ERRATA: ON ECKL'S PSEUDO-EFFECTIVE REDUCTION MAP

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Recall the main three definitions of numerical dimension: $\nu(L)$ due to [Bou04] and [BDPP04], $\kappa_\sigma(L)$ due to [Nak04], and $\kappa_\nu(L)$ due to [Nak04]. Since the main theorem of "Comparing numerical dimensions" is incorrect, one must be careful about which definition is being used.

Reduction maps. In sections 2-5 of "On Eckl's pseudo-effective reduction map", we are always using $\nu(L)$ in the sense of [BDPP04] (except in Theorem 5.7 and Corollary 5.8). This consistent usage means there are no errors in the statements.

However, a posteriori we can also extend the results to the other definitions of numerical dimension, using the following lemma:

Lemma 0.1. *Let X be a smooth projective variety and let L be a pseudo-effective divisor on X. Then the following conditions are equivalent:*

(1) $\nu(L) = 0$.
(2) $\kappa_\sigma(L) = 0$.
(3) $\kappa_\nu(L) = 0$.
(4) $P_\sigma(L) \equiv 0$.

Proof. The equivalence of (2)-(4) is proved by [Nak04, Proposition V.2.22.(2)]. So it suffices to prove (2) $\implies$ (1) $\implies$ (4).

The implication (2) $\implies$ (1) follows from the (corrected) main result of [Leh10].

Finally, assume (1). By [Leh10, Proposition 7.7] we see that $P_\sigma(L) \cdot C = 0$ where C is a very general intersection of very general ample divisors on X. Since the class of such a curve is in the interior of the nef cone of curves, we deduce that $P_\sigma(L)$ is numerically trivial. $\square$

Thus, when referring to "numerical dimension 0" it does not matter which version we use. This means that most of the results can be interpreted using any definition of the numerical dimension.

The statement of Theorem 5.7 is correct, although it does not follow immediately from the cited result. To be precise, the cited result shows that there is a divisor B such that

$$\mu^* L \sim_{\mathbb{Q}} g^* B + N_\sigma(\mu^* L; W/T).$$

However, since $N_\sigma(\mu^* L; W/T) \leq N_\sigma(\mu^* L)$, it is clear that B must be pseudo-effective and thus by taking positive parts of both sides we obtain the desired result. Corollary 5.8 is still valid even though it uses κ_σ in place of ν.

Abundance. In section 6, there could be three different definitions of abundance, depending on which version of the numerical dimension we use. Depending on which definition we choose, we might obtain different results.

The proof of the main theorem concerning abundance, Theorem 6.1, implicitly uses the strongest version of abundance, κ_ν-abundance. The corrected statement should read:

Theorem 0.2. *Let X be a smooth projective variety and let L be an $\mathbb{R}$-Cartier divisor with $\kappa(L) \geq 0$. Consider the following statements:*

1. $\kappa(L) = \kappa_\nu(L)$.
2. *If F denotes a general fiber of (a resolution of) the Iitaka fibration for L then the restriction of L to F has numerical dimension 0.*
3. *There is a smooth variety W admitting a birational map $\mu : W \to X$ and a morphism $g : W \to T$ with connected fibers such that $P_\sigma(\mu^*L) \sim_{\mathbb{Q}} P_\sigma(g^*B)$ for some big divisor B on T.*
4. $\kappa(L) = \kappa_\sigma(L)$.

Then (1) $\implies$ (2) $\Leftrightarrow$ (3) $\implies$ (4).

I do not know whether or not any of the missing implications is true.

All the remaining results in the section rely on property (3) of the theorem above.

References

[BDPP04] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[Bou04] S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. **37** (2004), no. 4, 45–76.

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[Nak04] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.