ASYMPTOTIC BEHAVIOR OF THE DIMENSION OF THE CHOW VARIETY

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Abstract. First, we calculate the dimension of the Chow variety of degree d cycles on projective space. We show that the component of maximal dimension usually parametrizes degenerate cycles, confirming a conjecture of Eisenbud and Harris. Second, for a numerical class α on an arbitrary variety, we study how the dimension of the components of the Chow variety parametrizing cycles of class $m\alpha$ grows as we increase m. We show that when the maximal growth rate is achieved, α is represented by cycles that are "degenerate" in a precise sense.

1. Introduction

Let X be an integral projective variety over an algebraically closed field. Fix a numerical cycle class α on X and let $\mathrm{Chow}(X, \alpha)$ denote the components of $\mathrm{Chow}(X)$ which parametrize cycles of class α. Our goal is to study the dimension of $\mathrm{Chow}(X, \alpha)$ and its relationship with the geometry and positivity of the cycles representing α. In general one expects that the maximal components of Chow should parametrize subvarieties that are "degenerate" in some sense, and our results verify this principle in a general setting.

We first consider the most important example: projective space. [EH92] analyzes the dimension of the Chow variety of curves on $\mathbb{P}^n$. Let ℓ denote the class of a line in $\mathbb{P}^n$. Then for $d > 1$,

$$\dim \mathrm{Chow}(\mathbb{P}^n, d\ell) = \max \left\{ 2d(n-1), \frac{d^2+3d}{2} + 3(n-2) \right\}.$$

The first number is the dimension of the space of unions of d lines on $\mathbb{P}^n$, and the second is the dimension of the space of degree d planar curves. Note the basic dichotomy: in low degrees the maximal dimension is achieved by unions of linear spaces, while for sufficiently high degrees the maximal dimension is achieved by "maximally degenerate" irreducible curves.

We prove an analogous statement in arbitrary dimension, establishing a conjecture of [EH92]:

The author is supported by NSF Award 1004363.

Theorem 1.1. *Let α denote the class of the k-plane on $\mathbb{P}^n$. Then for $d > 1$ the dimension of* $\mathrm{Chow}(\mathbb{P}^n, d\alpha)$ *is*

$$\max\left\{ d(k+1)(n-k), \binom{d+k+1}{k+1} - 1 + (k+2)(n-k-1) \right\}.$$

The first number is the dimension of the space of unions of d k-planes on $\mathbb{P}^n$ and the second is the dimension of the space of degree d hypersurfaces in $(k+1)$-planes. The same dichotomy seen for curves also holds in higher dimensions. The approach is to reduce to the result of [EH92] by cutting down by hyperplane sections. It would be interesting to develop analogous results for other varieties with simple structure, e.g. quadrics or rational normal scrolls.

For arbitrary varieties X, it is too much to hope for a precise relationship between $\dim \mathrm{Chow}(X, \alpha)$ and the "degeneracy" of cycles. Instead, by analogy with the divisor case, we obtain a cleaner picture by studying the asymptotics of $\dim \mathrm{Chow}(X, m\alpha)$ as m increases. For a Cartier divisor L on a smooth variety X of dimension n, the asymptotic behavior of sections is controlled by an important invariant known as the volume:

$$\mathrm{vol}(L) := \limsup_{m\to\infty} \frac{\dim H^0(X, \mathcal{O}_X(mL))}{m^n/n!}.$$

We formulate and study an analogous construction for arbitrary cycles.

The expected growth rate of $\dim \mathrm{Chow}(X, m\alpha)$ as m increases can be predicted by supposing that $m\alpha$ is a the pushforward of a divisor class on a fixed $(k+1)$-dimensional subvariety of X: Theorem 5.1 shows that $\dim \mathrm{Chow}(X, m\alpha) < Cm^{k+1}$ for some constant C. The variation function identifies the best possible constant C.

Definition 1.2. Let X be a projective variety and suppose $\alpha \in N_k(X)_{\mathbb{Z}}$ for $0 \leq k < \dim X$. The variation of α is

$$\mathrm{var}(\alpha) = \limsup_{m\to\infty} \frac{\dim \mathrm{Chow}(X, m\alpha)}{m^{k+1}/(k+1)!}.$$

Remark 1.3. It is important to formulate this function using the Chow variety and not the Hilbert scheme. For example, [Nol97, Corollary 1.6] shows that there are infinitely many components of $\mathrm{Hilb}(\mathbb{P}^3)$ parametrizing subschemes whose underlying cycle is a double line. Furthermore the dimensions of these components is unbounded. Thus it seems difficult to understand anything about the underlying cycle from the perspective of the Hilbert scheme.

It turns out that var is homogeneous and so extends naturally to $\mathbb{Q}$-numerical classes. In fact, Theorem 5.12 shows that var extends to a continuous function on the interior of $\overline{\mathrm{Eff}}_k(X)$.

Example 1.4. Using Theorem 1.1, we see that for the hyperplane class $\alpha \in N_k(\mathbb{P}^n)$ we have $\mathrm{var}(\alpha) = 1$.

Example 1.5. Let X be the blow-up of $\mathbb{P}^3$ at a closed point. Let E denote the exceptional divisor and let α be the class of a line in E. For any positive integer m, every effective cycle of class $m\alpha$ is contained in E. Thus, the variation coincides with the variation of the line class ℓ on $\mathbb{P}^2$, showing that that $\mathrm{var}(\alpha) = 1$.

The previous examples are typical: components of $\mathrm{Chow}(X)$ with large dimension tend to parametrize cycles that are as "degenerate" as possible. Note that, in contrast to the divisor case, a class may have positive variation but fail to be a big class as in Example 1.5.

Remark 1.6. [Leh16] defines a related function known as the mobility. The mobility measures how much a cycle moves using asymptotic point incidences. The variation and the mobility are complementary in the sense that they capture the behavior of Chow in different ranges (see Remark 5.5). The mobility function seems to be a closer analogue of the volume function for divisors, while the variation is more suitable for working with Chow.

To state our main theorem, we will need to recall some notions concerning the positivity of numerical cycle classes. Let $N_k(X)$ denote the vector space of numerical classes of k-cycles on X with $\mathbb{R}$-coefficients. The pseudo-effective cone $\overline{\mathrm{Eff}}_k(X) \subset N_k(X)$ is defined to be the closure of the cone generated by all effective k-cycles. Classes that lie in the interior of the cone are known as big classes.

The pushforward of a big class from a subvariety V of X will always have positive variation. Our main theorem shows that this is essentially the only way to construct classes of positive variation. A class can only have positive variation if it is represented by cycles which are "maximally degenerate" in the sense that they deform maximally in a fixed subvariety of dimension one higher.

Theorem 1.7. *Let X be a projective variety and suppose $\alpha \in N_k(X)_{\mathbb{Q}}$ for $0 \leq k < \dim X$. Then $\mathrm{var}(\alpha) > 0$ if and only if there is a $k+1$-dimensional integral subvariety $Y \subset X$ and a big class $\beta \in N_k(Y)_{\mathbb{Q}}$ such that some multiple of $\alpha - f_*\beta$ is represented by an effective cycle.*

The main conceptual advance used in the proof of Theorem 1.7 is an analysis of restriction maps. A key step in the proof of Theorem 1.7 is using an a priori bound on asymptotic growth to find a family of cycles whose restriction to a hyperplane H coincides. This replaces the use of exact sequences of sheaf cohomology for divisors.

1.1. **Organization.** Section 2 reviews background material on cycles. Section 3 describes several geometric constructions for families of cycles. In Section 4 we bound the dimension of components of $\mathrm{Chow}(\mathbb{P}^n)$. Finally, section 5 introduces the variation function and proves its basic geometric properties.

1.2. **Acknowledgements.** My sincere thanks to A.M. Fulger for numerous discussions and for his suggestions. Thanks to C. Voisin for recommending a number of improvements on an earlier draft. Thanks also to B. Hassett and R. Lazarsfeld for helpful conversations and to Z. Zhu and X. Zhao for reading a draft of the paper.

2. Preliminaries

Throughout we work over a fixed algebraically closed field K. A variety will mean a reduced irreducible quasiprojective scheme of finite type over K.

Lemma 2.1. *Let X be a variety. Suppose that $f : X \dashrightarrow Y$ is a rational map to a variety Y and $g : X \dashrightarrow Z$ is a rational map to a variety Z. Let F be a general fiber of f over a closed point of Y (so in particular F is not contained in the locus where g is not defined). Then* $\dim(\overline{g(X)}) \leq \dim(\overline{g(F)}) + \dim Y$.

Proof. Let U be an open subset of X where both f and g are defined and let $h : U \to Y \times Z$ be the induced map. Since F is a general fiber, $h(F \cap U)$ is dense in its closure in $Y \times Z$. By considering the first projection, we see that $\dim(\overline{h(U)}) \leq \dim(\overline{g(F \cap U)}) + \dim Y$. $\square$

We will often use the following special case of [RG71, Théorème 5.2.2].

Theorem 2.2 ([RG71], Théorème 5.2.2)**.** *Let $f : X \to S$ be a projective morphism of varieties such that some component of X dominates S. There is a birational morphism $\pi : S' \to S$ such that the morphism $f' : X' \to S'$ is flat, where $X' \subset X \times_S S'$ is the closed subscheme defined by the ideal of sections whose support does not dominate S'.*

2.1. **Cycles.** Suppose that X is a projective scheme. A k-cycle on X is a finite formal sum $\sum a_i V_i$ where the a_i are integers and each V_i is an integral closed subvariety of X of dimension k. The cycle is said to be effective if each $a_i \geq 0$. The group of k-cycles is denoted $Z_k(X)$ and the group of k-cycles up to rational equivalence is denoted $A_k(X)$. We will follow the conventions of [Ful84] in the use of various intersection products on $A_k(X)$.

[Ful84, Chapter 19] defines a k-cycle on X to be numerically trivial if its rational equivalence class has vanishing intersection with every weighted homogeneous degree-k polynomial in Chern classes of vector bundles on X. By quotienting $Z_k(X)$ by numerically trivial cycles, we obtain an abelian group $N_k(X)_{\mathbb{Z}}$ which is finitely generated by [Ful84, Example 19.1.4].

We also define

$$N_k(X)_{\mathbb{Q}} := N_k(X)_{\mathbb{Z}} \otimes \mathbb{Q}$$
$$N_k(X) := N_k(X)_{\mathbb{Z}} \otimes \mathbb{R}$$

Suppose that $f : Z \to X$ is an l.c.i. morphism of codimension d. Then [Ful84, Example 19.2.3] shows that the Gysin homomorphism $f^* : A_k(X) \to A_{k-d}(Z)$ descends to numerical equivalence classes. We will often use this fact when Z is a Cartier divisor on X to obtain maps $f^* : N_k(X) \to N_{k-1}(Z)$.

Convention 2.3. When we discuss k-cycles on a projective scheme X, we will always implicitly assume that $0 < k < \dim X$. This allows us to focus on the interesting range of behaviors without repeating hypotheses.

For a cycle Z on X, we let $[Z]$ denote the numerical class of Z, which can be naturally thought of as an element in $N_k(X)_{\mathbb{Z}}$, $N_k(X)_{\mathbb{Q}}$, or $N_k(X)$. If α is the class of an effective cycle Z, we say that α is an effective class. We write $\alpha \preceq \beta$ if the difference $\beta - \alpha$ is an effective class.

Definition 2.4. Let X be a projective scheme. The pseudo-effective cone $\overline{\mathrm{Eff}}_k(X) \subset N_k(X)$ is the closure of the cone generated by all classes of effective k-cycles. The big cone is the interior of the pseudo-effective cone.

[FL16, Theorem 1.4] shows that $\overline{\mathrm{Eff}}_k(X)$ is a full-dimensional cone which contains no lines. For any morphism of projective varieties $f : X \to Y$, there is a pushforward map $f_* : N_k(X) \to N_k(Y)$. [FL16, Theorem 1.4] shows that when f is surjective there is an induced equality $f_*(\overline{\mathrm{Eff}}_k(X)) = \overline{\mathrm{Eff}}_k(Y)$.

3. Families of cycles

In this section we set up a framework for discussing families of cycles (used also in [Leh16, Section 3]). Although there are several different notions of a family of cycles in the literature, the theory we will develop is somewhat insensitive to the precise choices. It will be most convenient to use a simple geometric definition.

Definition 3.1. Let X be a projective variety. A family of k-cycles on X consists of a variety W, a reduced closed subscheme $U \subset W \times X$, and an integer a_i for each component U_i of U, such that for each component U_i of U the first projection map $p : U_i \to W$ is flat dominant of relative dimension k. If each $a_i \geq 0$ we say that we have a family of effective cycles.

In this situation $p : U \to W$ will denote the first projection map and $s : U \to X$ will denote the second projection map unless otherwise specified. We will usually denote a family of k-cycles using the notation $p : U \to W$, with the rest of the data implicit.

For a closed point $w \in W$, the base change $w \times_W U_i$ is a subscheme of X of pure dimension k and thus defines a fundamental k-cycle Z_i on X. The cycle-theoretic fiber of $p : U \to W$ over w is defined to be the cycle $\sum a_i Z_i$ on X. We will also call these cycles the members of the family p.

Note that a family of k-cycles naturally determines a $(k + \dim W)$-cycle on $W \times X$. Conversely, the following construction shows how to construct a family of cycles from a cycle on $W \times X$.

Construction 3.2. Let X be a projective variety and let W be a variety. Suppose that $Z = \sum a_i V_i$ is a $(k + \dim W)$-cycle on $W \times X$ such that the first projection maps each V_i dominantly onto W. Let $W^0 \subset W$ be the (non-empty) open locus over which every projection $p : V_i \to W$ is flat and let $U \subset \mathrm{Supp}(Z)$ denote the preimage of W^0. Then the map $p : U \to W^0$ defines a family of cycles where we assign the coefficient a_i to the component $V_i \cap U$ of U.

Using Construction 3.2, we can translate cycle-theoretic operations into operations on families of cycles. (This is essentially the same as doing a cycle-theoretic operation to a general member of a family of cycles.) Note that the resulting family will usually only be defined over an open subset of the base W. In this way we can define:

- Proper pushforward families.
- Flat pullback families (which increase the dimension of the members of the family by the relative dimension of the map).
- Restrictions of families to subvarieties of W via base change of the flat map p.
- Family sums: given two families $p : U \to W$ and $q : S \to T$, we construct a family $p + q$ over an open subset of $W \times T$ whose cycle-theoretic fibers are sums of the fibers of p and q.
- Strict transform families: given a birational map $\phi : X \dashrightarrow Y$ of projective varieties, we first remove any components of U whose image is contained in the locus where ϕ is not an isomorphism, and then take the strict transform of the rest.
- Intersections against the members of a linear series $|L|$: this defines a family of $k-1$ cycles over an open subset of $W \times |L|$.

Since these constructions are usually only defined over an open subset of the base, it is useful to be able to pass to a projective completion using a flattening argument.

Lemma 3.3. *Let X be a projective variety and let $p : U \to W$ be a family of effective cycles on X. Then there is a normal projective integral variety W' that is birational to W and a family of cycles $p' : U' \to W'$ such that* $\mathrm{ch}(W') = \overline{\mathrm{ch}(W)}$.

3.1. **Chow varieties and the Chow map.** In this section we verify that the differences between Definition 3.1 and the construction of the Chow variety in [Kol96] can safely be neglected.

Fix a projective variety X and an ample divisor H on X. For any reduced scheme Z over the ground field, [Kol96, Chapter I.3] introduces a more refined definition of a family of k-cycles of X of H-degree d over Z. Kollár then constructs a semi-normal projective variety $\mathrm{Chow}_{k,d,H}(X)$ that parametrizes families of effective k-cycles of H-degree d. $\mathrm{Chow}(X)$ denotes the disjoint union over all k and d of $\mathrm{Chow}_{k,d,H}(X)$ for some fixed ample divisor H; it does not depend on the choice of H.

The precise way in which $\mathrm{Chow}(X)$ parametrizes cycles is somewhat subtle in characteristic p. For a discussion of the Chow functor and universal families, see [Kol96]. We will need the following properties of $\mathrm{Chow}(X)$:

- Any family of cycles in the sense of Definition 3.1 naturally yields a family of cycles in the refined sense of [Kol96, I.3.11 Definition] by applying [Kol96, I.3.14 Lemma] with the identity map (see also [Kol96, I.3.15 Corollary]).
- For any weakly normal integral variety W and any (refined) family of effective cycles $p : U \to W$, there is an induced morphism $\mathrm{ch}_p : W \to \mathrm{Chow}(X)$ by [Kol96, I.4.8-I.4.10]. (We will denote this map simply by ch when the family p is clear from the context.)

For any family of effective cycles $p : U \to W$ the base change to the normal locus $W^0 \subset W$ is still a family of cycles (where we assign the same coefficients). Thus there is an induced rational map $\mathrm{ch}_p : W \dashrightarrow \mathrm{Chow}(X)$ that is a morphism on the normal locus of W.

The following crucial lemma encapsulates the set-theoretical nature of the Chow functors constructed in [Kol96, Chapter I.3]. The point is simply that a non-trivial family of effective cycles, in the Chow sense, can not all have the same support (as opposed to the Hilbert scheme, where one allows variation of non-reduced structure).

Lemma 3.4. *Let X be a projective variety and let $p : U \to W$ be a family of effective k-cycles on X over a weakly normal W. A curve $C \subset W$ is contracted by* $\mathrm{ch} : W \to \mathrm{Chow}(X)$ *if and only if every cycle-theoretic fiber over C has the same support.*

We will freely use the notation of [Kol96] in the verification.

Proof. First suppose that every cycle-theoretic fiber of p over C has the same support. Since $\mathrm{Chow}_{k,d,H}(X)$ is constructed by taking a semi-normalization (which is set-theoretically bijective), we may instead consider the induced map to $\mathrm{Chow}'_{k,d,H}(X)$. This map factors through the map ch for a projective space containing an embedding of X; therefore it suffices to consider the case when $X = \mathbb{P}$. Then the construction following [Kol96, Ch. I Eq. (3.23.1.5)] shows that the Cartier divisors on $(\mathbb{P}^\vee)^{k+1}$ parametrized by the image of C in $\mathbb{H}$ must all have the same support. But this implies they are equal.

Conversely, suppose that C is contracted by ch. As discussed in [Kol96, I.3.27.3], C is also contracted by the morphism to $\mathrm{Hilb}((\mathbb{P}^\vee)^{k+1})$. Again comparing with [Kol96, Ch. I Eq. (3.23.1.5)], we see that the support of each of the cycles parametrized by C is the same. $\square$

3.2. **Chow dimension of families.** Let $p : U \to W$ be a family of effective k-cycles on a projective variety X. Then all the cycle-theoretic fibers of p are algebraically equivalent. Indeed, for any two closed points of W, let C be the normalization of a curve through those two points; since the base change of U to C is a union of flat families of effective cycles, we see that the corresponding cycle-theoretic fibers are algebraically equivalent.

Definition 3.5. Let $p : U \to W$ be a family of effective k-cycles on a projective variety X. We say that p represents $\alpha \in N_k(X)_{\mathbb{Z}}$ if the cycle-theoretic fibers of our family have class α.

Definition 3.6. Let X be a projective variety and let $p : U \to W$ be an effective family of k-cycles. We define the Chow dimension of p to be

$$\mathrm{chdim}_X(p) := \dim(\overline{\mathrm{Im}\ \mathrm{ch} \colon W \dashrightarrow \mathrm{Chow}(X)})$$

Note that $\mathrm{chdim}_X(p)$ is finite, since there are only finitely many components of $\mathrm{Chow}(X)$ parametrizing cycles of bounded degree. If $\alpha \in N_k(X)_{\mathbb{Z}}$, we define

$$\mathrm{chdim}_X(\alpha) = \max\{\mathrm{chdim}(p) | p : U \to W \text{ represents } \alpha\}.$$

We will usually omit the subscript when it is clear from the context.

Remark 3.7. When our ground field K has characteristic 0, [Kol96] constructs a universal family over any component of $\mathrm{Chow}(X)$. Using Construction 3.2 this can be turned into a family of effective cycles in the sense of Definition 3.1. Thus

$$\mathrm{chdim}(\alpha) = \max\{\dim(Y) \,|\, Y \text{ is a component of } \mathrm{Chow}(X) \text{ representing } \alpha\}.$$

Even when K has characteristic p, for any component of $\mathrm{Chow}(X)$ [Kol96, I.4.14 Theorem] constructs a family of cycles whose chow map ch is dominant, so that we still have the same interpretation.

Using Lemma 3.4, it is easy to understand how modifications of families of cycles change the Chow dimension. The most important examples are:

Lemma 3.8. *Let $f : X \to Y$ be a morphism of projective varieties. Let $p : U \to W$ be a family of effective k-cycles on X such that for every component U_i of U the image $\overline{s(U_i)}$ is not contracted to a variety of smaller dimension by f. Then* $\mathrm{chdim}(p) = \mathrm{chdim}(f_*p)$*.*

Proof. Let T be an integral curve through a general point of W that is not contracted by ch_p and set $S = p^{-1}(T)$. Then $\dim(\overline{s(S)}) = \dim(f(\overline{s(S)}))$. Thus the cycle-theoretic fibers parametrized by T do not pushforward to the same cycle on Y. We conclude by Lemma 3.4 that T is not contracted by ch_{f_*p}. □

Lemma 3.9. *Let X be a projective variety and let $p : U \to W$ and $q : S \to T$ be two families of effective k-cycles on X. Then* $\mathrm{chdim}(p+q) = \mathrm{chdim}(p) + \mathrm{chdim}(q)$*.*

Proof. A curve through a general point of $(W \times T)^0$ is contracted by ch_{p+q} if and only if its projection to W and to T are contracted by ch_p and ch_q respectively. We conclude by Lemma 3.4. □

Remark 3.10. When computing the Chow dimension of a class α, it suffices to consider families of cycles $p : U \to W$ such that the generic fiber of each component of U is integral. Indeed, since geometric integrality of fibers is a constructible property, we can modify each component of U by a finite base change so that the result has generically irreducible fibers. By taking a family sum of these various components, we then obtain a family of cycles representing α, whose components have integral generic fibers, and whose Chow dimension is at least as large as for our original family p.

4. Chow dimension of projective space

In this section we compute the dimension of the Chow variety of $\mathbb{P}^n$. [EH92] computes the dimension of the Chow variety of curves. The precise statement is as follows: let ℓ denote the class of a line on $\mathbb{P}^n$. Then for $d > 1$,

$$\dim \mathrm{Chow}(\mathbb{P}^n, d\ell) = \max \left\{ 2d(n-1), \frac{d^2 + 3d}{2} + 3(n-2) \right\}.$$

Note that the first number is the dimension of the space of unions of d lines on $\mathbb{P}^n$, and the second is the dimension of the space of degree d plane curves.

Remark 4.1. Although [EH92] does not specify the ground field, some of the references explicitly work only over $\mathbb{C}$. However, the results of [EH92] holds equally well over any algebraically closed field. The argument of [EH92] involves mainly estimates on the dimension of the space of sections of a normal sheaf. Since we are only working with embedded curves, we do not need to worry about pathologies of tangent spaces in characteristic p. The only additional verifications one needs to make are:

- The Halphen bounds on genus, and their extensions in [Har82], hold in arbitrary characteristic using the same arguments.
- The deformation theoretic results of [AC81] used in the paper are also true in arbitrary characteristic. Indeed, suppose that $f : C \to \mathbb{P}^n$ is a morphism from a smooth curve such that f restricts to an isomorphism from an open set of C onto its image. Using the deformation theory for maps as explained in [Ser06], one sees that deformations of f that change the image must be associated with first-order deformations which are not torsion sections of the normal sheaf (since they can not fix an open subset).

In this section we prove the analogue of [EH92, Theorem 3] in arbitrary dimension.

Theorem 4.2. *Let α denote the class of the k-plane on $\mathbb{P}^n$. Then for $d > 1$ the dimension of* $\mathrm{Chow}(\mathbb{P}^n, d\alpha)$ *is*

$$\max \left\{ d(k+1)(n-k), \binom{d+k+1}{k+1} - 1 + (k+2)(n-k-1) \right\}.$$

Again, the first number is the dimension of the space of a union of d k-planes on $\mathbb{P}^n$ and the second is the dimension of the space of degree d hypersurfaces in $(k+1)$-planes.

The basic tool in [EH92] is Castelnuovo theory for curves. It is unclear how best to formulate an analogue in higher dimension. Thus we take an alternative approach: the strategy is to reduce the general case to the calculation for curves using hyperplane sections. Our first lemma estimates the dimension of the "kernel" of restriction to a hyperplane.

Lemma 4.3. *Let $p : U \to W$ be a family of irreducible degree d effective k-cycles on $\mathbb{P}^n$. Suppose there is a fixed reduced subvariety Z of degree d and dimension $k-1$ such that every member of the family contains Z. Then*

$$\mathrm{chdim}(p) \leq (n-k-1) + \left(\begin{array}{c} d+k \\ k+1 \end{array} \right).$$

The strategy of the proof is to take a generic projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$. The pushforward family will still satisfy the hypotheses of the theorem, so by induction it is enough to estimate the dimension of the fibers of the induced map $\pi_* : \mathrm{Chow}(\mathbb{P}^n) \dashrightarrow \mathrm{Chow}(\mathbb{P}^{n-1})$. Since the projection is general, two irreducible cycles are identified by π_* only if they map to the same subvariety V; thus, we reduce to understanding the dimension of spaces of divisors on the cone $\pi^{-1}(V)$. This estimate is provided by Lemma 4.4 below.

Proof. The proof is by decreasing induction on the codimension $n-k$. For the base case, suppose first that p consists of a family of divisors. Then the family has dimension at most $h^0(\mathbb{P}^n, \mathcal{I}_Z(d)) - 1$. We can find an upper bound on this dimension by considering a deformation of Z to a degenerate subscheme Z' contained in a hyperplane H. More precisely, we can degenerate by rescaling a coordinate in $\mathbb{P}^n$, so that the general member of the family will be isomorphic to Z. Then by upper semicontinuity

$$h^0(\mathbb{P}^n, \mathcal{I}_Z(d)) \leq h^0(\mathbb{P}^n, \mathcal{I}_{Z'}(d)).$$

Note that the cycle $\tilde{Z}$ underlying Z' is a degree d divisor on the hyperplane $H \cong \mathbb{P}^{n-1}$. Thus, an element of $\mathcal{O}_{\mathbb{P}^n}(d)$ can contain Z' only if it contains H or its restriction to H agrees with $\tilde{Z}$. Thus

$$h^0(\mathbb{P}^n, \mathcal{I}_{Z'}(d)) \leq \dim \ker(H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(H, \mathcal{O}_H(d))) + 1$$

and we obtain the desired inequality for $n-k=1$.

Now suppose $n-k>1$. Consider the projection away from a general point in $\mathbb{P}^n$ and let $\phi : P \to \mathbb{P}^n$ and $\pi : P \to \mathbb{P}^{n-1}$ be the resolution given by blowing up the point. We may assume a general element of p does not contain the center of the projection, so that the pushforward family of p again satisfies the hypotheses of the theorem as a family on $\mathbb{P}^{n-1}$. We seek an upper bound on $\dim(F)$ where F is a component of a general fiber of the rational map $\mathrm{ch}_{\pi_* p} : W \dashrightarrow \mathrm{Chow}(\mathbb{P}^{n-1})$.

Let T be a general cycle parametrized by the component F. Note that T is a divisor in the projective bundle $\pi : \pi^{-1}\pi(T) \to \pi(T)$; we will estimate $\dim(F)$ using the geometry of divisors in this vector bundle. First we must normalize: let $\nu : R \to \pi(T)$ be the normalization of $\pi(T)$, and consider the pullback bundle

$$S := \mathbb{P}(\mathcal{O}_R \oplus \nu^* \mathcal{O}(1)).$$

Let $\psi : S \to \mathbb{P}^n$ denote the natural map taking S to a cone. We let $L = \psi^* H|_S$ for a general hyperplane H and let E denote the pull-back to S of the exceptional divisor for ϕ.

Let p' denote the family of divisors on S defined by F via strict transform. (Since the normalization ν happens over the base, we can take the strict transform of any divisor which maps surjectively onto $\pi(T)$, and furthermore the normalization does not affect the Chow dimension.) After replacing F by an open subset, by construction no element of p' intersects E. In other words, the divisors D parametrized by p' satisfy $L^k \cdot D = d$ and $E \cdot D = 0$. Thus any element of p' must actually lie in $|L|$ by Lemma 4.4.

Now fix a general element D of p'. Note that the intersection of any member of $|L|$ with D has degree d against L; by assumption the image of any such intersection in $\mathbb{P}^n$ must contain the degree d subset Z. It is easy to deduce that p' lies in a fiber of the restriction map $|L| \to |L|_D|$. Taking sections of the short exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(L) \to \mathcal{O}_D(L) \to 0$$

we see that $\mathrm{chdim}(p') \leq 1$. Then, arguing by induction we conclude

$$\mathrm{chdim}(p) \leq \mathrm{chdim}(\pi_* p) + 1 \leq (n-k-2) + \binom{d+k}{k+1} + 1.$$

$\square$

Lemma 4.4. *Let R be a normal projective variety of dimension $n \geq 1$. Suppose that the Cartier divisor A on R is the pullback of a very ample divisor under a birational map and set $d = A^n$. Define*

$$S = \mathbb{P}_R(\mathcal{O}_R \oplus \mathcal{O}_R(A))$$

and $\pi : S \to R$ the projection. Set L to be a Cartier divisor representing the relative dualizing sheaf $\mathcal{O}_{S/R}(1)$ and E to be the effective Cartier divisor corresponding to the unique section of $\mathcal{O}_{S/R}(1) \otimes \pi^*\mathcal{O}_R(-A)$. Then any irreducible effective Weil divisor D satisfying $L^n \cdot D = d$ and $E \cdot D = 0$ must lie in $|L|$ (and in particular must be Cartier).

Proof. Note that for any rank one reflexive sheaf $\mathcal{L}$ on R we have that $\pi^*\mathcal{L}$ is still reflexive. Furthermore, $\pi_*\pi^*\mathcal{L} \cong \mathcal{L}$, since upon restriction to the smooth locus $U \subset R$ both agree with $\pi_*\pi^*\mathcal{L}|_U$.

Consider the reflexive sheaf $\mathcal{O}_S(D)$. There is a unique integer q and reflexive rank one sheaf $\mathcal{O}_R(T)$ such that

$$\mathcal{O}_S(D) \cong \mathcal{O}_{S/R}(1)^{\otimes q} \otimes \pi^*\mathcal{O}_R(T).$$

(One can verify this over the smooth locus of R using the usual description of the Picard group of a projective bundle.) The numerical conditions on D imply that $q = 1$ and $[T] \in N_{n-1}(R)$ is the 0 class. Then by pushing forward we see that

$$H^0(S, \mathcal{O}_S(D)) \cong H^0(R, \mathcal{O}_R(T+A) \oplus \mathcal{O}_R(T)).$$

Suppose that $\mathcal{O}_R(T)$ is not the trivial bundle. Then $H^0(R, \mathcal{O}_R(T)) = 0$ and sections of $\mathcal{O}_S(D)$ are constructed by taking a flat pullback of an effective divisor in $H^0(R, \mathcal{O}_R(T + A))$ and adding on E. Such divisors can not be irreducible. Thus $T = 0$ and we have proved the statement. □

Now that we have an estimate on the dimension of the "kernel" of restriction to a hyperplane, we can estimate the total dimension via an inductive argument.

Theorem 4.5. *Let $p : U \to W$ be a family of irreducible degree d effective k-cycles on $\mathbb{P}^n$. Then*

$$\mathrm{chdim}(p) \leq \binom{d+k+1}{k+1} - 1 + (k+2)(n-k-1).$$

Proof. The proof is by induction on k. When $k = 1$, we have a family of curves on $\mathbb{P}^n$ and the statement is proved by [EH92].

For arbitrary k, fix a general hyperplane H. Consider the intersection family $p \cdot H$. By Lemma 2.1, we have

$$\mathrm{chdim}_{\mathbb{P}^n}(p) \leq \mathrm{chdim}_{\mathbb{P}^n}(p|_F) + \mathrm{chdim}_H(p \cdot H)$$

where F is a component of a general fiber of $\mathrm{ch}_{p \cdot H} : W \dashrightarrow \mathrm{Chow}(H)$. Since H is general, two cycles in our family are identified if and only if their intersection with H is the same. This is exactly the setting of Lemma 4.3, where Z is the common intersection of H with our cycles. The lemma shows that

$$\mathrm{chdim}_{\mathbb{P}^n}(p|_F) \leq (n-k-1) + \binom{d+k}{k+1}$$

and since $p \cdot H$ is a family of irreducible $(k-1)$-cycles in $\mathbb{P}^{n-1}$, by induction

$$\mathrm{chdim}_H(p \cdot H) \leq \binom{d+k}{k} - 1 + (k+1)(n-k-1).$$

Combining these two bounds gives the result. □

Now we have bounded the dimension of an irreducible family of cycles. However, in low degrees the maximal component of $\mathrm{Chow}(\mathbb{P}^n)$ will parametrize reducible cycles. To finish the proof, we just need to compare all possible ways of partitioning a degree d cycle into irreducible components of smaller degrees.

Proof of Theorem 4.2: Note that as $\lambda := (a_1, \ldots, a_q)$ varies over all partitions of d, we have

$$\dim \mathrm{Chow}(\mathbb{P}^n, d\alpha) = \sup_{\lambda} \left\{ \sum_{i=1}^{q} \dim \mathrm{Chow}_{irr}(\mathbb{P}^n, a_i\alpha) \right\}$$

where Chow_{irr} denotes the components of Chow parametrizing irreducible reduced subvarieties.

For simplicity, we define the functions $f(d) = d(k+1)(n-k)$ and

$$g(d) = \binom{d+k+1}{k+1} - 1 + (k+2)(n-k-1).$$

Note that $f(d)$ is the dimension of the component of $\mathrm{Chow}(\mathbb{P}^n)$ parametrizing a union of planes and $g(d)$ is the upper bound on the dimension of an irreducible family of cycles of degree d provided by Theorem 4.5. We want to show that the dimension of the degree d dimension k cycles on $\mathbb{P}^n$ is the maximum of $f(d), g(d)$.

Let r be the largest integer such that $f(r) \geq g(r)$. For $d \leq r$, it is easy to see by induction that the dimension of $\mathrm{Chow}(\mathbb{P}^n, d\alpha)$ is $f(d)$ since f is linear in d:

$$\dim \mathrm{Chow}(\mathbb{P}^n, d\alpha) = \sup_{\lambda} \left\{ \sum_{i=1}^{q} g(a_i) \right\} \leq \sup_{\lambda} \left\{ \sum_{i=1}^{q} f(a_i) \right\} = f(d).$$

Now suppose $d > r$. We prove by induction that the dimension of Chow is given by $g(d)$. For the base case $d = r+1$, we note that any reducible family will have dimension at most $f(r+1)$, which is strictly less than $g(r+1)$ by construction. For the induction step, it suffices to prove that

$$g(d) \geq \sup_{1 \leq q \leq d-r-1} \{g(q) + g(d-q)\} \qquad \text{and} \qquad g(d) \geq \sup_{d-r \leq q \leq d-1} g(q) + f(d-q).$$

These inequalities express the fact that we can do no better by splitting off a degree q irreducible component, where the remainder of degree $d-q$ falls in the range where g or f is the optimum respectively. In fact, we can replace the second inequality by the simpler inequality

$$g(d) \geq g(d-1) + f(1).$$

Indeed, if $d - q \leq r$ then the maximal dimensional component of $\mathrm{Chow}(\mathbb{P}^n, (d-q)\alpha)$ parametrizes a union of planes, so we may as well partition our cycles differently into a degree 1 irreducible piece and a degree $d-1$ piece, and by induction the degree $d-1$ piece has Chow dimension at most $g(d-1)$.

Since $d > r$ we must have

$$\binom{d+k}{k} \geq \binom{r+1+k}{k}.$$

Since we must have $g(r+1) - g(r) > f(r+1) - f(r) = (k+1)(n-k)$, the inequality $g(d) \geq g(d-1) + f(1)$ follows easily.

We still need to check the other inequality $g(d) \geq g(q) + g(d-q)$ where $1 \leq q \leq d - r - 1$. Then note that

$$\begin{aligned}
\binom{d+k+1}{k+1} - \binom{d-q+k+1}{k+1} - \binom{q+k+1}{k+1} \\
&= \sum_{i=0}^{q} \left(\binom{d-q+k+i}{k} - \binom{k+i}{k} \right) \\
&\geq (q+1)\left(\binom{d-q+k}{k} - 1 \right) \\
&\geq (q+1)\left(\binom{r+1+k}{k} - 1 \right) \\
&\geq (q+1)((k+1)(n-k)-1) \\
&\geq 2(k+1)(n-k) - 2 \\
&> (k+2)(n-k-1) - 1
\end{aligned}$$

and the conclusion follows by rearranging the inequality. □

5. The variation function

The variation of a class $\alpha \in N_k(X)_{\mathbb{Z}}$ measures the rate of growth of the dimensions of components of $\mathrm{Chow}(X)$ that represent $m\alpha$ as m increases. The main theorem in this section is Theorem 5.16 which shows that variation is in some sense a measure of bigness along subvarieties of X.

5.1. **Dimensions of families of cycles.** Before defining the variation, we need to find bounds for the dimension of components of $\mathrm{Chow}(X)$. The following theorem incorporates a suggestion of Voisin who pointed out that the coefficient in the original version could be improved by considering a generically finite map to projective space.

Theorem 5.1. *Let X be a projective variety of dimension n and let $\alpha \in N_k(X)_{\mathbb{Z}}$. Suppose that A is a very ample divisor on X and set $d = \alpha \cdot A^k$. Then we have*

$$\mathrm{chdim}(\alpha) \leq \binom{d+k+1}{k+1} + d(k+1)(n-k).$$

Proof. Suppose that $p : U \to W$ is a family of effective cycles representing α. Since the desired upper bound is superadditive in d, by Lemma 3.9 we may prove the bound for each irreducible component of U separately. Hence we assume that U is irreducible.

Let $\pi : X \to \mathbb{P}^n$ be a generically finite morphism defined by a general subspace of $|A|$. There are two possibilities:

(1) The general member of the family p is contracted by π. Then every member of p must be contained in the π-exceptional locus of X, and we conclude by induction on the dimension n.

(2) The general member of p is not contracted by π. By Lemma 3.8 we see that $\mathrm{chdim}(p) = \mathrm{chdim}(\pi_* p)$. Thus we obtain the upper bound by Theorem 4.2.

□

Theorem 5.1 shows that for any class $\alpha \in N_k(X)_{\mathbb{Z}}$, there is some positive constant C such that $\mathrm{chdim}(m\alpha) < Cm^{k+1}$. Furthermore, there is always a class on X achieving this growth rate as in the following example.

Example 5.2. Let $H_1, \ldots, H_{n-k}$ be general very ample divisors on X and set $\alpha = H_1 \cdot \ldots \cdot H_{n-k}$. Let V denote the scheme-theoretic intersection $H_1 \cap \ldots \cap H_{n-k-1}$. The linear series $|m(H_{n-k}|_V)|$ defines a rational family of k-cycles representing $m\alpha$. Thus

$$\mathrm{chdim}(m\alpha) \geq (H_1 \cdot \ldots \cdot H_{n-k-1} \cdot H_{n-k}^{k+1}) \frac{m^{k+1}}{(k+1)!} + O(m^k)$$

5.2. **Definitions.** Theorem 5.1 and Example 5.2 suggest that one should compare the growth rate of $\mathrm{chdim}(m\alpha)$ against m^{k+1}.

Definition 5.3. Let X be a projective variety. For any $\alpha \in N_k(X)_{\mathbb{Z}}$, we define the variation of α to be

$$\mathrm{var}(\alpha) := \limsup_{m\to\infty} \frac{\mathrm{chdim}(m\alpha)}{m^{k+1}/(k+1)!}.$$

The choice of the coefficient $(k+1)!$ is justified by our calculations for projective space.

Example 5.4. Let X be a normal projective variety of dimension n and suppose that X admits a resolution $\phi : Y \to X$ such that the kernel of $\phi_* : N_{n-1}(Y) \to N_{n-1}(X)$ is spanned by ϕ-exceptional divisors. (For example X could be smooth or normal $\mathbb{Q}$-factorial over $\mathbb{C}$.) Then for any Cartier divisor D on X we have $\mathrm{var}([D]) = \mathrm{vol}(D)$.

To verify this, note that since chdim is preserved by passing to strict transform families of divisors we have

$$\mathrm{chdim}(m[D]) = \max \left\{ \mathrm{chdim}(\beta) \,\middle|\, \begin{array}{l} \beta \text{ is a class on } Y \\ \text{with } \phi_*\beta = m[D] \end{array} \right\}.$$

Let L denote any divisor in the class β attaining this maximum value. We may write $L \equiv m\phi^* D + E$ where E is some ϕ-exceptional divisor. Since increasing the coefficients in E can only increase chdim, we may assume that E is effective. But then

$$h^0(Y, \mathcal{O}_Y(L)) = h^0(Y, \mathcal{O}_Y(L - E))$$

by the negativity of contraction lemma (see for example [Nak04, III.5.7 Proposition]). Thus

$$\begin{aligned} h^0(X, \mathcal{O}_X(mD)) - 1 &\leq \mathrm{chdim}(m[D]) \\ &\leq \dim \mathrm{Pic}^0(Y) + \max_{D' \equiv m\phi^* D} h^0(Y, \mathcal{O}_Y(D')) - 1. \end{aligned}$$

While the rightmost term may be greater than $h^0(X, \mathcal{O}_X(mD)) - 1$, the difference is bounded by a polynomial of degree $n-1$ in m (see the proof of [Laz04, Proposition 2.2.43]). Thus $\mathrm{var}([D])$ agrees with the volume.

Remark 5.5. It is also interesting to analyze the dimension of other components of $\mathrm{Chow}(X)$. For curves on $\mathbb{P}^3$, [Per87] conjectures that calculating the dimensions of components of $\mathrm{Chow}(X)$ parametrizing "general" curves of degree d – in the sense that the corresponding cycles are not contained in any hypersurface of degree $< d^{1/2}$ – will yield the values predicted by the mobility function of [Leh16].

Remark 5.6. One could also consider a "rational variation" function, where we maximize the dimension of subvarieties of $\mathrm{Chow}(X)$ parametrizing rationally equivalent subvarieties. Most of the theory developed in this paper goes through in this setting with no changes. For example, the rational variation is interesting already for 0-cycles, where it is closely related to the invariants of [Roĭ72].

5.3. **Basic properties.**

We next verify some of the basic properties of the variation.

Lemma 5.7. *Let X be a projective variety and let $\alpha \in N_k(X)_{\mathbb{Z}}$. Then for any positive integer c we have $\mathrm{var}(c\alpha) = c^{k+1}\,\mathrm{var}(\alpha)$.*

Proof. If $\mathrm{chdim}(\alpha) > 0$ then α is represented by an effective cycle Z. Thus $\mathrm{chdim}(\alpha+\beta) \geq \mathrm{chdim}(\beta)$ for any class β: if p is a family of effective cycles of class β then we can add the constant cycle Z to p (using the family sum construction) to obtain a family representing $\alpha + \beta$ with the same Chow dimension. We conclude by the following Lemma 5.8. □

Lemma 5.8 ([Laz04] Lemma 2.2.38)**.** *Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function. Suppose that for any $r, s \in \mathbb{N}$ with $f(r) > 0$ we have that $f(r+s) \geq f(s)$. Then for any $k \in \mathbb{R}_{>0}$ the function $g : \mathbb{N} \to \mathbb{R} \cup \{\infty\}$ defined by*

$$g(r) := \limsup_{m\to\infty} \frac{f(mr)}{m^k}$$

satisfies $g(cr) = c^k g(r)$ for any $c, r \in \mathbb{N}$.

Remark 5.9. Although [Laz04, Lemma 2.2.38] only explicitly address the volume function, the essential content of the proof is the more general statement above.

Lemma 5.7 allows us to extend the definition of variation to any $\mathbb{Q}$-class by homogeneity. Thus we obtain a function

$$\mathrm{var} : N_k(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}.$$

Lemma 5.10. *Let X be a projective variety. Suppose that $\alpha, \beta \in N_k(X)_{\mathbb{Q}}$ are classes such that some positive multiple of each is represented by an effective cycle. Then $\mathrm{var}(\alpha + \beta) \geq \mathrm{var}(\alpha) + \mathrm{var}(\beta)$.*

Proof. Note that we may check the inequality after rescaling α and β by the same positive integer c. Thus we may suppose that every multiple of α and β is represented by an effective cycle.

Suppose that $p : U \to W$ is a family representing $m\alpha$ and $q : S \to T$ is a family representing $m\beta$. Then the family sum $p + q$ represents $m(\alpha + \beta)$. Lemma 3.9 shows that

$$\mathrm{chdim}(p + q) = \mathrm{chdim}(p) + \mathrm{chdim}(q)$$

and the desired inequality follows. $\square$

By Example 5.2, we find:

Corollary 5.11. *Let X be a projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$ be a big class. Then* $\mathrm{var}(\alpha) > 0$.

As a consequence, we see that var is a continuous function on the big cone.

Theorem 5.12. *Let X be a projective variety. The function* $\mathrm{var} : N_k(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ *is locally uniformly continuous on the interior of $\overline{\mathrm{Eff}}_k(X)_{\mathbb{Q}}$.*

Proof. var verifies conditions (1)-(3) of the following Lemma 5.13. $\square$

Lemma 5.13 ([Leh16] Lemma 2.7)**.** *Let V be a finite dimensional $\mathbb{Q}$-vector space and let $C \subset V$ be a salient full-dimensional closed convex cone. Suppose that $f : V \to \mathbb{R}_{\geq 0}$ is a function satisfying*

1. *$f(e) > 0$ for any $e \in C^{int}$,*
2. *there is some constant $c > 0$ so that $f(me) = m^c f(e)$ for any $m \in \mathbb{Q}_{>0}$ and $e \in C$, and*
3. *for every $v \in C$ and $e \in C^{int}$ we have $f(v + e) \geq f(v)$.*

Then f is locally uniformly continuous on C^{int}.

The behavior of the variation along the pseudo-effective boundary is more subtle. Probably the most one can hope for is:

Question 5.14. Let X be a projective variety. Is the function $\mathrm{var} : \overline{\mathrm{Eff}}_k(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ upper semi-continuous?

Finally we note that variation behaves well with respect to inclusions of subvarieties.

Lemma 5.15. *Let X be a projective variety and $i : W \to X$ an integral closed subvariety. For any class $\beta \in N_k(W)_{\mathbb{Q}}$ we have* $\mathrm{var}(\beta) \leq \mathrm{var}(i_*\beta)$.

Proof. Let p be a family of effective cycles on W and consider the push-forward family q on X. Recall that for a general cycle-theoretic fiber Z of p the corresponding cycle in the push-forward family is just i_*Z; thus Lemma 3.4 shows that $\mathrm{chdim}(p) = \mathrm{chdim}(q)$ and the result follows. $\square$

5.4. **Variation and bigness.** Example 1.5 shows that a class may have positive variation even when it is not big. This class is constructed by pushing forward a big class on a subvariety. In this section we show that every class with positive variation arises in this way.

Theorem 5.16. *Let X be a projective variety and let $\alpha \in N_k(X)_{\mathbb{Q}}$. Then the following conditions are equivalent:*

(1) $\mathrm{var}(\alpha) > 0$.
(2) *There is a $(k+1)$-dimensional subvariety Y of X and a big class $\beta \in N_k(Y)_{\mathbb{Q}}$ such that some multiple of $\alpha - i_*\beta$ is represented by an effective cycle.*

Proof. We first show (1) $\implies$ (2). Suppose that $\mathrm{var}(\alpha) > 0$. We may rescale α so that $\alpha \in N_k(X)_{\mathbb{Z}}$ and every positive multiple of α is represented by an effective cycle. Fix a very ample Cartier divisor A, and choose some positive integer m sufficiently large so that

$$\mathrm{chdim}(m\alpha) > \binom{m\alpha \cdot A^k + k}{k} + (m\alpha \cdot A^k)k(n-k+1).$$

Let $p : U \to W$ denote a family of effective k-cycles that has maximal Chow dimension among all the families representing $m\alpha$. Denote the projection map to X by $s : U \to X$. By replacing A by a linearly equivalent divisor, we may suppose that A that does not contain any component of $s(U)$.

Let $q : R \to W^0$ denote the intersection family of p with A (where W^0 is an appropriately chosen open set of W). The family q has class $\beta := m\alpha \cdot A \in N_{k-1}(X)$. By Theorem 5.1, we have

$$\mathrm{chdim}(\beta) < \mathrm{chdim}(p).$$

Thus there is a curve $T \subset W^0$ through a general point of W that is contracted by ch_q but not by ch_p. Let $p_T : U_T \to T$ denote the restriction of the family p to (an open subset of) T. Using Lemma 3.3 we may extend the family p_T to a projective closure of T.

By Lemma 3.4 there is some irreducible component G of U_T whose s-image in X has dimension $k+1$. Set $Y = s(G)$ with the reduced induced structure and $i : Y \to X$ the corresponding closed immersion. Let $\rho : G \to T$ denote the induced family of divisors on Y and $\sigma : G \to Y$ the restriction of s to G. We claim that the class β' of the family ρ is big as a class on Y. Indeed, by construction every ρ-horizontal component of σ^*A on G has σ-image of dimension at most $k-1$. Thus, the support of $\sigma_*\sigma^*A$ is contained in the image of fibers of ρ. In other words, there is some multiple c of β' such that $c\beta' \succeq [\sigma_*\sigma^*A]$. Since the latter class is big, the former is as well.

Recall that $m\alpha$ is represented by cycles which are sums of fibers of ρ with effective cycles. Thus setting $\beta = \frac{1}{m}\beta'$ finishes the implication.

To show (2) $\implies$ (1), suppose that there is an inclusion $f : Y \to X$ from a $(k+1)$-dimensional integral projective subvariety Y and a big class $\beta \in N_k(Y)_{\mathbb{Q}}$ so that some multiple of $\alpha - f_*\beta$ is represented by an effective cycle. Since β is big we have $\mathrm{var}(\beta) > 0$, so by Lemma 5.15 $\mathrm{var}(i_*\beta) > 0$. Thus $\mathrm{var}(\alpha) > 0$ as well. $\square$

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