ERRATUM TO "ON THE GEOMETRY OF THIN EXCEPTIONAL SETS IN MANIN'S CONJECTURE"

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Abstract. The proof of Proposition 7.2 of [LT17] is not correct. In this note we give a corrected statement and explain its applications to [LT17].

1. Proposition 7.2 of [LT17]

[LT17, Proposition 7.2] is the following claim:

Claim 1.1. Let X be a projection $\mathbb{Q}$-factorial terminal threefold. Suppose we have an extremal K_X-negative divisorial contraction $f : X \to Y$ with exceptional divisor E. Then there is a family of rational curves Γ covering E and contracted by f such that

$$0 < -K_X \cdot \Gamma \leq 2.$$

The proof method of [LT17] follows a suggestion of Professor Kawakita: we work through the classification of divisorial contractions according to [Kaw05] and prove the statement using the classification.

Unfortunately, there is a mistake in our proof. (This mistake is of course our own and not due to Prof. Kawakita in any way.) While we constructed rational curves with the correct intersection number, we did not verify carefully that these curves moved on E. Indeed, even when H_X and S_X vary it is possible that $H_X \cap E$ or $S_X \cap E$ may not deform in E.

In this note we prove a slightly weaker statement of [LT17, Proposition 7.2] which suffices for our applications in [LT17].

Proposition 1.2. *Let $f : X \to Y$ be a K_X-negative extremal divisorial contraction between terminal $\mathbb{Q}$-factorial projective threefolds. We denote its exceptional divisor by E. Then there is a family of curves C covering E such that*

$$0 < -K_X.C \leq 2.$$

We are not sure if one can find rational curves with these properties as stated in [LT17, Proposition 7.2]. The proof below is a slight modification of the proof in [BLRT20] which proves some related statements:

Proof. We freely use the notations set up in [LT17, Proposition 7.2], [Kaw03], and [Kaw05]. When f is a divisorial contraction to a curve, general fibers will do the job. Thus we may assume that $f : X \to Y$ is a divisorial contraction to a point $P \in Y$.

We go through the numerical classification of terminal divisorial contractions by Kawakita in [Kaw05]. As in [Kaw05] we let $d(i, j)$ denote the Euler characteristic of the rank 1 S_2 sheaf obtained by restricting $iK_X + jE$ to E. [Kaw05] proves that the higher cohomology

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of this sheaf vanishes when $i, j \leq 0$. When this space of sections has positive dimension, the corresponding curves on E satisfy

$$-K_X \cdot \Gamma = -i \cdot \left(\frac{a}{n}\right)^2 E^3 + -j \cdot \left(\frac{a}{n}\right) bE^3.$$

It suffices to find negative values of i, j such that this intersection number is ≤ 2 but $d(i, j) \geq 2$.

We separate the proof into three parts. First suppose that P is an exceptional singularity that occurs in a finite set of types (that is, any exceptional type except e1, e2, e7, e13). Since there are only finitely many possible invariants for such singularities, one can just check the claim in a case-by-case manner. It turns out that in almost all cases there is some i with $1 \leq i \leq 6$ such that $d(-i, 0)$ will work. The values of i are recorded in Table 1.

Table 1. Finite families of exceptional singularities

singularity type	when $a = 1$	when $a > 1$
e3	$d(-1, 0)$ when $n = 1$, $d(-2, 0)$ when $n = 3$	$d(0, -2)$
e4	not possible	not possible
e5	$d(-2, 0)$ when $n = 1$, $d(-3, 0)$ when $n = 2$	$a = 2$, $n = 1 \implies d(-1, 0)$
e6	$d(-3, 0)$	not possible
e8	$d(-2, 0)$ when $n = 1$, $d(-4, 0)$ when $n = 3$	not possible
e9	$d(-3, 0)$ when $n = 1$, $d(-5, 0)$ when $n = 2$	$a = 2$, $n = 1 \implies d(-3, 0)$
e10	$d(-5, 0)$	not possible
e11	not possible	$a = 2$, $n = 2 \implies d(-4, 0)$
e12	$d(-4, 0)$	not possible
e14	$d(-3, 0)$	not possible
e15	$d(-4, 0)$	not possible
e16	$d(-6, 0)$	not possible

We next suppose that P is one of the exceptional singularity types that forms an infinite family (e1, e2, e7, e11). In most cases (i.e. when r is sufficiently large) there is some i such that $d(-i, 0) \geq 2$ and $i(\frac{a}{n})^2 E^3 \leq 1$ or $d(0, -i) \geq 2$ and $i(\frac{a}{n}) bE^3 \leq 2$. Such situations are summarized in Table 2.

Ordinary cases: Next we assume that P has ordinary type as in Table 1 on p. 59 of [Kaw05]. First we consider the case of o1. Suppose that $n \geq 2$. According to [Kaw05, Theorem 3.7] a and n are coprime. Then [Kaw05, Corollary 2.4] implies that $a = 1$. Thus we may conclude that $(a/n)^2 E^3 \leq 1$. Moreover we have $d(-1, 0) > 1$. Thus $S_X \cap E$ deforms in dimension 1. When $n = 1$, P is Gorenstein. According to [Kaw03, Theorem 1.4] we are in the situations of O or IV and this means that $b = 1$. When it is O, we have $\left(\frac{a}{n}\right)^2 E^3 = 2$ and $d(-1, 0) = 3$. When it is IV, we have $\frac{a}{n} bE^3 = 2$ and $d(0, -1) = 3$. Thus our assertion follows. In the remaining cases of o2 and o3, the discussion on p.81 of [Kaw05] implies that $b = 1$. In the case of o2, we have $\frac{a}{n} bE^3 \leq 2$ and $d(0, -1) > 1$.

Table 2. Infinite families of exceptional singularities

type	a value	n value	i value
e1	1	1,2	$d(-1,0)$ or $d(-2,0)$
		4	$d(-2,0)$
	2	1	$d(0,-2)$ when $r \geq 9$, $d(0,-i)$ with $i = 1,2,3$ when $r \leq 8$
		2	$d(-1,0)$ or $d(-2,0)$
	4	1	$d(-1,0)$ when $r \geq 9$, $d(0,-i)$ with $i = 1,2,3$ when $r \leq 8$
		2	$d(-1,0)$ or $d(-2,0)$
e2	1	1	$d(-1,0)$
		2	$d(-2,0)$
	2	1	$d(0,-2)$ when $r \geq 4$, $d(-1,0)$ when $r = 3$
		2	$d(-1,0)$ or $d(-2,0)$
e7	1	2	$d(-4,0)$
e13	1	1	$d(-2,0)$

Finally we consider the case of o3. We write $(r_1, 1), (r_2, 1)$ for the basket of singularities with $r_1 \leq r_2$. When $\frac{a}{n} \leq r_1$, then $d(-1,0) > 1$ and

$$\left(\frac{a}{n}\right)^2 E^3 = \frac{a}{n}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \leq 2.$$

Thus $S_X \cap E$ deforms in dimension at least 1. Now suppose that $\frac{a}{n} > r_1$. Then we have $r_1 \left(\frac{a}{n}\right) E^3 \leq 2$. Moreover in this setting one can prove that $d(0, -r_1) > 1$. Thus our assertion follows. $\square$

[LT17, Proposition 7.2] is only used in the proof of [LT17, Theorem 1.9]. However, the above proposition is sufficient for this proof to work. Note that in the case of any index 1 Fano threefold X of Picard rank 1, $-K_X$ is base point free so any curve C with $-K_X \cdot C = 1$ is a smooth rational curve. Thus no change in the proof is needed.

References

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