Lecture 3: Geometric Manin's Conjecture

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Introduction

Definition

A smooth projective variety X is called a Fano variety if $-K_X$ is ample.

As discussed last time, each Fano variety will contain "lots" of rational curves.

We are interested in classifying the irreducible components of $\mathrm{Mor}(\mathbb{P}^1, X)$ for a Fano variety X:

1. What are the components of $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$?
2. What are their dimensions?

In this lecture we will work over the ground field $\mathbb{C}$ unless otherwise specified.

Curves on Fano varieties

There are many examples of Fano varieties for which the irreducible components of $\mathrm{Mor}(\mathbb{P}^1, X)$ have been completely classified. These include:

- Most Fano hypersurfaces (Harris-Roth-Starr, Beheshti-Kumar, Riedl-Yang, Browning-Vishe)
- Homogeneous varieties (Thomson, Kim-Pandharipande)
- Del Pezzo surfaces (Testa)
- Toric varieties (Bourqui)
- Many Fano threefolds (Beheshti-Lehmann-Riedl-Tanimoto)

Most of these examples share an interesting feature: for any numerical class $\alpha \in N_1(X)_{\mathbb{Z}}$ there is at most one component of $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$ parametrizing free curves. (However there are also examples where this nice property does not hold.)

For example, the following result describes the rational curves on general Fano hypersurfaces of index ≥ 3.

Theorem (Riedl-Yang)

Let $X \subset \mathbb{P}^n$ be a general hypersurface of dimension ≥ 3 and degree $\leq n-2$. For every positive integer e, the space $\mathrm{Mor}(\mathbb{P}^1, X)_e$ parametrizing degree e curves is irreducible. In particular, this component generically parametrizes free curves and thus has the expected dimension.

The hypotheses of the theorem are crucial.

Example (Starr)

Let X be a cubic threefold. For every degree $e \geq 3$ the space $\mathrm{Mor}(\mathbb{P}^1, X)_e$ has two components. One component generically parametrizes free curves and has the expected dimension $e+3$. The other component parametrizes multiple covers of the lines on X and has dimension $2e+2$.

Next we consider (not necessarily Fano) smooth projective toric varieties X.

- Any rational curve which intersects the dense torus will deform in a dominant family.
- Any rational curve which does not intersect the dense torus will be contained in a boundary component.

Since curves of the second type can be analyzed by induction on dimension, it suffices to focus our attention on curves of the first type.

Theorem (Bourqui)

Let X be a smooth projective toric variety. Suppose $\alpha \in N_1(X)_{\mathbb{Z}}$ satisfies $D_i \cdot \alpha \geq \dim(X) + 1$ for every boundary divisor D_i on X. Then $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$ is irreducible and generically parametrizes free curves.

Manin's Conjecture

We will study the classification of rational curves by appealing to an analogy with a conjecture from arithmetic geometry. Manin's Conjecture for a Fano variety over a number field predicts the behavior of the rational points on X. Our goal is to translate Manin's Conjecture into a similar conjecture for rational curves on complex Fano varieties.

Mori proved that a Fano variety over $\mathbb{C}$ carries many rational curves due to the negativity of K_X. Manin's Conjecture quantifies this prediction: the "amount" of negativity of K_X predicts the "amount" of rational curves.

Manin's Conjecture

We will make the translation from number theory to complex geometry via the function field of an $\mathbb{F}_q$-curve.

Suppose that K is the function field of a 1-dimensional integral Noetherian scheme Z. Given a projective K-variety X, an integral model of X is a flat morphism $\pi : \mathcal{X} \to Z$ whose generic fiber is X. Using the valuative criteria for properness, we obtain a bijection between rational points $X(K)$ and sections of π.

$\mathbb{Q}$	$\mathbb{F}_q(t)$	$\mathbb{C}(t)$
$\pi : \mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$	$\pi : \mathcal{X} \to \mathbb{P}^1_{\mathbb{F}_q}$	$\pi : \mathcal{X} \to \mathbb{P}^1_{\mathbb{C}}$
count rational points of bounded height on Fano variety $X_{\mathbb{Q}}$	count sections of bounded degree of Fano fibration π	"count" sections of bounded degree of Fano fibration π

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How might we hope to solve Manin's Conjecture for a Fano fibration $\pi : \mathcal{X} \to \mathbb{P}^1_{\mathbb{F}_q}$?

Step 1: Classify all irreducible components of the parameter space of sections $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1_{\mathbb{F}_q})$.

Step 2: For each irreducible component $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1_{\mathbb{F}_q})$, count the $\mathbb{F}_q$-points on M.

In general we expect that $\#M(\mathbb{F}_q) \approx q^{\dim(M)}$. The error terms are controlled by the étale homology groups of M. In fact, one way to count rational points on M is to compute the étale homology groups and then to apply the Grothendieck-Lefschetz trace formula:

$$\#M(\mathbb{F}_q) = \sum_i (-1)^i \mathrm{Tr}(F^{n*}, H^i_c(X_{et}, \mathbb{Q}_\ell)).$$

We will formulate Geometric Manin's Conjecture by trying to carry out the same two steps over $\mathbb{C}$.

Step 1: Classify all irreducible components of the parameter space of sections $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1_{\mathbb{C}})$.

Step 2: Prove that the homology groups (or motive) of the irreducible components $M \subset \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1_{\mathbb{C}})$ "stabilize" as the degree increases. (Ellenberg-Venkatesh)

For simplicity, we will focus on the case where $\mathcal{X} \cong X \times \mathbb{P}^1$ for a complex Fano variety X. In this case, the parameter space of sections $\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1)$ is exactly the same as the moduli space $\mathrm{Mor}(\mathbb{P}^1, X)$.

In this lecture we will focus on Step 1: classifying components of $\mathrm{Mor}(\mathbb{P}^1, X)$ via the framework of Manin's Conjecture.

While Step 2 is also important, there are fewer examples for which this program has been carried out. The motivating conjecture is:

Conjecture (Cohen-Jones-Segal)

Let X be a smooth Fano variety over $\mathbb{C}$. Fix a general point p and let $M_{\alpha,p} \subset \mathrm{Mor}(\mathbb{P}^1, X)_\alpha$ denote the sublocus parametrizing morphisms f such that $f(\infty) = p$. As the degree of α increases, the homology of $M_{\alpha,p}$ stabilizes to the homology of the space of based continuous maps $\mathrm{Map}_*(S^2, X_{\mathbb{C}})$.

Known cases: $\mathbb{P}^n$ (Segal), toric varieties (Guest), homogeneous varieties (many authors), low degree hypersurfaces (Browning-Sawin, Starr-Tian).

Fujita invariant

Fujita invariant

The key input into Manin's Conjecture is the Fujita invariant, which we will now define. Recall that $N^1(X)_{\mathbb{Z}}$ denotes the numerical classes of Cartier divisors on X. We define $N^1(X)_{\mathbb{R}} := N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$.

Definition

The pseudo-effective cone of divisors $\overline{\mathrm{Eff}}^1(X)$ is the closure of the cone in $N^1(X)_{\mathbb{R}}$ generated by all effective Cartier divisors.

One should think of $\overline{\mathrm{Eff}}^1(X)$ as the "homological shadow" of the codimension 1 subvarieties of X. When X is a Fano variety $\overline{\mathrm{Eff}}^1(X)$ is a pointed polyhedral cone in the finite-dimensional vector space $N^1(X)_{\mathbb{R}}$.

Fujita invariant

Definition

Let X be a smooth projective variety and let L be a Cartier divisor on X. We define the Fujita invariant $a(X, L)$ as

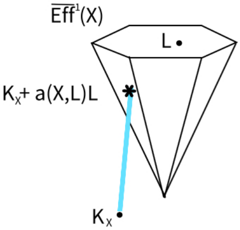
$$a(X, L) = \inf\{t \in \mathbb{R} \mid K_X + tL \in \overline{\mathrm{Eff}}^1(X)\}$$

or $a(X, L) = \infty$ if no such t exists.

When X is singular, we choose a resolution of singularities $\phi : X' \to X$ and define $a(X, L) = a(X', \phi^* L)$.

The Fujita invariant measures "how negative" K_X is with respect to the divisor L. In practice K_X will be an antiample divisor and L will be an ample divisor so that the Fujita invariant is positive.

Fujita invariant



Example

For a Fano variety X we have $a(X, -K_X) = 1$.

Example

If H is the hyperplane class on $\mathbb{P}^n$ then $a(X, H) = n + 1$.

Exceptional sets

Exceptional sets

Let X be a smooth Fano variety over a number field K. Fix an ample divisor L and choose an associated height function. Let $N(X, B)$ denote the number of K-points on X whose height is $\leq B$. Then Manin's Conjecture predicts that

$$N(X, B) \sim_{B \to \infty} cB^{a(X,L)} (\log B)^{b(X,L)-1}$$

where c denotes Peyre's constant and $b(X, L)$ is an invariant related to the Picard rank of X.

However, in order to obtain this expected growth rate, one might need to discount the rational points on special subvarieties $Y \subset X$ where the rational points grow too quickly. The subset of points we must remove is called the "exceptional set".

Exceptional sets

We can break Manin's Conjecture over a number field into two subproblems:

Subproblem 1: Identify the exceptional set
Subproblem 2: Bound the growth of the remaining points

Lehmann-Sengupta-Tanimoto propose a conjectural definition of the exceptional set using the Fujita invariant. Suppose that $f : Y \to X$ is a generically finite map such that $a(Y, f^*L) > a(X, L)$. Then the expected growth rate on Y is larger than the expected growth rate on X. Thus $f(Y(K))$ should be included in the exceptional set. We conjecture that the exceptional set consists of all the rational points of this type.

(We sometimes also need to discount the points for morphisms $f : Y \to X$ satisfying $a(Y, f^*L) = a(X, L)$.)

Exceptional sets

For Fano varieties X, we will always use the natural polarization $L = -K_X$. When classifying irreducible components of $\mathrm{Mor}(\mathbb{P}^1, X)$ we should also divide into two subproblems.

Subproblem 1: Identify the exceptional set
Let us say that a family of rational curves on X is "pathological" if it is "too large" in some way. We expect the existence of pathological families of rational curves on a Fano variety X to be controlled by the Fujita invariant.

Subproblem 2: Bound the growth of remaining curves
We would like to find some systematic structure for the remaining families of rational curves on X. Ideally this structure would allow us to use induction to classify the components parametrizing large degree rational curves.

For a Fano variety X there are several ways in which a family of rational curves can be "pathological". These include:

Possibility 1: M is a component of $\mathrm{Mor}(\mathbb{P}^1, X)$ which has larger than the expected dimension.

Possibility 2: for a numerical class $\alpha \in N_1(X)_{\mathbb{Z}}$ there are "too many" irreducible components of $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$. We hope that there is a universal upper bound on the number of components of $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$, but in pathological situations this hope can fail.

Remember, we hope that both types of pathologies can be "explained" by the presence of a morphism $f : Y \to X$ such that $a(Y, -f^*K_X) \geq a(X, -K_X)$.

Exceptional sets

It turns out that the Fujita invariant controls the expected dimension of components of $\mathrm{Mor}(\mathbb{P}^1, X)$.

Theorem (Lehmann-Tanimoto)

Let X be a Fano variety. Suppose that M is a component of $\mathrm{Mor}(\mathbb{P}^1, X)$ that does not generically parametrize free curves. (In particular this includes all components with larger than expected dimension.) Then the curves parametrized by M sweep out a closed subvariety $Y \subset X$ such that $a(Y, -K_X) > a(X, -K_X)$.

A result of Hacon-Jiang shows that the union of the subvarieties $Y \subset X$ with larger a-invariant is a proper closed subset of X. It is sometimes possible to compute this set explicitly using techniques from the Minimal Model Program.

The Fujita invariant also controls the fibers of the evaluation map.

Theorem (Lehmann-Tanimoto)

Let X be a Fano variety. Suppose that M is a component of $\mathrm{Mor}(\mathbb{P}^1, X)$ such that the evaluation map for the universal family over M is dominant and does not have connected fibers. Then this evaluation map factors rationally through a generically finite morphism $f : Y \to X$ of degree ≥ 2 such that $a(Y, -f^*K_X) = a(X, -K_X)$.

As we will see later it is useful to include such maps in the exceptional set.

Conjecturally the Fujita invariant also controls the existence of numerical classes with "too many" components of $\mathrm{Mor}(\mathbb{P}^1, X)$. We will demonstrate this with an example.

Example (Le Rudulier's example)

Let $X = \mathrm{Hilb}^2(\mathbb{P}^2)$. There is a degree 2 rational map

$$\phi : \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow X$$

which is not defined along the diagonal. We let W denote the blow-up of $\mathbb{P}^2 \times \mathbb{P}^2$ along the diagonal and denote by $f : W \to X$ the induced morphism.

Since $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) \cong \mathbb{Z}^2$, a numerical class of curves on this variety is determined by two integers (a, b). Note that $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^2 \times \mathbb{P}^2)_{(a,b)}$ is non-empty and irreducible whenever $a, b > 0$. We denote this irreducible component by $N_{a,b}$.

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Example

By composing with ϕ, each component $N_{a,b}$ yields a sublocus $\phi_* N_{a,b} \subset \mathrm{Mor}(\mathbb{P}^1, X)$. In fact, it turns out that $\phi_* N_{a,b}$ is dense in an irreducible component of $\mathrm{Mor}(\mathbb{P}^1, X)$.

However, $\phi_* N_{a,b}$ and $\phi_* N_{c,d}$ represent curves with the same numerical class whenever $a + b = c + d$. Thus, the number of components of $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$ will grow linearly as we increase the degree of α.

One can show that $a(W, -f^* K_X) = a(X, -K_X)$. Thus the presence of "too many components" of rational curves on X is explained by the existence of the map $f : W \to X$ which preserves the Fujita invariant. As predicted by Lehmann-Sengupta-Tanimoto, the contributions of W should be included in the exceptional set.

Structure of free curves

Structure of free curves

We next turn to the families of rational curves which are not in the exceptional set. The key conjecture is:

Conjecture

Let X be a Fano variety. For any "sufficiently positive" curve class α, the number of components of $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$ which are not contained in the exceptional set is exactly $|\mathrm{Br}(X)|$.

This conjecture is related to a heuristic of Batryrev for Manin's Conjecture over the function field of an $\mathbb{F}_q$-curve.

We have seen that every non-exceptional family of rational curves generically parametrizes free curves. In order to analyze this conjecture, we will identify some "inductive structure" for free rational curves.

Gluing-and-smoothing

Theorem

Let X be a smooth projective variety. Suppose $f_1 : \mathbb{P}^1 \to X$ and $f_2 : \mathbb{P}^1 \to X$ are free curves whose images intersect at a point. Let $f' : Z \to X$ denote the morphism whose domain Z is the union of two copies of $\mathbb{P}^1$ meeting at a single node and such that the restriction of f' to the two components of Z is f_1 and f_2.

Then there is a family of free curves $g_t : \mathbb{P}^1 \to X$ parametrized by an open subset T° of a curve T and a point $0 \in T \setminus T^\circ$ such that the limit of g_t as $t \mapsto 0$ is the morphism $f' : Z \to X$.

We refer to the construction of f' from f_1 and f_2 as "gluing" and the construction of g_t from f' as "smoothing". Thus, the "gluing and smoothing" operation allows us to construct new free curves from old ones.

Stable maps

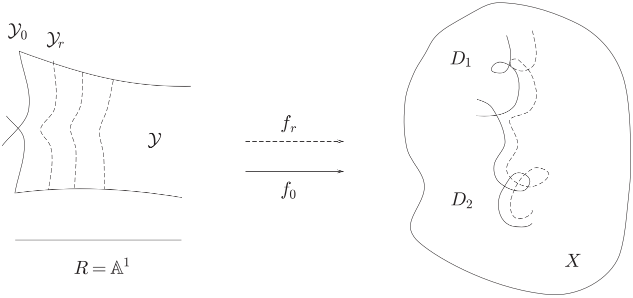

Fig. 2. Deforming the union of 2 free rational curves

Picture from Araujo and Kollár, "Rational curves on varieties"

Suppose that M_1, M_2 are components of $\mathrm{Mor}(\mathbb{P}^1, X)$ which generically parametrize free curves and such that the evaluation maps for the universal families have connected fibers. Then for any two free curves parametrized by M_1 and M_2 the result of "gluing and smoothing" lies in the same component $\widetilde{M}$ of $\mathrm{Mor}(\mathbb{P}^1, X)$. In other words:

Observation

The non-pathological components of $\mathrm{Mor}(\mathbb{P}^1, X)$ form a commutative monoid under the "gluing-and-smoothing" action.

One way to analyze the structure of free curves is to understand the properties of this monoid. Is it finitely generated? What is its rank? To answer these questions, the key question is: given a component $\widetilde{M}$, what are the possible ways of combining lower-degree components to get $\widetilde{M}$?

Using Bend-and-Break, one can show that a free curve with large degree deforms to a map with a reducible domain. However, as we discussed earlier Bend-and-Break does not give us much information about the resulting broken curve. The following conjecture predicts that we can "reverse" the gluing and smoothing operation.

Conjecture (Movable Bend-and-Break)

Let X be a Fano variety. There is some constant $Q = Q(X)$ such that for any component M of $\mathrm{Mor}(\mathbb{P}^1, X)$ that generically parametrizes free curves of anticanonical degree $\geq Q(X)$ the curves parametrized by M deform to a map $f : Z \to X$ such that Z has two components and the restriction of f to each component is a free curve.

In the case of Fano hypersurfaces (Riedl-Yang) and Fano threefolds (Beheshti-Lehmann-Riedl-Tanimoto) this conjecture has been verified and is a crucial part of the classification theory.