Lecture 1: Moduli spaces of rational curves

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Introduction

Set-up

Let X be a smooth projective variety over an algebraically closed field K.

Definition

A rational curve on X is an integral closed subscheme C that is birationally equivalent to $\mathbb{P}^1$.

Note that rational curves on X are allowed to be singular.

Why do we care about rational curves?

1. Connections with number theory
   rational curves $\leftrightarrow$ rational points
2. Connections with geometry
   rational curves on X $\leftrightarrow$ spheres in $X_{\mathbb{C}}$

Why do we care about rational curves?

There are many subdisciplines of algebraic geometry which study rational curves:

1. Minimal model program
2. Rationality questions
3. Gromov-Witten theory
4. Classification of Fano varieties

and so on.

Preliminaries

For a smooth projective variety X, the canonical line bundle ω_X is the top exterior power of the cotangent sheaf:

$$\omega_X := \bigwedge^{\dim X} \Omega_X.$$

The canonical bundle describes the "curvature" of Ω_X.

The canonical divisor K_X is any Cartier divisor satisfying the property

$$\mathcal{O}_X(K_X) = \omega_X.$$

We often speak of "the" canonical divisor even though really K_X is only unique up to linear equivalence.

Example

By taking top exterior powers in the Euler sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus n+1} \to \mathcal{O} \to 0$$

we see that $\omega_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$.

Example

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. By taking top exterior powers in the conormal bundle sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d)|_X \to \Omega_{\mathbb{P}^n}|_X \to \Omega_X \to 0$$

we see that $\omega_X \cong \mathcal{O}_{\mathbb{P}^n}(-n-1+d)|_X$.

Let X be a smooth projective variety. For any Cartier divisor L and any (projective) curve $C \subset X$, we let $\nu : B \to C$ denote the normalization map and define

$$L \cdot C = \deg_B(\nu^* \mathcal{O}_X(L)).$$

We say that two Cartier divisors L, L' are numerically equivalent if $L \cdot C = L' \cdot C$ for every curve $C \subset X$.

Definition

$N^1(X)_{\mathbb{Z}}$ denotes the quotient of the group of Cartier divisors by the subgroup of divisors which are numerically equivalent to the 0 divisor.

$N^1(X)_{\mathbb{Z}}$ is a finitely generated torsion-free abelian group. Since linearly equivalent divisors are numerically equivalent, there is a surjection $\mathrm{Pic}(X) \to N^1(X)_{\mathbb{Z}}$.

Let $Z_1(X)$ denote the free abelian group consisting of formal sums of curves (i.e. 1-cycles) on X. The pairing of Cartier divisors and curves extends linearly to $Z_1(X)$.

Definition

$N_1(X)_{\mathbb{Z}}$ denotes the quotient of $Z_1(X)$ by the subgroup of 1-cycles which have vanishing intersection against every line bundle on X.

$N_1(X)_{\mathbb{Z}}$ is also a finitely generated torsion-free abelian group. We have an intersection pairing

$$N^1(X)_{\mathbb{Z}} \times N_1(X)_{\mathbb{Z}} \to \mathbb{Z}$$

but this need not be a perfect pairing.

Suppose that $f : B \to X$ is a morphism from a curve. We can assign to f an element of $Z_1(X)$ (or $N_1(X)_{\mathbb{Z}}$) as follows.

Let $f(B)$ denote the reduced image of f. We associate to f the 1-cycle

$$f_* B := \deg(f) \cdot f(B)$$

as an element of $Z_1(X)$ or $N_1(X)_{\mathbb{Z}}$. This definition is chosen so that for any Cartier divisor L we have

$$L \cdot f_* B = \deg_B(f^* \mathcal{O}_X(L)).$$

Space of morphisms

Our first goal is to construct a moduli space of rational curves on X. However, instead of parametrizing closed subschemes of X, we will parametrize morphisms $f : \mathbb{P}^1 \to X$.

Note that a morphism $f : \mathbb{P}^1 \to X$ is almost the same thing as a rational curve $C \subset X$.

- Given any morphism $f : \mathbb{P}^1 \to X$ which does not contract $\mathbb{P}^1$ to a point, the image of f will be a rational curve on X.
- Conversely, given any rational curve $C \subset X$, its normalization map $\nu : \mathbb{P}^1 \to C \subset X$ gives us a morphism to X whose image is C.

However there are many different morphisms which have the same image curve; indeed, given any such map $\mathbb{P}^1 \to X$ we can precompose by any map $\mathbb{P}^1 \to \mathbb{P}^1$.

Definition

The space $\mathrm{Mor}(\mathbb{P}^1, X)$ parametrizes all morphisms from $\mathbb{P}^1 \to X$.

To construct this moduli space explicitly, we associate to any morphism $f : \mathbb{P}^1 \to X$ its graph in $\mathbb{P}^1 \times X$. Note that the possible graphs are the same as the sections of the first projection (i.e. closed subschemes $Z \subset \mathbb{P}^1 \times X$ such that $\pi_1 : Z \to \mathbb{P}^1$ is an isomorphism). In turn, these are the curves on $\mathbb{P}^1 \times X$ which have intersection 1 against the fibers of π_1. Therefore:

Lemma

The property that a curve is a section of π is an invariant of the Hilbert polynomial.

Thus we define $\mathrm{Mor}(\mathbb{P}^1, X)$ as an open sublocus of $\mathrm{Hilb}(\mathbb{P}^1 \times X)$.

Space of morphisms

Whenever we are working with the Hilbert scheme, we have a universal family equipped with an evaluation map.

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{ev} & X \\ \downarrow & & \\ \mathrm{Hilb}(X) & & \end{array}$$

Since we constructed $\mathrm{Mor}(\mathbb{P}^1, X)$ using graphs, the universal family has a particularly nice form: it is a product.

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X) & \xrightarrow{ev} & X \\ \downarrow & & \\ \mathrm{Mor}(\mathbb{P}^1, X) & & \end{array}$$

Often $\mathrm{Mor}(\mathbb{P}^1, X)$ will have infinitely many components. To obtain a finite type scheme, we need to fix some invariants of the morphism f.

Suppose that $f : \mathbb{P}^1 \to X$ is a morphism. As discussed earlier, we associate to f the class $f_*\mathbb{P}^1 = \deg(f) \cdot f(\mathbb{P}^1)$ in $N_1(X)_{\mathbb{Z}}$. This class does not change as we vary f in a connected family.

Definition

For any $\alpha \in N_1(X)_{\mathbb{Z}}$, we let $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$ denote the subset of $\mathrm{Mor}(\mathbb{P}^1, X)$ parametrizing morphisms with numerical class α.

It turns out that for any $\alpha \in N_1(X)_{\mathbb{Z}}$ the parameter space $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$ is a finite type scheme which is a union of connected components of $\mathrm{Mor}(\mathbb{P}^1, X)$.

Let's analyze $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n)$. Note that $N_1(\mathbb{P}^n)_{\mathbb{Z}} \cong \mathbb{Z}$. Indeed, since $\mathrm{Pic}(\mathbb{P}^n)$ is generated by $\mathcal{O}(1)$ the only numerical invariant of a curve is its degree.

A morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ is the same thing as the choice of a line bundle $\mathcal{L}$ on $\mathbb{P}^1$ together with $(n+1)$ sections $s_0, \ldots, s_n \in H^0(\mathbb{P}^1, \mathcal{L})$ which generate $\mathcal{L}$. The class of $f_*\mathbb{P}^1$ is the same as the degree of the line bundle $\mathcal{L}$. Thus:

Example

For any $d > 0$ the space $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n)_d$ is an open subscheme of

$$\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d))^{\oplus n+1}) \cong \mathbb{P}^{d(n+1)+n}.$$

Note that our analysis is much easier than trying to understand closed subschemes directly!

Example

For smooth projective varieties X, Y we have

$$\mathrm{Mor}(\mathbb{P}^1, X \times Y) \cong \mathrm{Mor}(\mathbb{P}^1, X) \times \mathrm{Mor}(\mathbb{P}^1, Y)$$

For example this determines $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1)$.

Example

Consider the Hirzebruch surface $\mathbb{F}_e := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$. Using the universal property of the Proj construction, we see that there is a bijection between sections of $\pi : \mathbb{F}_e \to \mathbb{P}^1$ and surjections $\mathcal{O} \oplus \mathcal{O}(-e) \twoheadrightarrow \mathcal{L}$ for a line bundle $\mathcal{L}$ on $\mathbb{P}^1$.

This describes some (but not all) of the components of $\mathrm{Mor}(\mathbb{P}^1, \mathbb{F}_e)$. To describe the other components, we can combine this analysis with a base change argument.

Examples

Advantages of working with space of morphisms:

- Clear geometric meaning.
- Easy to construct using Hilb.

Disadvantages of working with space of morphisms:

- The space is usually not compact.
- Does not literally parametrize rational curves.

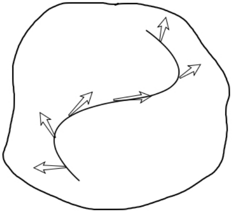
One common alternative is the space of stable maps $\overline{M}_{0,0}(X)$. The key advantage is that this space has proper components. The key disadvantage is that $\overline{M}_{0,0}(X)$ has irreducible components which do not parametrize any morphisms $f : \mathbb{P}^1 \to X$. (Instead, they parametrize morphisms from reducible curves.)

We next analyze the deformation theory of a morphism $f : \mathbb{P}^1 \to X$. The first step is to identify the tangent space of $\mathrm{Mor}(\mathbb{P}^1, X)$ at f. Fortunately, we already know how to do this for Hilbert schemes: the tangent space of the Hilbert scheme is determined by the global sections of the normal sheaf.

Suppose $Z \subset \mathbb{P}^1 \times X$ is the graph of $f : \mathbb{P}^1 \to X$. The normal bundle of Z in $\mathbb{P}^1 \times X$ is isomorphic to $f^* T_X$. Thus:

Theorem

The tangent space of $\mathrm{Mor}(\mathbb{P}^1, X)$ at a point $f : \mathbb{P}^1 \to X$ is $H^0(\mathbb{P}^1, f^* T_X)$.



A global section of $f^* T_X$ defines a deformation

In fact even more is true: the "space of obstructions" is $H^1(\mathbb{P}^1, f^*T_X)$.

In our setting, we interpret this to mean that in a neighborhood of f the scheme $\mathrm{Mor}(\mathbb{P}^1, X)$ is cut out in a space of dimension $h^0(\mathbb{P}^1, f^*T_X)$ by $h^1(\mathbb{P}^1, f^*T_X)$ equations.

In summary:

- The dimension of the tangent space of $\mathrm{Mor}(\mathbb{P}^1, X)$ at f is exactly $h^0(\mathbb{P}^1, f^*T_X)$.
- The dimension of $\mathrm{Mor}(\mathbb{P}^1, X)$ locally near f is at least

$$h^0(\mathbb{P}^1, f^*T_X) - h^1(\mathbb{P}^1, f^*T_X) = \chi(f^*T_X).$$

These two quantities give an upper and lower bound on the dimension of $\mathrm{Mor}(\mathbb{P}^1, X)$ near f. If they agree then f is guaranteed to be a smooth point (but this is not a necessary condition to be a smooth point).

We can compute the lower bound $\chi(f^* T_X)$ more explicitly using Riemann-Roch. If we let C denote the numerical class defined by f, then

$$\begin{aligned}\chi(f^* T_X) &= \deg(f^* T_X) + (1 - g(\mathbb{P}^1)) \cdot \mathrm{rk}(f^* T_X) \\ &= -K_X \cdot C + \dim(X)\end{aligned}$$

This quantity is known as the "expected dimension" of the component of $\mathrm{Mor}(\mathbb{P}^1, X)$ containing f. As suggested by the notation, in nice enough situations we "expect" this lower bound to agree with the actual dimension.

Example

Suppose that C is a degree d rational curve in $\mathbb{P}^n$. The expected dimension is

$$-K_{\mathbb{P}^{n+1}} \cdot C + \dim(\mathbb{P}^n) = (n+1)d + n.$$

This matches with the value we computed earlier.

Example

Let X denote the projective bundle $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2))$ equipped with the projection map $\pi : X \to \mathbb{P}^2$. It contains a “rigid” subvariety Y which is the section defined by the map $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(-2)$. By construction Y is isomorphic to $\mathbb{P}^2$ and we have $N_{Y/X} \cong \mathcal{O}_Y(-2)$. The adjunction formula states that $(K_X + Y)|_Y = K_Y$ so that

$$\omega_X|_Y = \mathcal{O}_Y(-1).$$

Consider the irreducible component M of $\mathrm{Mor}(\mathbb{P}^1, X)$ defining isomorphisms onto the lines in Y. This is an open subset of $\mathbb{P}^5$. However, the expected dimension of M is

$$-K_X \cdot C + \dim(X) = 1 + 3.$$

Thus the dimension of M is larger than the expected dimension.

Free curves

Let's analyze the bundle $f^* T_X$ in more detail. Recall:

Theorem (Grothendieck-Birkhoff)

Every vector bundle on $\mathbb{P}^1$ splits into a direct sum of line bundles.

Thus for any map $f : \mathbb{P}^1 \to X$ we can write $f^* T_X \cong \oplus_{i=1}^{\dim(X)} \mathcal{O}_{\mathbb{P}^1}(a_i)$ where the a_i form a non-increasing sequence of integers.

We have $H^1(\mathbb{P}^1, f^* T_X) = 0$ if and only if $a_i \geq -1$ for every i. The maps with this property will satisfy $h^0(\mathbb{P}^1, f^* T_X) = \chi(f^* T_X)$ and thus will be smooth points of $\mathrm{Mor}(\mathbb{P}^1, X)$.

The following definition identifies the morphisms with the best behavior.

Definition

A non-trivial map $f : \mathbb{P}^1 \to X$ is said to be a free rational curve if $f^* T_X \cong \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$ where each $a_i \geq 0$.

Note that free curves are smooth points of $\mathrm{Mor}(\mathbb{P}^1, X)$. They have a number of other important properties; for us, the most important property is the following one.

Theorem

If $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ is an irreducible component that generically parametrizes free curves, then the evaluation map for the universal family over M is dominant.

Conversely, assume $char(K) = 0$. Suppose that $T \subset \mathrm{Mor}(\mathbb{P}^1, X)$ is an irreducible closed subvariety. Consider the evaluation map on the universal family $ev : \mathbb{P}^1 \times T \to X$. If ev is dominant then the general map parametrized by T is a free curve.

If *ev* generically induces a surjection of tangent spaces then it is dominant, and if we are in characteristic 0 then the converse is also true. Thus both parts of the previous theorem amount to a tangent space calculation for the universal family.

Proof: Let $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ be an irreducible component and consider the map $ev : \mathbb{P}^1 \times M \to X$. Recall that a free curve is contained in the smooth locus of M. Thus if we fix a free curve f and a closed point $p \in \mathbb{P}^1$, then the tangent map for ev at (p, f) is

$$\begin{aligned} T_{\mathbb{P}^1,p} \oplus H^0(\mathbb{P}^1, f^* T_X) &\to T_{X,f(p)} \\ (v, \sigma) &\mapsto (Tf)_p(v) + \sigma|_p \end{aligned}$$

Since $f^* T_X$ is globally generated the map $H^0(\mathbb{P}^1, f^* T_X) \to T_X|_{f(p)}$ is already surjective. Thus the map above is also surjective and thus ev is dominant.

The converse direction is a similar calculation. □

The previous result has an interesting corollary.

Corollary

Assume $char(K) = 0$. Let X be a smooth projective variety and let $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ be an irreducible component that parametrizes a dominant family of curves. Then M has the expected dimension.

This means that if M has larger than the expected dimension then the corresponding curves must sweep out a proper closed subset of X. Later on we will study how to describe this set explicitly.

We will be particularly interested in those varieties which admit a dominant family of rational curves.

Definition

A smooth projective variety X is uniruled if there is an irreducible component $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ such that the evaluation map for the universal family over M is dominant.

If $\mathrm{char}(K) = 0$ then X is uniruled if and only if it admits a free curve.
Examples of uniruled varieties include:

- rational varieties
- unirational varieties
- low degree hypersurfaces

It is sometimes helpful to recast the uniruled condition as follows.

Theorem

Assume $K = \mathbb{C}$. A smooth projective variety X is uniruled if and only if there is a rational curve through every point of X.

To prove the forward implication, we need to know that we can compactify the space of rational curves on X in such a way that every irreducible component of a "boundary curve" is still rational. We will not prove this.

To prove the reverse implication, suppose for a contradiction that no irreducible component $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ parametrizes free curves. We showed that the image of the evaluation map for the universal family over M must be a proper closed subvariety of X. As we vary over all components M, these images define a countable union of proper closed subvarieties of X. But since $\mathbb{C}$ is uncountable this countable union cannot cover all of the points of X.

References:

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