ERRATA: COMPARING NUMERICAL DIMENSIONS

Theorem 1.1 of the paper is incorrect. The error is introduced in Proposition 5.3. The argument in the first paragraph implicitly uses the (incorrect) claim that the image of a closed cone under a linear map is closed. I use this incorrect claim to show that $\kappa_\nu(L) = \min\{\dim(W)|\zeta(L, I_W) = 0\}$; this is also incorrect. The proof of Theorem 1.1 correctly shows that $\min\{\dim(W)|\zeta(L, I_W) = 0\} \le \nu(L)$ but incorrectly deduce that $\kappa_\nu(L) \le \nu(L)$.

Corrected results. Let X be a smooth projective complex variety and let L be an $\mathbb{R}$-Cartier divisor on X. The main three definitions of the numerical dimension are as follows:

$$\nu(L) = \max\{k \in \mathbb{Z}_{\ge 0} | \langle L^k \rangle \ne 0\}.$$

$$\kappa_\sigma(L) = \max\left\{ k \in \mathbb{Z}_{\ge 0} \,\middle|\, \limsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(\lfloor mL \rfloor + A))}{m^k} > 0 \right\}.$$

$$\kappa_\nu(L) = \min\left\{ \dim W \,\middle|\, \begin{array}{l} \phi^* L - \epsilon E \text{ is not pseudo-effective} \\ \text{for any } \epsilon > 0 \text{ where } \phi : Bl_W X \to X \\ \text{and } \mathcal{O}_{Bl_W X}(-E) = \phi^{-1}\mathcal{I}_W \cdot \mathcal{O}_{Bl_W X} \end{array} \right\}.$$

The definition of $\nu(L)$ is due to [Bou04] and [BDPP04], while the second two are due to [Nak04]. The paper "Comparing numerical dimenions" also gives many alternative definitions of the numerical dimension, most of which are variants of these but a couple which could potentially be different.

Theorem 0.1. *We have*

$$\nu(L) \le \kappa_\sigma(L) \le \kappa_\nu(L).$$

The second inequality is due to [Nak04]; the first is the contribution of the paper. [Les19] shows that the second inequality can be strict.

Question 0.2. Do we always have $\nu(L) = \kappa_\sigma(L)$? (One can also ask whether equality holds for the $\mathbb{R}$-versions of these constants.)

The various definitions of numerical dimension satisfy many nice properties, as verified by [Nak04, Propositions V.2.7 and V.2.22]. These properties also hold for ν:

- $0 \le \nu(L) \le \dim(X)$.
- $\nu(L) = \dim(X)$ iff L is big and $\nu(L) = 0$ iff $P_\sigma(L) \equiv 0$.
- $\kappa(L) \le \nu(L)$.
- If $\phi : Y \to X$ is a surjective morphism then $\nu(\phi^* L) = \nu(L)$.

In particular, we have $\nu(L) = 0 \Leftrightarrow \kappa_\sigma(L) = 0 \Leftrightarrow \kappa_\nu(L) = 0$. We also prove two additional basic properties:

- $\nu(L) = \nu(P_\sigma(L))$.
- Fix some sufficiently ample $\mathbb{Z}$-divisor A. Then there are positive constants C_1, C_2 such that
$$C_1 m^{\nu(L)} < h^0(X, \mathcal{O}_X(\lfloor mL \rfloor + A)) < C_2 m^{\kappa_\nu(L)}$$
for every sufficiently large m.

The final section of the paper defines a restricted numerical dimension (similar in spirit to the restricted volume of [ELM+09]). I am not aware of any errors in this section.

References

[BDPP04] S. Boucksom, J.P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, 2004, http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/coneduality.pdf, submitted to J. Alg. Geometry.

[Bou04] S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. **37** (2004), no. 4, 45–76.

[ELM+09] ———, *Restricted volumes and base loci of linear series*, Amer. J. Math. **131** (2009), no. 3, 607–651.

[Les19] J. Lesieutre, *Notions of numerical Iitaka dimension do not coincide*, arXiv:1904.10832 [math.AG]

[Nak04] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.