac{|U|-1}{2}$ objects and $\frac{|U|+1}{2}$ agents in U.

Suppose there are $\frac{|U|-1}{2}$ agents and $\frac{|U|+1}{2}$ objects. By the GED Lemma, under any maximal market assignment, all agents in $|U|$ are matched (to be able to match $|U|-1$ nodes within U), contradicting to the fact that these agents are members of $UD(t)$. Therefore, there cannot be any agents in U, implying that $\frac{|U|-1}{2} = 0 \Longrightarrow |U| = 1$, i.e. U consists of a single object.

The case with $\frac{|U|-1}{2}$ objects and $\frac{|U|+1}{2}$ agents is symmetric. ♦

In Lemma 3, we showed that objects in $UD(t)$ cannot supply any agent in $US(t)$ (and hence, agents in $US(t)$ cannot demand any object in $UD(t)$). The following corollary is straightforward to state given Lemma 3, GED Lemma for general graphs and the fact that objects in $PD(t)$ cannot supply any agent in $US(t)$ and agents in $PS(t)$ cannot demand any object in $UD(t)$ (by definition):

Corollary 2: For any money distribution $t \in \mathcal{T}$,

- For each $i \in US(t)$ we have $D_i(t) \subseteq OD(t)$.
- For each $a \in UD(t)$ we have $S_a(t) \subseteq OS(t)$.

Corollary 2 and the GED Lemma for general graphs imply the GED Lemma for bipartite graphs.

References

[1] A. Abdulkadiroğlu, T. Sönmez and M. U. Ünver (2004) "Room assignment-rent division: A market approach," Social Choice and Welfare, 22: 515-538.

[2] A. Alkan, G. Demange and D. Gale (1991) "Fair allocation of indivisible goods and criteria of justice," Econometrica, 59: 1023-1039.

[3] E. Aragones (1995) "A derivation of the money Rawlsian solution," Social Choice and Welfare, 12: 267-276.

[4] C. Berge (1957) "Two theorems in graph theory." Proceedings of the National Academy of Science of the U.S. 43: 842-844.

[5] A. Bogomolnaia and H. Moulin (2004) "Random matching under dichotomous preferences." Econometrica 72: 257-279.

[6] S. J. Brams and D. M. Kilgour (2001) "Competitive fair allocation," Journal of Political Economy, 109: 418-443.

[7] S. de Vries, J. Schummer and R. Vohra (2005) "On Ascending Vickrey Auctions for Heterogeneous Objects." Working paper.

[8] G. Demange, D. Gale and M. Sotomayor (1986) "Multi-item auctions." Journal of Political Economy 94: 863-872.

[9] J. Edmonds (1965) "Paths, trees, and flowers." Canadian Journal of Mathematics 17: 449-467.

[10] D. Foley (1967) "Resource allocation and the public sector," Yale Economics Essays, 7: 45-98.

[11] D. Gale and L. S. Shapley (1962): "College admissions and the stability of marriage," American Mathematical Monthly, 69: 9-15.

[12] T. Gallai (1963) "Kritische Graphen II." Magyar Tud. Akad. Mat. Kutató Int. Közl. 8: 373-395.

[13] T. Gallai (1964) "Maximale Systeme unabhängiger kanten." Magyar Tud. Akad. Mat. Kutató Int. Közl. 9: 401-413.

[14] M. X. Goemans (2004) Lecture notes on topics in combinatorial optimization, Massachusetts Institute of Technology Lecture Notes.

[15] C.-J. Haake, M. G. Raith and F. E. Su (2002) "Bidding for envy-freeness: A procedural approach to n-player fair-division problems," Social Choice and Welfare, 19: 723-749.

[16] P. Hall (1935) "On representatives of subsets." J Lond Math Soc, 10: 26-30.

[17] F. Klijn (2000) "An algorithm for envy-free allocations in an economy with indivisible objects and money." Social Choice and Welfare, 17: 201-216.

[18] B. Korte and J. Vygen (2000) Combinatorial Optimization: Theory and Algorithms. 2nd Edition. Springer: Berlin.

[19] E. Maskin (1987) "On the fair allocation of indivisible goods." G. Feiwel (ed.) Arrow and the foundations of the theory of economic policy. London: MacMillan, 341-349.

[20] R. F. Potthoff (2002) "Use of linear programming to find an envy-free solution closest to the Brams-Kilgour Gap solution for the housemates problem." Group Decision and Negotiation, 11: 405-414.

[21] M. Quinzii (1984) "Core and competitive equilibria with indivisibilities." International Journal of Game Theory, 13: 41-60.

[22] A.E. Roth, T. Sönmez and M.U. Ünver (2005) "Pairwise kidney exchange." Journal of Economic Theory, 125: 151-188.

[23] F. E. Su (1999): "Rental harmony: Sperner's lemma in fair allocation," American Mathematical Monthly, 106: 930-942.

[24] L.-G. Svensson (1983) "Large indivisibles: An analysis with respect to price equilibrium and fairness," Econometrica, 51: 939-954.

[25] L.-G. Svensson and B. Larsson (2002) "Strategy-proof and nonbossy allocation of indivisible goods and money," Economic Theory, 20: 483-502.

[26] K. Tadenuma and W. Thomson (1991) "No-envy and consistency in economies with indivisible goods," Econometrica, 59: 1755-1767.

[27] K. Tadenuma and W. Thomson (1995) "Refinements of the no-envy solution in economies with indivisible goods," Theory and Decision, 39: 189-206.